INCLUSION OF TREES IN THEIR ITERATED LINE GRAPHS

P.D. Chawathe
Center of Advanced Study in Mathematics
University of Bombay
Vidyanagari, Bombay
and
N.A. Joshi
Department of Mathematics
D.G. Ruparel College
Mahim, Bombay
INDIA

Abstract. Bauer and Tindell defined the graph invariant $\Lambda(G)$, for graphs G other than paths and the star $K_{1,3}$, to be the least n for which G embeds in the nth iterated line graph of G. They also proposed the problem of determining $\Lambda(T)$ for all trees T. In this note we completely solve this problem by showing that $\Lambda(T) = 3$ for any proper homeomorph T of $K_{1,3}$ and that $\Lambda(T) = 2$ for all trees T which are neither paths nor homeomorphs of $K_{1,3}$.

Unless otherwise noted, we assume the notation of the book by Harary [2]. A graph is a pair $G = (V, E)$ of sets with E a set of 2-element subsets of V; the elements of V are called vertices (or points) of G and the elements of E are called edges (or lines) of G. The edge with endpoints u and v is denoted by uv. By an embedding of graph G into graph H we mean an injective mapping from the vertex set of G into that of H such that adjacent vertices of G are mapped to adjacent vertices of H. Recall that the line graph $L(G)$ is defined to have as vertex set the edge set of G, with two distinct edges being adjacent if and only if they have a common endpoint. The nth iterated line graph is denoted $L^n(G)$. Bauer and Tindell [1] noted that for every graph G other than a path and $K_{1,3}$, there is an integer n for which G embeds in $L^n(G)$, and defined $\Lambda(G)$ to be the least such n. They then determined all graphs G with $\Lambda(G) = 1$, and proposed that $\Lambda(T)$ be determined for all trees T other than paths and $K_{1,3}$. Herein we solve this problem by first showing that $\Lambda(T) = 2$ for all trees T which are neither paths nor homeomorphs of $K_{1,3}$. If T is a proper homeomorph of $K_{1,3}$, then $L^2(T)$ has fewer vertices than T and thus $\Lambda(T) > 2$. We conclude by showing that $\Lambda(T) = 3$ in this case.

If $G = (V, E)$ is a graph, we will denote by $P_i(G)$ the set of all subgraphs of G isomorphic to the path of length i. Note that we may identify in a natural way $P_0(G)$ with V and $P_1(G)$ with E. We will denote an element of $P_i(G)$ by

---

During the preparation of this paper, the second author was a teacher fellow at the Centre of Advanced Study in Mathematics, University of Bombay, under a University Grants Commission (New Delhi) Scheme.

writing the vertices (without commas) in order of traversal from one endpoint to the other: $u_0 u_1 \dots u_i$. Notice that since we are considering the elements as subgraphs (rather than walks), $u_0 u_1 \dots u_i$ and $u_i u_{i-1} \dots u_0$ denote the same element of $P_i(G)$. The line graph $L(G)$ has vertex set equal to the edge set of G, with two such edges being adjacent in $L(G)$ if and only if they have a common endpoint in G. Thus, two edges of G are adjacent as vertices of $L(G)$ if and only if their union (as paths) is a path of length 2. Since a path of length 2 has a unique representation as the union of two length-1 paths, we may view the line graph as having vertex set $P_1(G)$ and edge set $P_2(G)$. By the preceding, we see that the vertices of $L^2(G)$ may be identified with $P_2(G)$, and two paths are adjacent as vertices of $L^2(G)$ precisely when they intersect in an edge. Notice that the edges of $L^2(G)$ may not be identified with length-3 paths of G, since the union of two length-2 paths of G which intersect in an edge of G may be isomorphic to $K_{1,3}$.

A graph embedding $\varphi: V(G) \to V(L^2(G)) = P_2(G)$ of G into $L^2(G)$ is an *incidence embedding* of G into $L^2(G)$ if every $u \in V(G)$ lies on the length-2 path $\varphi(u)$.

