A note on distar-factorizations

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Abstract. An $S_{s,t}$ distar-factorization of DK_m is an edge partitioning of the complete symmetric directed graph DK_m into subdigraphs each of which is isomorphic to the distar $S_{s,t}$ (the distar $S_{s,t}$ being obtained from the star $K_{1,s+t}$ by directing s of the edges into the centre and t of the edges out of the centre). We consider the question, "When can the arcs of DK_m be partitioned into arc-disjoint subgraphs each isomorphic to $S_{s,t}$?" and give necessary and sufficient conditions for $S_{s,t}$ distar-factorizations of DK_m in the cases when either $m \equiv 0$ or $1 \pmod{s+t}$.

1. Introduction

Over the last few years there has been considerable work done on the existence of $CD(m, n, c, \lambda)$-designs. These designs are edge- partitions of $\lambda K_m(n)$, the complete m-partite multigraph with n vertices in each part and in which each edge has multiplicity λ, into complete bipartite subgraphs $K_{1,c}$ (called claws or stars). (The acronym CD stands for claw-decomposition.)

The case first considered was $n = 1$ (when $\lambda K_m(1)$ is simply λK_m). Early progress on the problem was made by Cain [1] and later Yamamoto et al [12] gave necessary and sufficient conditions for the existence of $CD(m, 1, c, 1)$-designs (as did Huang [2]) and for the existence of $CD(m, 2, c, 1)$. These results were extended by Ushio, Tazawa and Yamamoto [11] who found necessary and sufficient conditions for the existence of $CD(m, n, c, 1)$-designs and by Tarsi [5] who gave necessary and sufficient conditions for the existence of $CD(m, 1, c, \lambda)$-designs.

There have been two generalizations of the problem; one of which has been completely solved. A $BCD(m, n, c, \lambda)$-design is a balanced $CD(m, n, c, \lambda)$-design where balance refers to the fact that each vertex lies in the same number of stars in the design. The question of the existence of these designs has been resolved by Ushio [9], [10] who gives necessary and sufficient conditions for their existence. (However, other authors [3], [4] had previously obtained results for special cases of the problem.)

A $PCD(m, n, c, \lambda)$-design is a partite $CD(m, n, c, \lambda)$-design if each subgraph $K_{1,c}$ of the design has the property that no two of its vertices lie in the same

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part of $\lambda K_m(n)$. Tazawa, Ushio and Yamamoto [7] have given necessary and sufficient conditions for the existence of $PCD(m, n, c, 1)$-designs.

We wish to propose yet another generalization, but first we require two definitions.

The directed star $S_{s,t}$ is obtained from the star $K_{1,s+t}$ by directing t of the edges out of the centre and s of the edges into the centre of the star. We refer to this directed graph as a distar. If A is a graph we will denote by DA the symmetric directed graph obtained from G by replacing each edge $\{x, y\}$ by the two arcs (x, y) and (y, x).

We are interested in the following question: When can the arcs of DK_m be partitioned into arc-disjoint subgraphs each isomorphic to $S_{s,t}$? We call such a partitioning an $S_{s,t}$ distar-factorization of DK_m. (More generally we might ask for the existence of $DCD(m, n, s, t, \lambda)$-designs; that is, for $S_{s,t}$ distar-factorizations of $\lambda DK_m(n)$.) Observe that we may assume $s \geq t$ since there is an $S_{s,t}$ distar-factorization of DK_m if and only if there is an $S_{t,s}$ distar-factorization of DK_m.

Necessary conditions are easily determined.

Theorem 1.1. *If DK_m has an $S_{s,t}$ distar-factorization, then*

(a) $m(m-1) \equiv 0 \pmod{s+t}$,

(b) $m \geq s+t+1$, *and*

(c) $\lfloor \frac{m-1}{s} \rfloor \geq \lceil \frac{m-1}{s+t} \rceil$.

