MORE ON DECOMPOSABLE 3-(12 , 6 , 4) DESIGNS

D.R. Breach
Department of Mathematics
and
A.R. Thompson
Computer Services Centre

University of Canterbury
Christchurch, New Zealand

Abstract. In a previous paper all non-isomorphic decomposable 3-(12 , 6 , 4) designs without repeated blocks were determined. These results are extended here by allowing repeated blocks. Under this condition there are 26 non-isomorphic decomposable 3-(12 , 6 , 4) designs of which 14 have repeated blocks. Key blocks and point permutations for models of these designs are given along with descriptions of their automorphism groups.

Introduction

A t-(v, k, λ) design D has as *blocks* subsets of size k taken from a *point set* of size v. The blocks must be such that each t-subset of the v points is also a subset of exactly λ blocks. A permutation on the points which maps the family of blocks onto itself is called an *automorphism* of D. The set of all automorphisms under successive applications forms the *automorphism group* of D for which we write *Aut* D.

If D is such that the family of blocks contains a proper sub-family corresponding to a t-(v, k, μ) design (with $\mu < \lambda$) then D is said to be *decomposable* (or *reducible*). In a previous paper [1] we listed all non-isomorphic decomposable 3-(12 , 6 , 4) designs without repeated blocks. Such designs can be decomposed into two 3-(12 , 6 , 2) designs. We found 12 such decomposable 3-(12 , 6 , 4) designs. In this paper we allow repeated blocks and in consequence we find a further 14 decomposable 3-(12 , 6 , 4) designs. We provide a model of each of the 26 decomposable 3-(12 , 6 , 4) designs and give descriptions of their automorphism groups by providing sets of generators in each case. To help with the identification of these groups we have referred to Coxeter and Moser [2] and to Hall and Senior [3].

Block types

A 3-(12 , 6 , 4) design has 44 blocks which always occur in complementary (and therefore disjoint) pairs [1], [4]. For a given block B_o let n_i be the number of

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blocks containing precisely i points of B_o. Then each block of the design is of one of three types [1], [4];

	n_0	n_1	n_2	n_3	n_4	n_5	n_6
Type AC :	1	1	5	30	5	1	1
Type B :	1	0	9	24	9	0	1
Type R :	2	0	0	40	0	0	2.

Thus blocks of Type R are always repeated. Blocks of Type AC can yield blocks of two different types if a restriction is made to a 2-(11,5,4) design by deleting a point χ and all blocks not on χ from the 3-design.

The methods used to determine the non-isomorphic designs with repeated blocks are similar to those used in [1]. In addition a computer program was used to list for each design all the point permutations of its automorphism group. Thus the order of *Aut D* was known in all cases *before* an attempt to describe it was made.

Models of decomposable 3-(12,6,4) designs

For each decomposable 3-(12,6,4) design the block set (of 44 blocks) can be partitioned into two sets of 22 blocks with each set corresponding to a 3-(12,6,2) design. All 3-(12,6,2) designs are isomorphic and have point cycles of length 11 in their automorphism groups. Since the blocks occur in complementary pairs any 3-(12,6,2) design can be constructed if just one block and one 11-point cycle are known. Thus to construct a decomposable 3-(12,6,4) design at most two distinct blocks and two 11-cycles need be specified. In Table I there are listed the key blocks and key permutations needed to produce models of the decomposable 3-(12,6,4) designs. In addition to the information given in this table it should be noted that;

- (i) designs 1 to 12 inclusive all have [1 3 4 5 9 11] as a second key block under the action of the second key permutation (0 1 2 3 4 5 6 7 8 9 10);
- (ii) designs 13 to 26 inclusive all have [1 2 3 4 5 12] as a repeated key block and all have σ =(1 9 3 4 5 6 7 8 2 10 11) as a second key permutation;
- (iii) if a block is in the design then its complement is also;
- (iv) key permutations are not necessarily elements of the automorphism group of the relevant design.

