Divisibility Properties of Some Number Arrays[1]

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Abstract. For all nonnegative integers i, j let $q(i,j)$ denote the number of all lattice paths in the plain from $(0,0)$ to (i,j) with steps $(1,0)$, $(0,1)$, and $(1,1)$. In this paper it is proved that

$$q(i_n p^n + \ldots + i_0, j_n p^n + \ldots + j_0) \equiv q(i_n, j_n) \ldots q(i_0, j_0) \pmod p$$

where p is an odd prime and $0 \leq i_k < p$, $0 \leq j_k < p$. This relation implies a remarkable pattern to the divisibility of the array of numbers $q(i,j)$.

1. Motivation.

Suppose that each square of the chessboard is represented by an ordered pair (i,j) of nonnegative integers. The chessboard is infinitely large in the sense that the coordinates i and j can be arbitrary nonnegative integers. The starting position of the king is the point $(0,0)$. Suppose that on such a chessboard the king can move in three directions only:

$$(i,j) \to (i+1,j), (i,j) \to (i,j+1), (i,j) \to (i+1,j+1) \qquad (1)$$

Let $q(i,j)$ denote the number of all different paths in which the king can reach the point (i,j). We will solve the following problems (compare [1, 2, 3, 7]):

1 In how many ways can the king reach the square (i,j) in exactly k moves?
2 In how many ways can the king reach the square (i,j) in general? (find the number $q(i,j)$).
3 Display the divisibility properties of numbers $q(i,j)$.

2. Recurrence relations.

Let $M(i,j,k)$ denote the number of all different ways in which the king can move from the square $(0,0)$ to the square (i,j) in exactly k steps of the form (1). If we put $m = \max\{i,j\}$ then the following relations are clear: $M(i,j,k) = M(j,i,k)$, $q(i,j) = q(j,i)$, $M(i,0,k) = \delta_{ik}$ and $q(i,j) = \sum_{k=m}^{i+j} M(i,j,k)$.

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If $k < m$ or $k > i+j$ then $M(i,j,k) = 0$, where δ_{ik} denotes the Kronecker delta. An easy combinatorial argument gives the following recurrence formulas:

$$M(i,j,k) \\ = M(i-1,j,k-1) + M(i,j-1,k-1) + M(i-1,j-1,k-1) \tag{2}$$

$$q(i,j) = q(i-1,j) + q(i,j-1) + q(i-1,j-1) \tag{3}$$

for $i \geq 1, j \geq 1, k \geq 1$. For each nonnegative integer i we introduce the generating function $G_i(x) = \sum_{j=0}^{\infty} q(i,j)x^j$ and from (3) we get (see [1] for details)

$$G_i(x) = (1+x)^i(1-x)^{-i-1}. \tag{4}$$

The generating function $G(x,y)$ in two formal variables x, y of the numbers $q(i,j)$ is defined by

$$G(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q(i,j)x^i y^j \tag{5}$$

and from (4) we get

$$G(x,y) = \frac{1}{1-x-y-xy}. \tag{6}$$

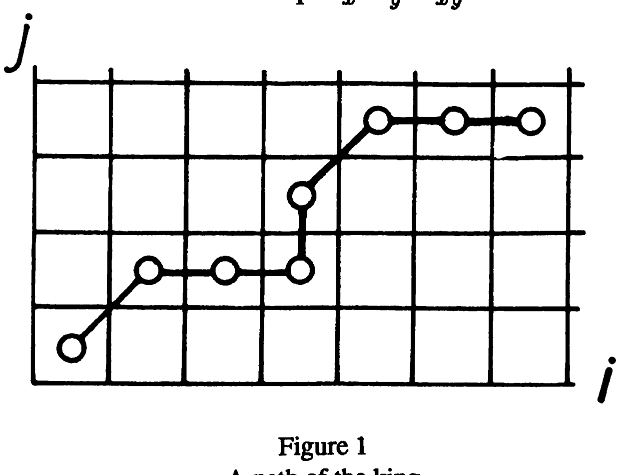

Figure 1
A path of the king

If we use the same indexing scheme also for array of numbers $M(i,j,k)$ then we can construct step by step from the relation (2) the tables for $M(i,j,k)$:

$$M(i,j,2): \begin{pmatrix} 1 & 2 & 1 \\ & 2 & 2 \\ & & 1 \end{pmatrix} \quad M(i,j,3): \begin{pmatrix} 1 & 3 & 3 & 1 \\ & 3 & 6 & 3 \\ & & 3 & 3 \\ & & & 1 \end{pmatrix}.$$

In the same manner we obtain the table for the numbers $q(i,j)$:

1	9	41	129	321
1	7	25	63	129
1	5	13	25	41 .
1	3	5	7	9
1	1	1	1	1

Explicit formulas for $M(i,j)$ and $q(i,j)$ can be derived, for instance, by means of the formal power series and the umbral calculus (see, for example, [**4**, **6**, **8**]), but we give a simple combinatorial proof.

Therefore we adopt the convention: if n, m are integers then $\binom{n}{m} = 0$ for $0 < n < m$ or for $n > 0$, $m < 0$.

The king must arrive from $(0,0)$ at (i,j) in k moves, namely in $i + j - k$ diagonal steps, in $i - (i + j - k) = k - j$ horizontal, and in $j - (i + j - k) = k - i$ vertical steps. Therefore we obtain

$$M(i,j,k) = \frac{k!}{(i+j-k)!(k-j)!(k-i)!} = \binom{k}{i}\binom{i}{k-j}. \tag{7}$$

Since $q(i,j) = \sum_{k=m}^{i+j} M(i,j,k)$ we have an explicit formula for $q(i,j)$:

$$q(i,j) = \sum_{k=m}^{i+j} \binom{k}{i}\binom{i}{k-j}. \tag{8}$$

Formulas (7) and (8) give solutions of Problem 1 and 2 presented at the beginning. By a simple summation-index manipulation we get from (8) an attractive formula:

$$q(i,j) = \sum_{k=0}^{\mu} \binom{i}{k}\binom{i+j-k}{i} = \sum_{k=m}^{i+j} \binom{k}{i}\binom{i}{i+j-k},$$

where $m = \max\{i,j\}$ as before and $\mu = \min\{i,j\}$.

Remark: If a, b, c are natural numbers, then we can find also the number of colored paths $x(i,j;a,b,c)$, where the king marks each horizontal, vertical, and diagonal step, choosing respectively any one of a colors, b colors, and c colors (cf. [1]). By the same combinatorial argument as by derivation of (3) we get the recurrence relation

$$\begin{aligned} &x(i,j;a,b,c) \\ &\quad = ax(i-1,j;a,b,c) + bx(i,j-1;a,b,c) + cx(i-1,j-1;a,b,c) \end{aligned} \tag{9}$$

for $i \geq 1, j \geq 1$ and similarly as (7) the explicit formula

$$x(i,j;a,b,c) = \sum_{k=m}^{i+j} \binom{k}{i}\binom{i}{k-j} a^{k-j} b^{k-i} c^{i+j-k}. \tag{10}$$

It is clear that $q(i,j) = x(i,j;1,1,1)$, moreover the solution of the difference equation (9) is given by (10) which can be derived also for arbitrary positive numbers a, b, c by using formal power series (see, for example, [1]).