Proof: Condition (a) comes from counting arcs in the distar and arcs in DK_m. Since each distar contains $s+t+1$ vertices, (b) follows. Each vertex has indegree $m-1$, and hence is the centre of at most $\lfloor \frac{m-1}{s} \rfloor$ distars. On the other hand there are $\frac{m(m-1)}{s+t}$ distars in the factorization and thus the average number of centres on a given vertex is $\frac{m(m-1)}{(s+t)m} = \frac{m-1}{s+t}$. Inequality (c) now follows. ∎

In this paper we give necessary and sufficient conditions for $S_{s,t}$ distar-factorizations of DK_m when $m \equiv 0$ or $1 \pmod{s+t}$.

2. $m \equiv 1 \pmod{s+t}$

In this section we will deal with the case $m \equiv 1 \pmod{s+t}$. The first results require no proof.

Lemma 2.1. *There is an $S_{s,0}$ distar-factorizations of DA if and only if A is regular of degree ks.*

Corollary 2.2. *There is an $S_{s,0}$ distar-factorizations of DK_m if and only if $m \equiv 1 \pmod{s}$.*

Lemma 2.3. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of DK_{s+t+1} if and only if K_{s+t+1} has an s-factor.*

Proof: Supppose that DK_{s+t+1} has an $S_{s,t}$ distar-factorization and consider the subdigraph H whose arcs are precisely those arcs which were directed into the centre of some distar in the factorization. This subdigraph is symmetric and each vertex has indegree a multiple of s. The number of distars in the factorization is $s+t+1$ and we will show that each vertex is the centre of exactly one distar implying that H is regular of indegree s and so corresponds to an s-factor in K_{s+t+1}.

Each vertex of DK_{s+t+1} has indegree $s+t \leq 2s$ and except when $t = s$ there can be at most one distar at each vertex (and hence exactly one). When $t = s$, if a vertex is the centre of two distars, then the remaining digraph is DK_{s+t} which has no $S_{s,t}$ subdigraph, a contradiction .

If K_{s+t+1} has a degree s subgraph H, put a distar at each vertex x of DK_{s+t+1} so that the endpoints of its in-arcs are the vertices adjacent to x in H. ∎

Corollary 2.4. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of DK_{s+t+1} if and only if s or t is even.*

Proof: By Lemma 2.3 we need only show that K_{s+t+1} has an s-factor if and only if at least one of s and t is even. The necessity of the conditions is obvious. For the sufficiency: if $s + t + 1$ is even take the union of s 1-factors in K_{s+t+1}, and if $s + t + 1$ is odd take the union of $s/2$ Hamilton cycles in K_{s+t+1}. ∎

Observe that Corollary 2.4 shows that the necessary conditions given in Theorem 1.1 are not always sufficient.

Lemma 2.5. *The complete symmetric bipartite digraph $DK_{q(s+t),r(s+t)}$ has an $S_{s,t}$ distar-factorization, $q \geq 1, r \geq 1$.*

Proof: Consider first the digraph $DK_{s+t,s+t}$. Since $K_{s+t,s+t}$ contains an s-factor (take the union of s 1-factors) and is regular of degree $s + t$, then by the argument of Lemma 2.3 it follows that $DK_{s+t,s+t}$ has an $S_{s,t}$ distar-factorization. The result now follows as $DK_{q(s+t),r(s+t)}$ is the union of qr copies of $DK_{s+t,s+t}$. ∎

Lemma 2.6. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of $DK_{3(s+t)+1}$ for all values of s and t.*

Proof: Label the vertices of $K_{3(s+t)+1}$ by the integers $0, 1, 2, \cdots, 3(s+t)$. Let T_i, $0 \leq i \leq t-1$, be the 6-regular subgraph defined by $E(T_i) = \{\{j, j+3i+1\}, \{j, j+3i+2\}, \{j, j+3i+3\} : 0 \leq j \leq 3(s+t)\}$ where all addition is modulo $3(s+t)+1$. Let $A = K_{3(s+t)+1} - \cup_{0 \leq i \leq t-1} E(T_i)$. Clearly A is a regular graph of degree $3(s-t)$ and so DA has an $S_{0,s-t}$ distar-factorization in which each vertex j is the centre of 3 distars which we denote by X_{1j}, X_{2j} and X_{3j}. To each of these distars we add exactly one in-arc and exactly one out-arc from each DT_i, $1 \leq i \leq t-1$, as follows: To X_{nj} add the arcs $(j, j+3i+n)$ and $(j+3i+n+1, j)$ where arithmetic is modulo 3 on the residues $j+3i+1$, $j+3i+2$, $j+3i+3$. This yields an $S_{s,t}$, $s \geq t$, distar-factorization of $DK_{3(s+t)+1}$ for all values of s and t. ∎