In Table II we list for each design the number of blocks of each type, the order of the automorphism group *Aut D* and the numbers and sizes of the point orbits. If a design has m orbits of size a, n orbits of size b, etc., then the entry in the point orbits column is $a^m b^n \ldots$. Thus $am + bn + \cdots = 12$ always and $m + n + \ldots$ is the number of non-isomorphic ways of restricting each 3-design to a 2-(11,5,4)

Table I

Design Number	Key blocks	Key permutations
1	[0 1 2 6 8 11]	(0 1 2 3 6 5 7 8 10 9 4)
2	[0 1 2 8 10 11]	(0 1 2 3 4 7 10 6 5 8 9)
3	[0 1 6 8 10 11]	(1 2 3 0 7 10 8 6 4 5 9)
4	[0 1 6 8 10 11]	(1 2 3 0 7 6 8 10 4 5 9)
5	[0 1 7 8 10 11]	(1 2 3 9 8 10 0 4 7 6 5)
6	[1 6 7 8 10 11]	(1 2 3 6 0 7 8 10 4 5 9)
7	[0 1 2 3 4 11]	(1 2 3 6 0 9 8 4 7 10 5)
8	[0 1 6 8 10 11]	(1 2 3 6 7 10 0 8 5 4 9)
9	[0 1 6 7 8 11]	(1 2 3 9 6 8 0 5 7 10 4)
10	[0 1 6 7 8 11]	(1 2 3 6 10 0 8 7 4 5 9)
11	[1 2 3 4 5 11]	(1 2 3 7 9 6 5 8 0 4 10)
12	[0 1 3 4 5 11]	(0 1 2 5 10 8 4 6 7 9 3)
13	[1 2 3 4 5 12]	(1 2 3 7 5 10 8 4 11 9 6)
14	[1 2 3 4 5 12]	(1 2 3 8 4 9 7 5 11 10 6)
15	[1 2 3 4 5 12]	(1 2 3 7 5 9 8 4 11 10 6)
16	[1 2 3 4 5 12]	(1 2 3 7 4 9 8 5 11 10 6)
17	[1 2 3 4 5 12]	(1 2 3 8 5 10 7 4 11 9 6)
18	[1 2 3 4 5 12]	(1 2 3 7 4 10 8 5 11 9 6)
19	[1 2 3 4 5 12]	(1 2 3 7 5 11 8 4 10 9 6)
20	[1 2 3 4 5 12]	(1 2 3 7 4 11 8 5 10 9 6)
21	[1 2 3 4 5 12]	(1 2 3 7 4 11 8 5 9 10 6)
22	[1 2 3 4 5 12]	(1 2 3 8 4 9 6 5 11 10 7)
23	[1 2 3 4 5 12]	(1 2 3 8 4 11 6 5 9 10 7)
24	[1 2 3 4 5 12]	(1 2 3 7 4 11 6 5 9 10 8)
25	[1 2 3 4 5 12]	σ; all blocks repeated
26	[1 2 3 4 5 12]	(1 2 3 7 4 11 10 5 9 6 8)

design. The sum of all the indices in the point orbits column is the number of non-isomorphic decomposable 2-$(11,5,4)$ designs arising as restrictions on the 3-designs. The total number of these is 95.

Notes on the automorphism groups

In these descriptions the term automorphism group refers to the set of point permutations only. For those designs with repeated blocks strictly speaking there are block permutations mapping repeated blocks onto each other but fixing every point. Automorphism groups having these block permutations as elements can be made from the groups described below by taking direct products with cyclic groups of order two. Dots between cycles help to show the point orbits.

Table II

Design Number	#AC Blocks	#B Blocks	#R Blocks	$\|Aut\ D\|$	Point Orbits
1	24	20	0	40	$2^1 10^1$
2	32	12	0	4	$2^4 4^1$
3	36	8	0	4	$1^2 2^3 4^1$
4	36	8	0	4	$1^2 2^3 4^1$
5	36	8	0	2	$1^2 2^5$
6	40	4	0	2	$1^2 2^5$
7	40	4	0	2	$1^4 2^4$
8	40	4	0	16	$4^1 8^1$
9	40	4	0	2	$1^4 2^4$
10	40	4	0	16	$4^1 8^1$
11	44	0	0	10	$2^1 5^2$
12	44	0	0	110	$1^1 11^1$
13	36	0	8	12	$1^1 2^1 3^1 6^1$
14	32	0	12	32	$4^1 8^1$
15	32	8	4	16	$4^1 8^1$
16	36	4	4	6	$1^1 2^1 3^1 6^1$
17	40	0	4	10	$1^2 5^2$
18	40	0	4	10	$2^1 10^1$
19	36	0	8	36	$3^2 6^1$
20	40	0	4	8	$2^2 4^2$
21	36	4	4	18	$3^2 6^1$
22	24	0	20	240	$2^1 10^1$
23	32	8	4	32	$4^1 8^1$
24	24	16	4	48	$2^1 4^1 6^1$
25	0	0	44	7920	12^1
26	0	40	4	1440	12^1