3. The numbers $q(i,j)$ modulo p.

Here the divisibility properties of the numbers $q(i,j)$ are studied. Since $q(i,j)$ are odd numbers, some interesting results can be obtained for moduli $p > 2$. We suppose that p here is always an odd prime. The array of $q(i,j)$ with the boundary conditions $q(0,j) = q(i,0) = 1$ can be produced very quickly from the recurrence formula (3). For $p = 3$ we have the array of remainders of dividing $q(i,j)$ by 3 for $0 \le i \le 17$, $0 \le j \le 17$:

$$
\begin{array}{cccccccccccccccccc}
1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 \\
1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\
1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Denote by $q(i,j)_p$ the numbers $q(i,j)$ modulo p and introduce for $p = 3$ the matrix

$$
A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.
$$

The standard tensor (Kronecker product) in Z_3, where Z denotes the ring of integers, gives

$$
A \otimes A = \begin{bmatrix} 1A & 2A & 1A \\ 1A & 0A & 2A \\ 1A & 1A & 1A \end{bmatrix} = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
1 & 2 & 1 & 0 & 0 & 0 & 2 & 1 & 2 \\
1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\
1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.
$$

This is exactly the array of $q(i,j)_3$ for $0 \leq i \leq 8, 0 \leq j \leq 8$. In the same way we construct $A \otimes A \otimes A, A \otimes A \otimes A \otimes A, \ldots$ and we get the arrays $q(i,j)_3$ for $0 \leq i \leq 3^k - 1,\ 0 \leq j \leq 3^k - 1$ for $k = 3, 4, \ldots$ respectively. We shall generalize this construction for an arbitrary odd prime p.

Theorem 1. *Let p be an odd prime. Then the following relations hold over Z_p:*

1. $q(p,i) = 1,\ 0 \leq i < p;$
2. $q(p-1,i) + q(p-1,i-1) = 0,\ 1 \leq i < p;$
3. $\sum_{k=0}^{p-1} q(k,j) = 0,\ 0 \leq j < p-1;$
4. $\sum_{k=0}^{p-1} q(k,p-1) = 1.$

Proof: During the proof we always calculate in Z_p.

1. We need the fact that

$$\binom{p+k}{i} = \binom{k}{i}$$

for each prime p and each nonnegative integer i, $i < p$, (see, for example, [9]). Suppose that $0 \leq i < p$, where p is an odd prime. Because of (8) we have

$$q(p,i) = \sum_{k=p}^{i+p} \binom{k}{i}\binom{i}{k-p} = \sum_{k=0}^{i} \binom{p+k}{i}\binom{i}{k} = \sum_{k=0}^{i} \binom{k}{i}\binom{i}{k} = 1.$$

2. From the recurrence formula (3) we obtain for $1 \leq i < p$:

$$q(p,i) = q(p,i-1) + q(p-1,i) + q(p-1,i-1).$$

The assertion 1 implies $q(p,i) = 1$ and $q(p,i-1) = 1$. Thus $q(p-1,i) + q(p-1, i-1) = 0$.

3. Denote by S_j the sum

$$\sum_{k=0}^{p-1} q(k,j),\ 0 \leq j \leq p-2.$$

From the recursion formula for $q(i,j)$ we get by an easy calculation the relation

$$S_j + S_{j-1} = S_j - S_{j-1}.$$

This relation implies $2S_{j-1} = 0$ and finally

$$S_0 = S_1 = S_2 = \ldots = S_{p-2} = 0.$$

4. Since p is an odd prime, we have

$$q(0,p-1) + q(1,p-1) + q(2,p-1) + \ldots + q(p-2,p-1) + q(p-1,p-1)$$
$$= 1+(q(1,p-1)+q(2,p-1))+\ldots+(q(p-2,p-1)+q(p-1,p-1))=1$$

because of 2. ∎

The consequence of this theorem is the following form of the array of $q(i,j)_p$ for $0 \leq i < p,\ 0 \leq j < p$:

$$\begin{array}{cccccc} 1 & p-1 & 1 & p-1 & \ldots & 1 \\ 1 & \star & \star & \star & \ldots & p-1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 3 & \star & \star & \ldots & p-1 \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{array}.$$

By a fixed prime p the whole array of $q(i,j)_p$ is determined. Note that we choose always $0 \leq q(i,j)_p < p$. The origin for the subscripts i,j is at the lower left corner. We introduce the matrix A with the entries a_{ij} by

$$a_{ij} = q(p-1-i,j)_p \tag{11}$$

for $0 \leq i < p,\ 0 \leq j < p$.