Theorem 2.7. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of $DK_{q(s+t)+1}$ for all values of s, t and q except in the case when $q = 1$ and both s and t are odd.*

Proof: If one of s and t is even observe that the arcs of $DK_{q(s+t)+1}$ can be partitioned into $q(q-1)/2$ copies of $DK_{s+t,s+t}$ and q copies of DK_{s+t+1} By Lemmas 2.3 and 2.5 we know that in this case both $DK_{s+t,s+t}$ and DK_{s+t+1} have $S_{s,t}$ distar-factorizations, and hence so too does $DK_{q(s+t)+1}$.

Suppose now that both s and t are odd. We know by Lemma 2.3 that there is no $S_{s,t}$ distar-factorization when $q = 1$. Lemma 2.3 also tells us that there is an $S_{2s,2t}$ distar-factorization of $DK_{2s+2t+1}$ and, since there is an $S_{s,t}$ distar-factorization of $S_{2s,2t}$, we have an $S_{s,t}$ distar-factorization of $DK_{2s+2t+1}$. So the case $q = 2$ is also resolved. The case $q = 3$ has already been taken care of in Lemma 2.6.

If q is even ($q = 2r$), then $DK_{q(s+t)+1} = DK_{r2(s+t)+1}$ the arcs of which can be partitioned into $r(r-1)/2$ copies of $DK_{2(s+t),2(s+t)}$ and r copies of $DK_{2(s+t)+1}$. Since each of these subdigraphs has an $S_{2s,2t}$ distar-factorization, the result follows.

If q is odd ($q = 2r+1$), then $DK_{q(s+t)+1} = DK_{(2r+1)(s+t)+1}$ the arcs of which can be partitioned into $(r-1)(r-2)/2$ copies of $DK_{2(s+t),2(s+t)}$, $r-1$ copies of $DK_{3(s+t),2(s+t)}$, $r-1$ copies of $DK_{2(s+t)+1}$ and one copy of $DK_{3(s+t)+1}$. Since each of these subdigraphs has an $S_{2s,2t}$ distar-factorization (Lemmas 2.5 and 2.6), the result follows. ∎

3. $m \equiv 0 \pmod{s+t}$.

We now turn our attention to the case $m \equiv 0 \pmod{s+t}$. In this case all will be dealt with in one theorem.

Theorem 3.1. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of $DK_{q(s+t)}$ for all values of s, t and q except when $q = 1$, and when $t = 0$.*

Proof: From Theorem 1.1(b) we know that $q \geq 2$, and from Theorem 1.1(c), if $t = 0$, then we must have $qs - 1 \equiv 0 \pmod{s}$ which is impossible.

Consider first the case when one of s and t is odd. Observe that the arcs of $DK_{q(s+t)}$ can be partitioned into $q(q-1)/2$ copies of $DK_{s+t,s+t}$ and q copies of each of DK_{s+t}. Observe that each copy of $DK_{s+t,s+t}$ has an $S_{s,t}$ distar-factorization so that each vertex of the $DK_{s+t,s+t}$ lies in exactly one $S_{s,t}$. Thus, so far, each vertex of $DK_{q(s+t)}$ lies in $q-1$ distars and the arcs remaining are exactly the arcs of the q copies of DK_{s+t}. Now $DK_{s+t} = DK_{s+t-1} + \{y\}$, where y is a vertex of DK_{s+t}. Since $DK_{s+t-1} = DK_{(s-1)+(t-1)+1}$ and one of s and t is odd, then DK_{s+t-1} has an $S_{s-1,t-1}$ distar-factorization. Let X be one of the $S_{s-1,t-1}$ distars so defined and let x be its centre. From the above discussion x is also the centre of some $S_{s,t}$ distar X' in one of the $DK_{s+t,s+t}$. Then in that distar there is an arc (x, z) where $z \notin V(X)$. Remove the arc (x, z) from X' and replace it with the arc (x, y). At the same time add the arcs (x, z) and (y, x) to X. Repeat this procedure in each copy of DK_{s+t} to obtain an $S_{s,t}$ distar-factorization of $DK_{q(s+t)}$.