Design 1: Here $|Aut\ D| = 40$. $Aut\ D$ contains $\alpha = (0\ 2\ 8\ 9\ 1\ 11\ 5\ 10\ 7\ 4)$ and $\beta = (3\ 6).(1\ 4)(0\ 2\ 7\ 10)(5\ 9\ 8\ 11)$ for which $\alpha^{10} = \beta^4 = e$ with $\beta\alpha^5 = \alpha^5\beta$. If $S = \alpha^2$ and $T = \beta^2$ then $S^5 = T^2 = (ST)^2 = e$. Thus S and T generate a subgroup H order 20. In the notation of Coxeter and Moser [2, p. 134], $H = \langle 2,2,5 \rangle$ and H is ZS metacyclic. Since $a^5 \notin H$ and a^5 commutes with all elements of $Aut\ D$ we have$Aut\ D \cong H \times C_2$.

Designs 2,3,4: For each of these $|\ Aut\ D\ | = 4$ with $Aut\ D \cong C_2 \times C_2$. Pairs of generators are respectively

$$(4\ 7).(6\ 11).(0\ 9)(1\ 8), \quad (0\ 8)(1\ 9).(2\ 3)(5\ 10);$$
$$(7\ 9).(1\ 2).(4\ 8)(5\ 6), \quad (0\ 11).(1\ 2).(4\ 5)(6\ 8);$$
$$(1\ 2).(4\ 8)(5\ 6), \quad *(0\ 11).(7\ 9).(4\ 6)(5\ 8).$$

Designs 5,6,7,9: For each we have $|\ Aut\ D\ | = 2$. The non-trivial elements are $(0\ 1).(2\ 3).(4\ 7).(5\ 10).(6\ 11)$, $(0\ 1).(2\ 8).(3\ 5).(4\ 10).(6\ 9)$, $(0\ 5).(1\ 8).(3\ 7).(6\ 10)$ and $(0\ 11).(1\ 4).(2\ 5).(6\ 8)$ respectively.

Design 8: With $|\ Aut\ D\ | = 16$ the group is generated by $\alpha = (3\ 6\ 10\ 8).(0\ 2\ 5\ 11\ 1\ 9\ 7\ 4)$ and $\beta = (3\ 6)(8\ 10).(2\ 4)(5\ 7)(9\ 11)$. These satisfy $\alpha^8 = \beta^2 = (\alpha\beta)^2 = e$ so $Aut\ D \cong D_8$.

Design 10: The 16 elements of *Aut D* are generated by $\alpha = (0\ 11\ 8\ 6).(1\ 4\ 2\ 3\ 9\ 5\ 10\ 7)$ and $\beta = (6\ 11).(1\ 5\ 9\ 4)(2\ 3\ 10\ 7)$ which satisfy $\alpha^8 = \beta^4 = e$ and $\alpha^4\beta = \beta\alpha^4$. Also $(\alpha\beta)^2 = e$ and $(\alpha\beta)\alpha(\alpha\beta) = \alpha^3$ so in the notation of Coxeter and Moser [2, p. 134], $Aut\ D \cong \langle -2, 4\,|\,2 \rangle$.

Further *Aut D* has $1,5,6,4$ elements of order $1,2,4,8$ respectively. There is only one group of order 16 having this pattern of orders and that is #13 of the list of Hall and Senior [3]. In their notation $Aut\ D \cong \Gamma_3 a_2$.

Design 11: Here $|\ Aut\ D\ | = 10$. Generators are $\alpha = (0\ 1\ 4\ 11\ 10).(2\ 5\ 9\ 3\ 8)$ and $\beta = (1\ 10)(4\ 11).(2\ 3)(5\ 9)(6\ 7)$. Since $\alpha^5 = \beta^2 = (\beta\alpha)^2 = e$ we have $Aut\ D \cong D_5$.

Design 12: With ∞ as a fixed point and working modulo 11 the whole design can be developed cyclically from the starter blocks $[1\ 3\ 4\ 5\ 9\ \infty]$, $[2\ 6\ 7\ 8\ 10\ \infty]$ and their complements. *Aut D* is the group of linear transformations $\chi \to a\chi + b$, $a \not\equiv 0 \pmod{11}$. Also *Aut D* is 2-transitive on the points other than ∞.