Theorem 2. *For every odd prime p the following assertions hold in Z_p:*

1. $\sum_{k=0}^{p-1} a_{ik} = \sum_{k=0}^{p-1} a_{kj} = 0,\ 1 \leq i \leq p-2,\ 0 \leq j \leq p-2;$
2. $\sum_{k=0}^{p-1} a_{0k} = 1,\ \sum_{k=0}^{p-1} a_{k\,p-1} = 1;$
3. $A^2 = I,$ *where I denotes the unit matrix.*

Proof:

1. Assertions 1 and 2 are direct consequences of Theorem 1. It is easy to prove that over Z_p $q(i,j) = q(i,p+j) = q(i,2p+j) = \ldots$. If $B = A^2$ then for $0 \leq i < p,\ 0 \leq j < p$

$$b_{ij} = \sum_{k=0}^{p-1} a_{ik}a_{kj} = \sum_{k=0}^{p-1} q(p-1-i,k)q(p-1-k,j).$$

For integers u and v using (4) we derive in the sense of formal power series

$$(1+x)^{u+v}(1-x)^{-u-v-2} = (1-x)^{-1}\sum_{k=0}^{\infty} q(u+v,k)x^k$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j} q(u,k)q(v,j-k)x^j$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{j} q(u+v,k)x^j.$$

Thus

$$\sum_{k=0}^{j} q(u,k)q(v,j-k) = \sum_{k=0}^{j} q(u+v,k)$$

for every integer j. The substitution $j = p-1, \quad u = p-1-i, \quad v = j$ for $0 \le i < p, 0 \le j < p$ yields

$$\sum_{k=0}^{p-1} q(p-1-i,k)q(j,p-1-k) = \sum_{k=0}^{p-1} q(p-1-i+j,k) = \delta_{ij}$$

using Theorem 1. It follows that $b_{ij} = \delta_{ij}$ or equivalently $B = I$. ∎

For each odd prime p the minimal polynomial of A over Z_p, is $p(\lambda) = \lambda^2 - 1 = \lambda^2 + p - 1$.

Theorem 3. *For each odd prime* p *the following relation holds for every pair of nonnegative integers* i, j:

$$q(i,j) \equiv q(i_n,j_n)q(i_{n-1},j_{n-1})\dots q(i_0,j_0) \pmod{p}, \qquad (12)$$

where $i = i_n p^n + i_{n-1}p^{n-1} + \dots + i_0$, $j = j_n p^n + j_{n-1}p^{n-1} + \dots + j_0$, $0 \le i_k < p$, $0 \le j_k < p$.

Proof: We write $i = ap+b$, $j = cp+d$, where a, c are quotients and b, d remainders of dividing i, j by p respectively. Since $0 \le b < p$, $0 \le d < p$ the numbers $q(b,d)$ are elements of the matrix A. We will prove that in this case over Z_p $q(i,j) = q(a,c)q(b,d)$. Since in the field Z_p $(a+b)^p = a^p + b^p$, then $(f(x,y) + g(x,y))^p = f(x,y)^p + g(x,y)^p$ for any two formal power series $f(x,y), g(x,y)$ with coefficients in Z_p.

