Finding Nice Crystallizations for Handle Free N-Manifolds

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Abstract. The paper deals with combinatorial structures (pseudo-complexes, crystallizations) giving a direct link between the topology of triangulated manifolds and the theory of edge-coloured multigraphs. We define the concept of regular crystallization of a manifold and prove that every non-trivial handle free closed n-manifold has a regular crystallization. Then we study some applications of regular crystallizations and give a counter-example to a conjecture of Y. Tsukui [20] about strong frames of the 3-sphere.

1. Introduction.

K. Kobayashi and Y. Tsukui introduced the concept of ball coverings for connected compact n-manifolds in [12]. The second author showed in [20] how to represent ball coverings of closed 3-manifolds by means of edge-coloured graphs, called *generalized graphs*. Some properties of special generalized graphs (named *weak-* and *strong-frames*, respectively) were also studied in [20]. The main purpose of the quoted paper was to construct ball coverings with nice intersection properties for any handle free closed 3-manifold (see also Section 2). As a direct consequence of these results, Y. Tsukui stated a conjecture ([20], Conjecture 5.4) about strong-frames which represent the 3-sphere (Section 4).

In the present paper we relate frames to another graphical representation of manifolds (named *crystallization theory*), first introduced by M. Pezzana in [17].

Then we introduce the definition of *regular crystallization* for a closed connected n-manifold (Section 4). It partially extends to dimension n the notion of strong-frame of a closed 3-manifold. Regular crystallizations are proved to exist for any closed non-trivial handle free n-manifold. The proof, performed by using a constructive procedure, is completely different from the one given in [20] for the dimension 3. Therefore our proposition also implies the main theorem of [20] as a simple corollary. Then we can construct a counter-example to the above mentioned conjecture of Y. Tsukui by starting from a Heegaard diagram of the 3-sphere shown in [16]. Finally we study some applications of the concept of regular crystallization to minimum crystallizations, the Heegaard genus of a closed 3-manifold (Section 4) and combinatorial pseudo-handles in graphs (Section 5).

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2. Notation.

Let Δ_n be the set $\{0, 1, \ldots, n\}$ and $N_n = \Delta_n - \{0\}$. The symbol $\# X$ means the cardinality of the finite set X. Throughout the paper, we work in the piecewise linear (PL) category (see [8], [18]). All manifolds are connected and compact. The prefix PL will always be omitted.

If S^n denotes the n-sphere, then we will write $S^{n-1} \otimes S^1$ to indicate either the topological product $S^{n-1} \times S^1$ or the "twisted" sphere bundle $S^{n-1} \underset{\sim}{\times} S^1$ (see [19]). A closed n-manifold M is said to have a *(topological) handle* if M is homeomorphic to the connected sum $N \# (S^{n-1} \otimes S^1)$ for a suitable closed n-manifold N. Otherwise M is said to be a *handle free* n-manifold. We say that M is *trivial* (resp. *non-trivial*) if M is (resp. is not) the n-sphere S^n. A closed n-manifold M is said to be *prime* if $M = M_1 \# M_2$ implies that either M_1 or M_2 is trivial.

Let $G = (V, E)$ be a graph where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G, respectively. The term *graph* will be used instead of finite undirected multigraph without loops (multiple edges are allowed). As general reference for graph theory see [9].