Lemma 1. *If u is a leaf of tree T and φ is an incidence embedding of $T-u$ into $L^2(T-u)$, then φ extends to an incidence embedding of T into $L^2(T)$.*

Proof: Let v be the unique point of $T-u$ adjacent in T with u. By the definition of incidence embedding, there is a vertex w of $T-u$ such that vw is an edge of $\varphi(v)$. If we extend φ by defining $\varphi(u)$ to be uvw, then the result is the desired incident embedding of T into $L^2(T)$. ∎

We now define a **minimal tree** to be a tree which is neither a path nor a homeomorph of $K_{1,3}$, but is such that the removal of any leaf results in a tree which is either a path or a homeomorph of $K_{1,3}$. It is obvious that removal of a leaf from a minimal tree cannot, in fact, result in a path, and, hence, must result in a homeomorph of $K_{1,3}$. We will refer to the tree on 6 vertices with two vertices of degree 3 and 4 vertices of degree 1 as the H-graph. The following lemma is easily established.

Lemma 2. *T is a minimal tree if and only if T is $K_{1,4}$ or a homeomorph of the H-graph in which every leaf is adjacent to a degree-3 point.*

Theorem 1. *If T is a tree which is not a path and not a homeomorph of $K_{1,3}$, then there is an incidence embedding φ of T into $L^2(T)$.*

Proof: In view of Lemma 1, we need only prove that the theorem holds for minimal trees. By Lemma 2, we need consider only two cases. For the first case, assume $T = K_{1,4}$ and let c, u, v, w, x be the vertices of T, with c having degree 4. Then the desired incidence embedding is defined by $\varphi(c) = ucw$, $\varphi(u) = ucv$, $\varphi(v) = vcw$, $\varphi(w) = wcx$, and $\varphi(x) = xcu$. The remaining case is where T is a homeomorph of the H-graph in which every leaf is adjacent to a degree-3 point of T. Let c_1 and c_2 be the degree-3 points of T. Next let u_i, v_i be the leaves of T, and w_i the nonleaf of T, adjacent to c_i in T, $i = 1, 2$. We define the desired incidence embedding φ as follows. For each degree-2 point x, $\varphi(x)$ is the unique length-2 path of T in which x has degree 2. As for the other points, $\varphi(u_i) = u_i c_i v_i$, $\varphi(v_i) = v_i c_i w_i$, $\varphi(c_i) = u_i c_i w_i$ $(i = 1, 2)$. It is straightforward to verify that φ is an incidence embedding of T into $L^2(T)$, so the proof is complete. ∎

Theorem 2. *If T is a homeomorph of $K_{1,3}$ other than $K_{1,3}$, then there is an embedding of T into $L^3(T)$.*

Proof: Let c be the vertex of T with degree 3. Since $T \neq K_{1,3}$, we may choose a degree-2 vertex x adjacent to c in T; let u and v be the other vertices adjacent to c in T. Let y be the vertex adjacent to x with $y \neq c$. To define an embedding φ of T into $L^3(T)$, we need to map the vertices of T into the edges of $L^2(T)$, which are the vertices of $L^3(T)$. Recall that an edge of $L^2(T)$ consists of a pair of distinct length-2 paths of T which intersect in an edge; two such path pairs are adjacent as vertices of $L^3(T)$ if there is exactly one path common to the two pairs.

We first specify the value of φ on the special vertices named so far: $\varphi(c) = ucx, vcx$; $\varphi(u) = ucx, ucv$; $\varphi(v) = vcu, vcx$; $\varphi(x) = ucx, cxy$; and $\varphi(y) = cxy, vcx$. Now consider a vertex $z \notin \{u, v, c, x, y\}$ such that the unique path $zz_2 \ldots z_{t-1}c$ in T from z to c contains x. Then z has distance at least three from c, and we define $\varphi(z) = zz_2z_3, z_2z_3z_4$; $\varphi(z)$ is clearly outside the set $\varphi(\{u, v, c, x, y\})$, so the map as defined so far is injective. Now consider a vertex $z \notin \{u, v, c, x, y\}$ such that the path from z to c does not contain x. Then z has distance at least 3 from x, so we may write the unique $z - x$ path in T as $zz_2 \ldots z_{t-1}x$ and as before define $\varphi(z) = zz_2z_3, z_2z_3z_4$. This completes the definition of $\varphi: V(T) \to V(L^3(T))$. It is a straightforward matter to verify that φ is injective and preserves adjacencies, so that φ is an embedding of T into $L^3(T)$ as desired. ∎

Acknowledgement.

The authors wish to express their sincere thanks to the referee for the improvements in the presentation of the paper.

References

1. D. Bauer and R. Tindell, *Graphs isomorphic to subgraphs of their line graphs*, Discrete Mathematics **41** (1982), 1-6.
2. F. Harary, "Graph Theory," Addison-Wesley, 1969.