Design 13: Here $|\ Aut\ D\ | = 12$. Since $\alpha = (10\ 11).(2\ 3\ 4).(1\ 7\ 5\ 6\ 12\ 9)$ and $\beta = (10\ 11).(3\ 4).(5\ 6)(7\ 12)(1\ 9)$ are generators with $\alpha^6 = \beta^2 = (\alpha\beta)^2 = e$ we have $Aut\ D \cong D_6$.

Design 14: The group *Aut D* of order 32 is generated by three generators $\alpha = (1\ 3\ 11\ 9).(2\ 5\ 6\ 8\ 4\ 12\ 7\ 10)$, $\beta = (1\ 3\ 11\ 9).(2\ 5\ 7\ 10\ 4\ 12\ 6\ 8)$, and $\gamma = (1\ 3)(11\ 9).(2\ 5)(4\ 12)(6\ 10)(7\ 8)$. For these $\alpha^8 = \beta^8 = \gamma^2 = e$, $\beta^2 = \alpha^6$ and $\beta\gamma\alpha = e$. Also *Aut D* has $1,15,8,8$ elements of order $1,2,4,8$ respectively. This pattern uniquely determines the group which is #44 of the table of Hall and Senior [3]. For this group they write $\Gamma_6\alpha_1$.

Design 15: The 16 elements of *Aut D* are generated by $\alpha = (3\ 5)(8\ 11).(1\ 2\ 4\ 12)(6\ 9\ 10\ 7)$, $\beta = (3\ 8)(5\ 11).(1\ 6\ 4\ 10)(2\ 9\ 12\ 7)$ and $\gamma = (3\ 8)(5\ 11).(1\ 7\ 4\ 9)(2\ 10\ 12\ 6)$. These satisfy $\alpha^4 = \beta^4 = \gamma^4 = e$, $\alpha^2 = \beta^2 = \gamma^2$, $\alpha\beta = \beta\alpha$ and $\gamma\beta = \beta\gamma$. There are 7 elements of order 2 and 8 elements of order 4; an element of order 4, β, commutes with all elements. The group #8 of Hall and Senior [3] is uniquely defined by these properties.

Also if $R = \beta\gamma$, $S = \beta\gamma\alpha$, $T = \beta\alpha^3$ then $R^2 = S^2 = T^2 = (ST)^4 = (TR)^4 = (RS)^4 = e$. Therefore $Aut\ D \cong <2,2,2>_2$ in the notation of Coxeter and Moser [2, p. 30].

Design 16: Here $Aut\ D$, of order 6, permutes the points $6, 8, 10$ in all possible ways so $Aut\ D \cong S_3$. Generators are $\alpha = (6\ 8\ 10).(1\ 4\ 5)(2\ 12\ 3)$ and $\beta = (8\ 10).(9\ 11).(1\ 2)(3\ 4)(5\ 12)$ for which $\alpha^3 = \beta^2 = e$ and $\beta\alpha\beta = \alpha^2$.

Design 17: In this case $|Aut\ D| = 10$. Two generators are $\alpha = (1\ 2\ 5\ 3\ 4).(6\ 9\ 7\ 11\ 8)$ and $\beta = (2\ 4)(3\ 5).(6\ 8)(9\ 11)$. For these $\alpha^5 = \beta^2 = (\alpha\beta)^2 = e$. It follows that $Aut\ D \cong D_5$.

Design 18: Here $|Aut\ D| = 10$ also, but $\alpha \in Aut\ D$ where $\alpha = (4\ 7).(1\ 6\ 12\ 10\ 2\ 9\ 3\ 8\ 5\ 11)$. Therefore $Aut\ D \cong C_{10}$.

Design 19: The group $Aut\ D$ has three point orbits two of which are $\{2,3,4\}$ and $\{8,10,11\}$. Any permutation on the first of these appears with any permutation on the second just once. Since $|Aut\ D| = 36$ it follows that $Aut\ D \cong S_3 \times S_3$. Generators are $\alpha = (3\ 4).(8\ 11\ 10).(1\ 6\ 12\ 9\ 5\ 7)$, $\beta = (2\ 4).(8\ 10\ 11).(1\ 6\ 5\ 7\ 12\ 9)$ and $\gamma = (2\ 3\ 4).(8\ 10).(1\ 6\ 5\ 9\ 12\ 7)$. These satisfy $\alpha^6 = \beta^6 = \gamma^6 = e$, $\alpha^4 = \beta^2$, $\gamma\alpha = \beta\gamma$, $\alpha\beta = \gamma^4$.