If $e = (u, v)$ is a *directed edge* in E, then we also write $u = e(0)$ and $v = e(1)$. A *coboundary* of G is a set of edges $H \subset E(G)$ for which there exists a partition (V', V'') of V so that the elements of H are exactly the unique edges between V' and V''. In particular, the graph G is bipartite if and only if $E(G)$ is a coboundary. Given a non-empty finite set C, an *edge-colouration of G with colour set C* is a map $c: E(G) \longrightarrow C$ such that $c(e) \neq c(f)$ for any two adjacent edges $e, f \in E$. An $(n+1)$*-coloured graph with boundary* is a pair (G, c) where G is a graph and $c: E \longrightarrow \Delta_n$ is an edge-colouration of G with colour set Δ_n. A *boundary vertex* of G is a vertex of V whose degree is strictly less than $n+1$. If G has no boundary vertices, then (G, c) is simply called an $(n+1)$*-coloured graph*. For every subset $\Gamma \subset \Delta_n$, G_Γ is the subgraph $(V, c^{-1}(\Gamma))$. Each connected component of G_Γ is said to be a Γ*-residue*. We will call *i-coloured edges* the $\{i\}$-residues of G for each $i \in \Delta_n$. An $\{i, j\}$*-coloured cycle* is a cycle of the subgraph G_Γ, where $\Gamma = \{i, j\} \subset \Delta_n$. For each $i \in \Delta_n$, we set $\hat{i} = \Delta_n - \{i\}$. The definition of *colour isomorphic graphs* is the standard one. An *n-pseudo-complex K* is a homogeneous n-dimensional ball complex in which every h-ball, considered with all its faces, is abstractly isomorphic to the closure of a standard n-simplex (see [11], p. 49). Thus we will always call the balls of K *simplexes*. If s is a simplex of K, then the *disjoined star $std(s, K)$* is defined as the disjoint union of the n-simplexes of K containing s, with re-identification of the $(n-1)$-faces containing s and of their faces. Given an $(n+1)$-coloured graph (G, c), with or without boundary, an n-pseudo-complex $K = K(G)$ can be *associated* with (G, c) so that $|G|$ becomes its dual 1-skeleton (see [4]). The construction can be easily reversed (see [5]) so that $(G(K(G)), c_{K(G)})$ and (G, c) are colour isomorphic graphs. Furthermore, if $|K|$ is an n-manifold, then $K(G(K))$ is abstractly isomorphic to K and the graph $(G(K), c_K)$ will be said to *represent* $|K|$. If, for every colour $i \in \Delta_n$, the subgraph $G_{\hat{i}}$ is connected, then $K(G)$ has exactly $n+1$ vertices and both $K(G)$ and G are said to be *contracted*. A *crystallization* of a closed n-manifold M is a contracted $(n+1)$-coloured graph (G, c) representing M. In this case, the pseudo-complex $K(G)$ is called a *contracted triangulation* of M. Every closed n-manifold admits a crystallization (see [17]). For a survey on crystallizations we refer to [4].

3. Graph moves.

In order to prove our statements we need some definitions given in [1], [2]. Given an $(n+1)$-coloured graph (G, c), let H be a subgraph of G induced by $n+1$ edges coloured with distinct colours. Then H is said to be a *(combinatorial) handle* in (G, c) if any two edges of H are members of a bicoloured cycle of G (see [1]).

This concept generalizes the analogous one given in [7]; there the combinatorial handle is defined as a subgraph θ of G formed by two vertices P, Q joined by $n-1$ edges $t_1, \dots, t_{n-1}$ with distinct colours $c_1, \dots, c_{n-1} \in \Delta_n$ such that P, Q belong to the same $\{i, j\}$-coloured cycle C of G, where $\{i, j\} = \Delta_n - \{c_1, \dots, c_{n-1}\}$. If t_n (resp. t_{n+1}) is the i-coloured (resp. j-coloured) edge of C containing P (resp. Q) as its vertex, then the sub-graph of G induced by the edge set $\{t_r / r \in N_{n+1}\}$ is a special case of a combinatorial handle as defined in the present paper.

The graph $(\overline{G}, \overline{c})$ obtained from (G, c) by *cutting the handle* H is defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(G) - E(H)$. Obviously $(\overline{G}, \overline{c})$ is an $(n+1)$-coloured graph with boundary. If $\partial K(\overline{G})$ is the disjoint union of two triangulated $(n-1)$-spheres T_1, T_2, then T_1, T_2 have exactly $n+1$ $(n-1)$-simplexes each. This condition is always verified whenever (G, c) represents a closed n-manifold, $n \geq 3$, (see [1], p. 81) so that throughout the paper, we will be interested in this case. Let $a_0, \dots, a_r$, $r \leq n$, (resp. $b_0, \dots, b_s$, $s \leq n$) be the boundary vertices of $\overline{G}$ representing all the n-simplexes of $K(\overline{G})$ that have at least an $(n-1)$-face in T_1 (resp. T_2). Then the graph $(\tilde{G}, \tilde{c})$ obtained from (G, c) by *cancelling the handle* H is the $(n+1)$-coloured graph defined by the following rules:

(1) $V(\tilde{G}) = V(G) \cup \{X, Y\}$, where $X, Y \notin V(G)$;
(2) two vertices $u, v \in V(G)$ are joined by an i-coloured edge if and only if u, v were joined in $\overline{G}$ by such an edge;
(3) there is an i-coloured edge e_i (resp. f_i), $i \in \Delta_n$, joining a_j (resp. b_j) with X (resp. Y) for each i-coloured free edge incident to a_j (resp. b_j).

The following Proposition 1 and Proposition 2 proved in [1] show the topological meaning of the above graph moves.

Proposition 1. *Let (G, c) be an $(n+1)$-coloured graph representing the n-sphere S^n and let H be a handle in G. Then the graph $(\tilde{G}, \tilde{c})$ has two connected components both representing the n-sphere S^n.*

Proposition 2. *Let M be a closed n-manifold, (G,c) be an $(n+1)$-coloured graph representing M and H a handle in G. Then we have*

(1) *If $E(H)$ is not a coboundary, then M is homeomorphic to the connected sum $\widetilde{M} \# (S^{n-1} \otimes S^1)$, where $\widetilde{M} = |K(\widetilde{G})|$.*

(2) *If $E(H)$ is a coboundary, then $(\widetilde{G},\widetilde{c})$ has two connected components representing two n-manifolds $\widetilde{M}_1, \widetilde{M}_2$ such that $M \simeq \widetilde{M}_1 \# \widetilde{M}_2$.*

Given an $(n+1)$-coloured graph (G,c), let D be a subgraph of G formed by two vertices u, v joined by h edges $(h \in N_n)$ coloured by distinct colours $c_1, c_2, \dots, c_h \in \Delta_n$. If we set $\Gamma = \Delta_n - \{c_1, c_2, \dots, c_h\}$, then D will be called a *dipole of type* h if u, v belong to distinct components of G_Γ (see [2]). The $(n+1)$-coloured graph (G', c') obtained from (G,c) by *cancelling a dipole* D is defined by the following rules: $(1')$ delete u, v from G; $(2')$ paste together the pairs of free edges (the ones which had an end-point in the deleted vertices) with the same colour.

In [2] the following proposition is proved:

Proposition 3. *Let M be a closed n-manifold, (G,c) an $(n+1)$-coloured graph representing M and D a dipole in G. Then the graph (G',c') also represents M.*

4. Regular crystallizations.

Now we recall some definitions given in [20] for a closed 3-manifold M^3. A crystallization (G,c) of M^3 is said to be a *strong frame* of M if G has no multiple edges (that is, G is *simple*) and any $\{i,j\}$-coloured cycle C_{ij} and $\{i,h\}$-coloured cycle C_{ih} of G meet at most one edge for $\{i,j,h\} \subset \Delta_3$.

Actually the original definition of strong frame is stated in terms of "minimal ball coverings" of 3-manifolds (see [20]); the equivalence of the two definitions in an easy exercise.

The main result of [20], stated with our notation, is the following

Proposition 4. *Let M^3 be a closed non-trivial handle free 3-manifold. Then there exists a strong frame (G,c) representing M.*

As a direct consequence of Proposition 4, Y. Tsukui stated in [20] the following conjecture: *there is no strong frame which represents the 3-sphere S^3.*

Now we introduce regular crystallizations for closed n-manifolds which partially extend strong frames to dimension n. Indeed the concepts of regular crystallization and strong frame coincide in dimension 3. However, there exists a regular crystallization of $S^2 \times S^2$ which still has multiple edges (see [3], Figure 6). Regular crystallizations are proved to exist for any closed non-trivial handle free n-manifold (see Proposition 5). Our techniques are completely different from the ones used in [20] so that Proposition 5 also gives an alternative simple proof of the main theorem of [20] (Proposition 6).