For $i = 0, 1, \dots, p-1$ we introduce the polynomials $S_i(x)$ by

$$S_i(x) = \sum_{j=0}^{p-1} q(i,j)x^j;$$

specially, from this definition we get

$$S_0(x) = 1 + x + x^2 + \dots + x^{p-2} + x^{p-1}$$

and from Assertion 2 of Theorem 1

$$S_{p-1}(x) = 1 - x + x^2 - \dots - x^{p-2} + x^{p-1}.$$

From (3) we get for $1 \le i \le p-1$ a new recurrence formula

$$(1-x)S_i(x) = (1+x)S_{i-1}(x).$$

Introducing the sum $S(x,y)$ by

$$S(x,y) = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} q(i,j)x^i y^j$$

we see that

$$S(x,y) = \sum_{i=0}^{p-1} S_i(y)x^i$$

and an easy calculation gives

$$S(x,y) = S_0(y) + \frac{x(1+y)}{1-y}S(x,y) - \frac{x^p(1+y^p)}{1-y}.$$

Over the field Z_p we finally get

$$S(x,y) = \sum_{k=0}^{p-1}(x+y+xy)^k.$$

From the generating function (5) we find that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q(i,j)x^i y^j = \sum_{n=0}^{\infty}(x+y+xy)^n = \sum_{j=0}^{\infty}\sum_{k=0}^{p-1}(x+y+xy)^{jp+k}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{p-1}(x^p+y^p+x^p y^p)^j (x+y+xy)^k$$

$$= \sum_{d=0}^{\infty}\sum_{c=0}^{\infty} q(a,c)x^{cp}y^{pc}\sum_{b=0}^{p-1}\sum_{d=0}^{p-1} q(b,d)x^b y^d.$$

Equating coefficients of $x^i y^j$ shows that if $i = ap + b$ and $j = cp + d$ then in Z_p $q(i,j) = q(a,c)\,q(b,d)$. By induction we finally obtain the relation (12). ∎

This theorem implies that all information about the numbers $q(i,j)_p$ lies in the matrix A given by (11). Denote by $A^{(1)}, A^{(2)}, A^{(3)}, \ldots$ the matrices A, $A \otimes A$, $A \otimes A \otimes A, \ldots$ respectively. They are square-matrices of orders p, p^2, $p^3, \ldots$ respectively and their entries are $q(i,j)_p$ determined from (12). In particular, for $p = 7$, we obtain:

$$A = A^{(1)} = \begin{bmatrix} 1&6&1&6&1&6&1\\ 1&4&5&0&2&3&6\\ 1&2&6&3&6&2&1\\ 1&0&4&0&3&0&6\\ 1&5&6&4&6&5&1\\ 1&3&5&0&2&4&6\\ 1&1&1&1&1&1&1 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 1A&6A&1A&6A&1A&6A&1A\\ 1A&4A&5A&0A&2A&3A&6A\\ 1A&2A&6A&3A&6A&2A&1A\\ 1A&0A&4A&0A&3A&0A&6A\\ 1A&5A&6A&4A&6A&5A&1A\\ 1A&3A&5A&0A&2A&4A&6A\\ 1A&1A&1A&1A&1A&1A&1A \end{bmatrix}.$$

We introduce in $A^{(2)}$ over Z_p the indexing of the blocks $B(i,j)$ as follows: $B(i,j) = q(i,j)A$. For $B(i,j)$ the same rule as for the numbers $q(i,j)$ is valid:

$$B(i,j) = B(i-1,j) + B(i,j-1) + B(i-1,j-1), \quad i \geq 1,\ j \geq 1,$$

with the boundary conditions $B(0,j) = B(i,0) = A$. This can be generalized for the matrices $A^{(k)}$ for $k \geq 2$. We say that the array of $q(i,j)_p$, is *self-similar*. If we mark black the points (i,j) for which $q(i,j)_p = 0$ (or equivalently $q(i,j)$ is divisible by p) we obtain pictures of a remarkable design. The picture was constructed by a computer for $0 \leq i \leq 175,\ 0 \leq j \leq 175$. For small p each point can be coloured differently according with the number $q(i,j)_p$. The obtained picture displays the divisibility properties of the numbers $q(i,j)$.

Let $z(k,p),\ k = 1,2,\ldots$ denote the number of zeros in the matrix $A^{(k)}$ for a given odd prime p.