Designs 20: The 8 elements of $Aut\ D$ are generated by $\alpha = (1\ 12)(7\ 11).(2\ 3\ 5\ 4).(6\ 9\ 10\ 8)$ and $\beta = (1\ 12).(2\ 3)(4\ 5).(6\ 9)(8\ 10)$. Since $\alpha^4 = \beta^2 = (\alpha\beta)^2 = e$ we have $Aut\ D \cong D_4$.

Design 21: For this design $|Aut\ D| = 18$. Two of the three point orbits are $\{6,8,10\}$ and $\{7,9,11\}$. On the first $Aut\ D$ acts as C_3; on the second as S_3. Each permutation needed for the direct product of C_3 and S_3 is present. Consequently $Aut\ D \cong S_3 \times C_3$. To generate the group use $\alpha = (6\ 10).(7\ 9\ 11).(1\ 2\ 4\ 3\ 5\ 12)$ and $\beta = (6\ 8).(7\ 9\ 11).(1\ 3\ 4\ 12\ 5\ 2)$. For these $\alpha^6 = \beta^6 = e$ and $\alpha^2 = \beta^2$.

Design 22: Here $|Aut\ D| = 240$. The transposition (2 4) belongs to $Aut\ D$. Two elements fixing 2 and 4 are $R = (1\ 6\ 12\ 8\ 7)(5\ 11\ 10\ 9\ 3)$ and $T = (1\ 3\ 9)(5\ 7\ 8\ 12\ 6\ 10)$ for which $T^6 = R^5 = (RT)^2 = e$. Therefore R and T generate S_5 (Coxeter and Moser [2, p. 137]). Consequently $Aut\ D \cong S_5 \times C_2$.

Design 23: In this case $Aut\ D$, of order 32, can be generated by $\alpha = (2\ 9\ 4\ 11).(1\ 6\ 5\ 7\ 3\ 10\ 12\ 8)$, $\beta = (2\ 9\ 4\ 11).(1\ 7\ 12\ 6\ 3\ 8\ 5\ 10)$ and $\gamma = (2\ 9)(4\ 11).(1\ 6\ 3\ 10)(5\ 8\ 12\ 7)$. These satisfy $\alpha^8 = \beta^8 = \gamma^4 = e$, $\alpha^4 = \gamma^2$, $\alpha^2 = \beta^6$, $\alpha\beta = \beta\alpha$ and $\alpha\beta\gamma = \gamma\alpha\beta$. The centre of the group is generated by γ^2 and $\alpha\beta$ and has order 4. There are $1, 11, 12, 8$ elements of order $1, 2, 4, 8$ respectively. From these properties we find that $Aut\ D$ is isomorphic to #27 on the list of Hall and Senior [3]. They label it $\Gamma_3 c_1$.

Design 24: The 48 elements of *Aut D* are generated by $\alpha = (7\ 9\ 8).(1\ 2\ 3\ 12\ 4\ 5)$ and $\beta = (6\ 10).(7\ 9\ 11\ 8).(2\ 3\ 4\ 5)$. Here $\alpha^6 = \beta^4 = e$ and $\alpha^3\beta = \beta\alpha^3$. If $T = \alpha^2\beta$ and $S = \beta$ then $S^4 = T^2 = (ST)^3 = e$ so S_4 is a subgroup (Coxeter and Moser [2, p. 134]). This subgroup does not contain α^3 which commutes with all elements. Therefore $Aut\ D \cong S_4 \times C_2$.

Design 25: Every block in the design is repeated. The group is that of the well-known 3-transitive 3-(12, 6, 2) design which is the unique extension of the unique Hadamard 2-(11, 5, 2) design. For the 3-design $|\ Aut\ D\ | = 7920$.

Design 26: Here $|\ Aut\ D\ | = 1440$. Two elements are (3 9 4 10)(1 6 2 7 5 8 12 11) and (1 2)(3 4 5)(6 10 9 8 7) so *Aut D* is 1-transitive on points. The stabilizer $(Aut\ D)_{(12)}$ has two point orbits one of which is $\{1, 2, 3, 4, 5\}$. Each element of $(Aut\ D)_{(12)}$ contains a unique permutation of this set. Since $|\ (Aut\ D)_{(12)}| = 120$ we must have $(Aut\ D)_{(12)} \cong S_5$.

References

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