Definition 1: Let M^n be a closed n-manifold and let (G, c) be a crystallization of M. Then (G, c) is said to be a *regular crystallization* of M if G satisfies the following properties:

(1) G has neither dipoles nor handles whose edge-sets are not coboundaries;

(2) no two edges of the same colour belong to $n-1$ distinct bicoloured cycles.

The main purpose for introducing the concept of regular crystallization is to provide a combinatorial method to recognize the topological handle $S^{n-1} \otimes S^1$ among closed n-manifolds. More precisely, we try to characterize the crystallizations of $S^{n-1} \otimes S^1$ among coloured-graphs which represent closed n-manifolds. Similar and partially successful attempts have been performed for the 3-sphere by several authors (see [10], [21]). Their point of view consisted in constructing algorithms to recognize the Heegaard diagrams of genus two of the 3-sphere.

Our approach, valid in all dimensions, seems to characterize (modulo Conjectures 1, 2), instead, the topological handle $S^{n-1} \otimes S^1$ as the unique closed prime n-manifold whose crystallizations should always have a degeneracy condition with respect to the intersections of the 2-residues (see Propositions 5 and 8). Indeed it seems true (but it is an open question) that $S^{n-1} \otimes S^1$ does not admit a regular crystallization (compare also [15], Section 4). This leads to an algorithm for recognizing (modulo Conjectures 1, 2) $S^{n-1} \otimes S^1$ among closed n-manifolds as follows. Let M be a closed prime non-trivial n-manifold which has no regular crystallizations. Then the procedure of regularization, described in Proposition 5 below, does not completely work for an arbitrary crystallization of M. Thus we must obtain a new crystallization of M containing a combinatorial handle whose edge-set is not a coboundary. Now Proposition 2, Section 1 implies that M is homeomorphic to $S^{n-1} \otimes S^1$.

Further applications to standard spines of 3-manifolds will appear in a future paper.

Proposition 5. *Let M^n be a closed non-trivial handle free n-manifold. Then there exists a regular crystallization (G, c) which represents M.*

Proof: We may suppose that M is prime because the general case is easily obtained by using suitable connected sums (see also [20]). By abuse of language, we will call by the same name each part of a graph which is left unchanged by graphical moves considered in this and all subsequent proofs. Let $(G^{(1)}, c^{(1)})$ be a crystallization of M. Since M is a non-trivial handle free n-manifold, the graph $G^{(1)}$ will be assumed without dipoles and handles whose edge-sets are not coboundaries. If $G^{(1)}$ is not regular, then there exist two i-coloured edges $e, f \in E(G^{(1)}_{\hat{j}})$, for some $i, j \in \Delta_n$ $(i \neq j)$, which belong to the same $\{i, h\}$-coloured cycle of $G^{(1)}$ for every $h \in \Delta_n - \{i, j\}$. We may assume that $i = 0$ and $j = n$ without loss of generality. Let C_{0h} $(h \in N_{n-1})$ be the $\{0, h\}$-coloured cycle of $G^{(1)}_{\hat{n}}$ containing the edges e, f. By $x_h \in C_{0h}$ we denote the h-coloured edge adjacent to e. Let $H^{(1)}$ be the subgraph of $G^{(1)}$ induced by the n edges $f, x_1, \ldots, x_{n-1}$ coloured by the colours $0, 1, \ldots, n-1$, respectively. Since any two edges of $H^{(1)}$ are members of a bicoloured cycle of $G^{(1)}_{\hat{n}}$, the subgraph $H^{(1)}$ is a handle in the n-coloured graph $(G^{(1)}_{\hat{n}}, c^{(1)}_{\hat{n}})$, where $c^{(1)}_{\hat{n}}$: $E(G^{(1)}_{\hat{n}}) = (c^{(1)})^{-1}(\Delta_{n-1}) \longrightarrow \Delta_{n-1} \subset \Delta_n$ is the restriction of $c^{(1)}$ to $E(G^{(1)}_{\hat{n}})$. Further $G^{(1)}_{\hat{n}}$ is the unique Δ_{n-1}-residue of $G^{(1)}$ which corresponds to the vertex v_n of the contracted triangulation $K^{(1)} = K(G^{(1)})$ of M. By construction, the pseudo-complex $K(G^{(1)}_{\hat{n}})$ is abstractly isomorphic to the disjoined star $std(v_n, K^{(1)})$ so that $G^{(1)}_{\hat{n}}$ represents the $(n-1)$-sphere $S^{n-1} \simeq \partial|std(v_n, K^{(1)})|$. By Proposition 1, the graph $(\widetilde{G}^{(1)}_{\hat{n}}, \widetilde{c}^{(1)}_{\hat{n}})$ obtained from $(G^{(1)}_{\hat{n}}, c^{(1)}_{\hat{n}})$ by cancelling the handle $H^{(1)}$ has two connected components (G_1, c_1), (G_2, c_2) both representing S^{n-1}. Let us consider the edges e, f, x_h $(h \in N_{n-1})$ as directed edges so that $e(0)$, $f(0)$ (resp. $e(1)$, $f(1)$) belong to the same component of $C_{0h} - \{e, f\}$ in $G^{(1)}$ and $e(1) = x_h(0)$ for every $h \in N_{n-1}$ (see Figure 1).