Theorem 4. *For every odd prime p and $k \geq 1$ the number $z(k,p)$ of all zero entries of the matrix $A^{(k)}$ is given by the formula*

$$z(k,p) = p^{2k} - [p^2 - z(1,p)]^k. \tag{13}$$

Proof: Since $A^{(k+1)} = A \otimes A^{(k)}$ over Z_p we obtain a simple recurrence formula

$$z(k+1,p) = z(k,p)[p^2 - z(1,p)] + z(1,p)p^{2k}$$

and by induction we verify now (13) very easily.

For $p = 7$ we get $z(1,7) = 5,\ z(2,7) = 465$. Theorem 7 says that $A^2 = I$, moreover, we can prove that over Z_p in general

$$[A^{(k)}]^2 = I$$

for all $k \geq 0$. Specially, $[A^{(2)}]^2 = (A \otimes A)\,(A \otimes A) = A^2 \otimes A^2 = I \otimes I = I.$ ∎

The author found the ideas for the study of number array modulo a prime in [9, 10].

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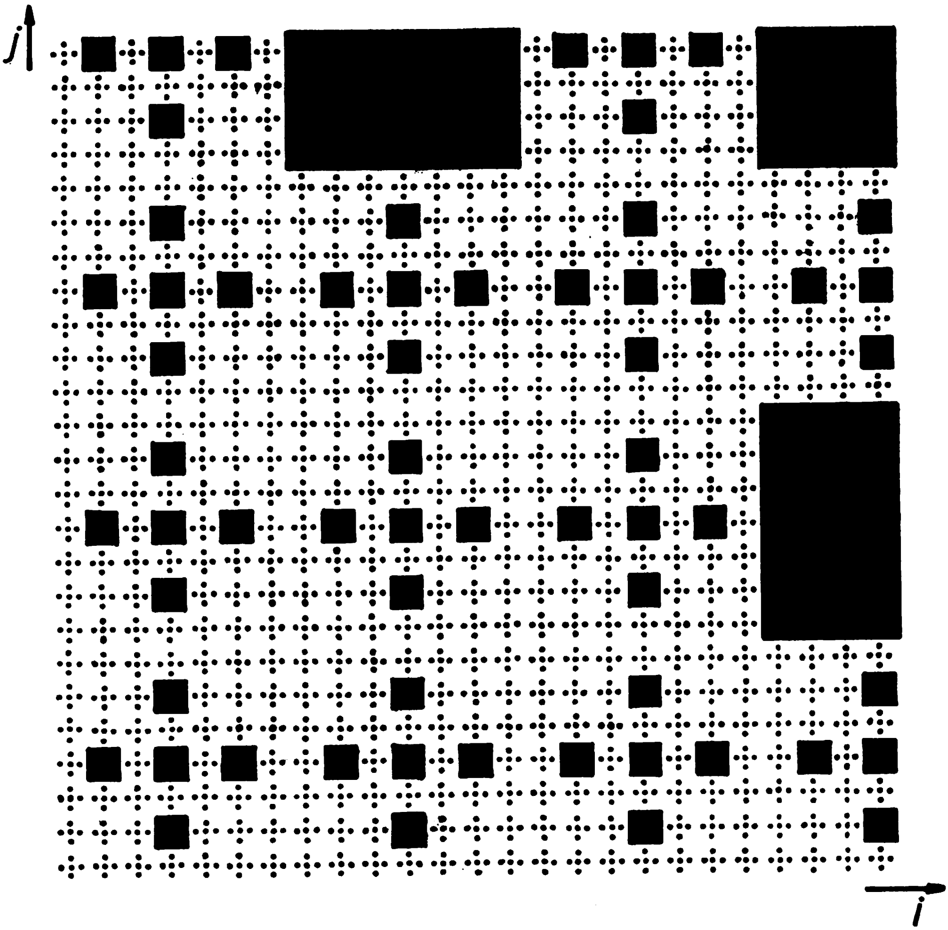

Figure 2
The pattern to the divisibility of numbers $q(i, j)$ by $p = 7$

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