A family of D-optimal designs

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Abstract. A construction is given of a family of D-optimal designs of order $n = 2v \equiv 2 \pmod 4$, where $v = 2q^2 + 2q + 1$ and q is an odd prime power. For $q > 3$ all the orders of D-optimal designs produced by this construction are new.

Ehlich [6] has shown that if $n \equiv 2 \pmod 4$, $v = n/2$ and M, N are $v \times v$ commuting matrices with elements ± 1 such that

$$MM^T + NN^T = (2v - 2) I_v + 2 J_v,$$

then the $n \times n$ matrix

$$D = \begin{bmatrix} M & N \\ -N^T & M^T \end{bmatrix}$$

has the maximum determinant among all $n \times n$ ± 1 matrices.

Such matrices D are called D-optimal designs of order n. As of the year 1987 their construction was known for the following values of n: 2, 6, 10, 14, 18, 26, 30, 38, 42, 46, 50, 54, 62, 66, 82, 86 (Ehlich [6], Yang [11], [12], [13], [14], [15], Chadjipantelis and Kounias [4], Chadjipantelis, Kounias and Moyssiadis [5], Kharaghani [7]).

In 1988 Koukouvinos, Kounias and Seberry [8] constructed the infinite family of D-optimal designs summarized in the following theorem.

Theorem 1. *There exist D-optimal designs of order $n \equiv 2 \pmod 4$, where*

$$n = 2v = 2(q^2 + q + 1)$$

and q is a prime power.

The cases $q = 2, 3, 4$ and 5 of this construction produce the already known orders $n = 14, 26, 42$ and 66. However, beginning with the case $q = 7$ which produces the order $n = 114$ all the other orders are new.

For further information on D-optimal designs see the interesting account in [8].

The purpose of this note is to establish the following supplement to Theorem 1.

Theorem 2. *There exist D-optimal designs of order* $n \equiv 2 \pmod 4$, *where*

$$n = 2v = 2(2q^2 + 2q + 1)$$

and q *is an odd prime power.*

Theorem 2 closely resembles Theorem 1 and produces additional orders of D-optimal designs. The case $q = 3$ produces the already known order $n = 50$ (see the constructions of Yang [14], and Chadjipantelis, Kounias and Moyssiadis [5]). Beginning with the case $q = 5$ which produces the order $n = 122$ all the other orders are new.

Although the case when q is a power of 2 is excluded in the statement of Theorem 2 it should be noted that the cases $q = 2$ and $q = 4$ produce the already known orders $n = 26$ and $n = 82$. The case $q = 8$ produces $n = 290$. There is no D-optimal design known of this order.

The proof of Theorem 1 in [8] makes use of supplementary difference sets whereas the proof of Theorem 2 in this note is based on a remarkable construction of symmetric block designs due to A. E. Brouwer [3]. A symmetric balanced incomplete block design SBIBD with parameters v, k, λ can be defined as a square $(0, 1)$-matrix of order v with k 1's in each row and column and with the inner product of a pair of distinct rows equal to λ. For details about the properties of such designs see the book by W. D. Wallis [10]. The construction of Brouwer is summarized in the following theorem.

Theorem 3. *There exist SBIBD's with parameters*

$$v = 2(q^h + q^{h-1} + \cdots + q) + 1,$$
$$k = q^h$$
$$\lambda = \frac{1}{2}q^{h-1}(q-1)$$

whenever q *is an odd prime power and* $h \geq 1$.

Corollary. *For* $h = 2$ *Theorem 3 states that there exist SBIBD's with parameters*

$$v = 2q^2 + 2q + 1, \quad k = q^2, \quad \lambda = \frac{1}{2}q(q-1)$$

whenever q *is an odd prime power.*

The possibility that the statement of the Corollary is also valid when q is a power of 2 is not precluded. The case $q = 2$ produces the parameters $(13, 4, 1)$ of the familiar finite projective plane of order 3. The case $q = 4$ produces the parameters $(41, 16, 6)$. Symmetric block designs with these parameters have been constructed by Bridges, Hall and Hayden [2] and by Trung [9]. The case $q = 8$

produces the parameters $(145, 64, 28)$. There is no SBIBD known with these parameters (see the book by Beth, Jungnickel and Lenz [1, p. 627]).

We now deduce Theorem 2 from the Corollary. Our method is motivated by the construction of Kharaghani [7] of a D-optimal design of order 82.

Let A be the incidence matrix of the design in the Corollary. Then A is a $(0,1)$-matrix of order v which satisfies the equation

$$AA^T = (k - \lambda) I + \lambda J.$$

The $(-1,1)$-incidence matrix of the design is given by $S = 2A - J$ where

$$SS^T = 4(k - \lambda) I + (v - 4(k - \lambda)) J$$

which reduces to

$$SS^T = 2(q^2 + q) I + J.$$

Thus the matrix

$$D = \begin{bmatrix} S & S \\ -S & S \end{bmatrix}$$

is a D-optimal design of order

$$n = 2v = 2(2q^2 + 2q + 1).$$

This completes the proof of Theorem 2.

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