By construction, we have $V(\widetilde{G}^{(1)}_{\hat{n}}) = V(G^{(1)}_{\hat{n}}) \cup \{X, Y\} = V(G^{(1)}) \cup \{X, Y\}$, where $X, Y \notin V(G^{(1)})$. If we set $E' = E(G^{(1)}) - \{f, x_h / h \in N_{n-1}\}$, then the subgraph $(V(G^{(1)}), E')$ of $(G^{(1)}_{\hat{n}} c^{(1)}_{\hat{n}})$ is left unchanged in $(\widetilde{G}^{(1)}_{\hat{n}}, \widetilde{c}^{(1)}_{\hat{n}})$. Let $u, v \in V(G^{(1)})$ be two vertices which are no end-points of the edges e, f, x_h $(h \in N_{n-1})$. Then u, v are joined in $\widetilde{G}^{(1)}_{\hat{n}}$ by an i-coloured edge $(i \in N_{n-1})$ if and only if u, v were joined in $G^{(1)}_{\hat{n}}$ by such an edge. There is an h-coloured edge $y_h \in E(\widetilde{G}^{(1)}_{\hat{n}})$ (resp. z_h), $h \in N_{n-1}$, joining $x_h(1)$ (resp. $x_h(0) = e(1)$) with X (resp. Y). There are two 0-coloured edges y_0, z_0 joining $f(1)$ with X and $f(0)$ with Y, respectively (see Figure 2).

Let $(G^{(2)}, c^{(2)})$ be the $(n+1)$-coloured graph obtained from $(\widetilde{G}^{(1)}_{\hat{n}}, \widetilde{c}^{(1)}_{\hat{n}})$ by the following rules:

(1) $V(G^{(2)}) = V(G^{(1)}) \cup \{X, Y\}$;
(2) $(G^{(2)}_{\hat{n}}, c^{(2)}_{\hat{n}}) = (\widetilde{G}^{(1)}_{\hat{n}}, \widetilde{c}^{(1)}_{\hat{n}})$;
(3) Let $u, v \in V(G^{(2)})$ be two arbitrary vertices which are no endpoints of the edges e, f, x_h $(h \in N_{n-1})$. Then u, v are joined in $G^{(2)}$ by an n-coloured edge if and only if u, v were joined in $G^{(1)}$ by such an edge;
(4) put an n-coloured edge a_n between X and Y.