The remaining case is when both s and t are even. First we will show that $DK_{2(s+t)}$ has an $S_{s,t}$ distar-factorization. Since $DK_{s+t} = DK_{(s-1)+t+1}$ then it has an $S_{s-1,t}$ distar-factorization. Again, the arcs of $DK_{2(s+t)}$ can be partitioned into two copies of DK_{s+t}, which we denote by G and H and one copy of $DK_{s+t,s+t}$. In each of G and H distinguish a vertex g and a vertex h respectively. Then the arcs of $DK_{s+t,s+t}$ can be partitioned into one copy of $DK_{s+t-1,s+t-1}$ denoted B, and all of the arcs to and from g and h. Observe also that $DK_{s+t-1,s+t-1}$ has an $S_{s,t-1}$ distar-factorization. Now, consider vertex $x \neq g$ in G. This vertex is the centre of an $S_{s-1,t}$ in G and the centre of an $S_{s,t-1}$ in B. To the $S_{s-1,t}$ add the arc (h, x) and to the $S_{s,t-1}$ add the arc (x, h). Thus both distars become $S_{s,t}$. We do the same for each vertex $y \neq h$ in H but use the arcs (g, y) and (y, g). There still remains an $S_{s-1,t}$ centred at g and another centred at h. To the one centred at g add the arc (h, g) and to the one centred at h add the arc (g, h). We now have an $S_{s,t}$ distar-factorization of $DK_{2(s+t)}$.

We next show that there is an $S_{s,t}$ distar-factorization of $DK_{3(s+t)}$ as these two results are then, as in Theorem 2.7, sufficient to obtain the more general result. Now $DK_{3(s+t)} = DK_{3(s+t)-1} + \{x\}$. From $K_{3(s+t)-1}$ delete a Hamilton cycle H. Then $K_{3(s+t)-1} - H$ is regular of degree $3(s+t) - 4$ and so has an $S_{3s-2,3t-2}$ distar-factorization (by the same argument as in Lemma 2.3) with one such distar centred at each vertex. The arcs of each $S_{3s-2,3t-2}$ can be partitioned into an $S_{s,t}$, an $S_{s-2,t}$ and an $S_{s,t-2}$. To each vertex of $DK_{3(s+t)-1}$ we associate exactly one of the two edges of the Hamilton cycle H which are incident with it. Then if vertex y has associated edge $\{y, y'\}$, we add the arcs (y', y) and (x, y) to the $S_{s-2,t}$ centred at y and the arcs (y, y') and (y, x) to the $S_{s,t-2}$ centred at y. We now have an $S_{s,t}$ distar-factorization of $DK_{3(s+t)}$. ∎

Using Theorem 3 .1 and the fact that $K_2(n)$ has a 1 -factorization we obtain the following corollary.

Corollary 3.2. *There is an $S_{s,t}$, $s \geq t$, distar-factorization of $DK_{q(s+t)}(n)$ for all values of n, s, t and q except when $q = 1$, and when $t = 0$.*

4. Remaining questions

The original question of the existence of $CD(m, n, c, \lambda)$-designs remains unresolved when both n and λ are at least 2. In only one of the generalizations has the problem been completely solved and that is for the existence of $BCD(m, n, c, \lambda)$-designs. The question of the existence of $PCD(m, n, c, \lambda)$-designs is unsolved for $\lambda \geq 2$, and the existence of $DCD(m, n, s, t, \lambda)$-designs is largely unresolved.

There is one other generalization which should be mentioned. This is the problem of finding edge-partitions of $\lambda K_m(n_1, n_2, \cdots, n_m)$, the complete m-partite multigraph with n_i vertices in the i'th part and in which each edge has multiplicity λ, into stars $K_{1,c}$. Yamamoto et al [12] resolved the problem for $\lambda = 1$ and $m = 2$ and Truszczyński [8] resolved it for $\lambda = 1$ and $m = 3$. Both Truszczyński [8] and Tazawa [6] have obtained further results on this particular problem.

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