Since $G^{(2)}_{\hat{n}} = \widetilde{G}^{(1)}_{\hat{n}}$ has two connected components (G_1, c_1), (G_2, c_2) with $X \in V(G_1)$ and $Y \in V(G_2)$, the n-coloured edge a_n induces a dipole of type 1 between the vertices X, Y in $G^{(2)}$. By cancelling such a dipole from $G^{(2)}$ we get the initial crystallization $(G^{(1)}, c^{(1)})$. Thus $(G^{(2)}, c^{(2)})$ is an $(n+1)$-coloured graph (non contracted) representing M, that is, $|K(G^{(2)})| \simeq |K(G^{(1)})| \simeq M$. Since M is a prime handle free n-manifold, the subgraph $D^{(2)}$, formed by the two vertices $e(1), Y$ joined by the $n-1$ edges $z_1, \ldots, z_{n-1}$, is a dipole of type $n-1$ in $G^{(2)}$. By cancelling $D^{(2)}$ from $G^{(2)}$ we get a new $(n+1)$-coloured graph $(G^{(3)}, c^{(3)})$ which represents M. The sequence of the above moves which

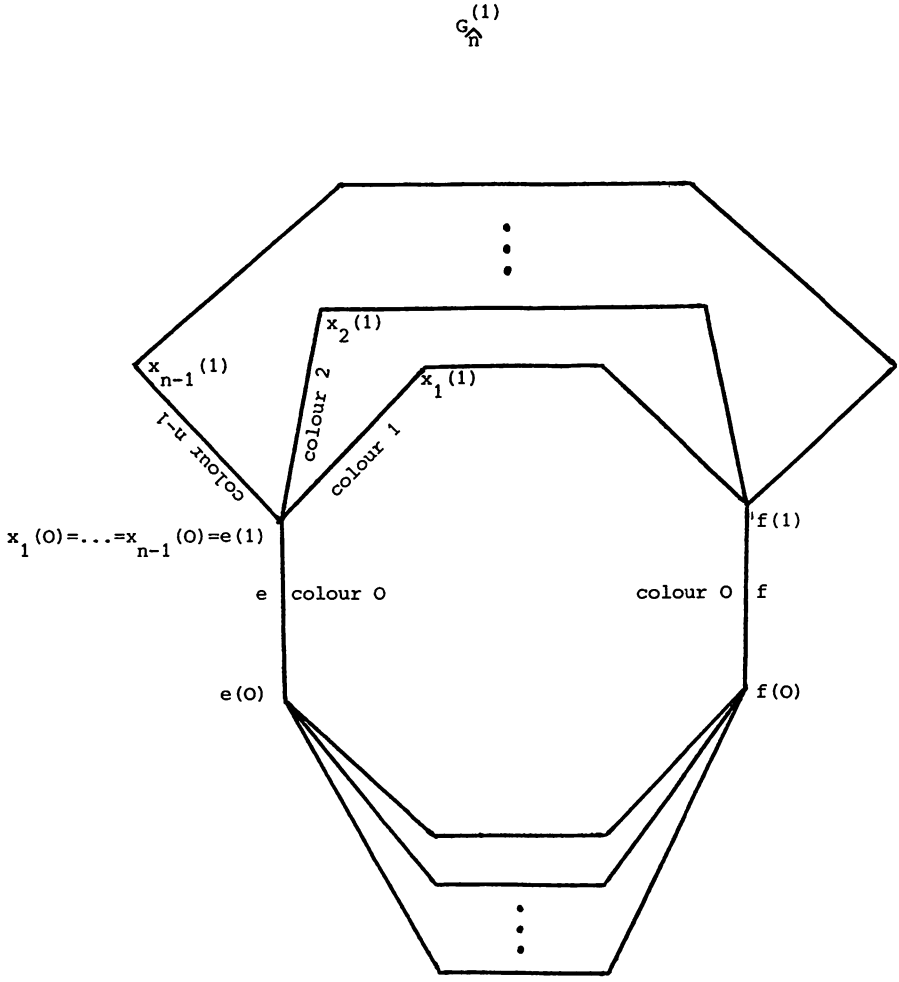

Figure 1
The handle $H^{(1)}$ in the n-coloured subgraph $G_{\hat{n}}^{(1)}$.

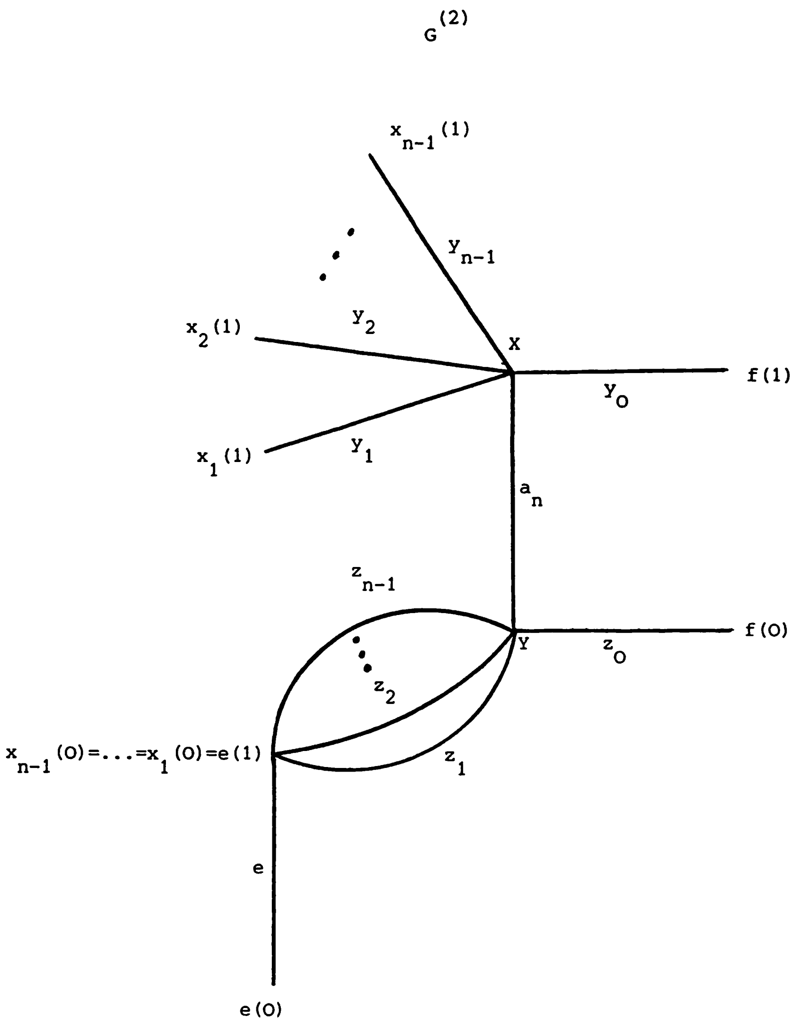

Figure 2
Graphical moves taking the graph $G^{(1)}$ into $G^{(2)}$.

takes the graph $(G^{(1)}, c^{(1)})$ to $(G^{(3)}, c^{(3)})$ can be resumed as follows: the graph $(G^{(3)}, c^{(3)})$ is obtained from $(G^{(1)}, c^{(1)})$ by cancelling the edges e, f and joining the vertex $f(1)$ (resp. $f(0)$) with $e(1)$ (resp. $e(0)$) by a 0-coloured edge a_0 (resp. b_0). Obviously $G^{(3)}_{\hat{n}}$ consists of two connected components, (G_3, c_3), (G_4, c_4), since so does $G^{(2)}_{\hat{n}}$. By the connectedness of $G^{(3)}$, there exists an n-coloured edge b_n joining the components G_3 and G_4. Such an edge b_n induces a dipole $D^{(3)}$ of type 1 in $(G^{(3)}, c^{(3)})$. Let $(G^{(4)}, c^{(4)})$ be the $(n+1)$-coloured graph obtained from $(G^{(3)}, c^{(3)})$ by cancelling the dipole $D^{(3)}$. Then we have $|K(G^{(4)})| \simeq M$ and $\# V(G^{(4)}) = \# V(G^{(1)}) - 2$. By successively cancelling every other dipole (if there is any) from $(G^{(4)}, c^{(4)})$ we get a new crystallization $(G^{(5)}, c^{(5)})$ of M with $\# V(G^{(5)}) \leq \# V(G^{(1)}) - 2$. If $G^{(5)}$ is regular, then the proof is completed. Otherwise we repeat the above constructions by replacing $G^{(1)}$ with $G^{(5)}$. Going on like this, we can easily obtain a final crystallization (G, c) of M which is regular. The sequence of moves must end because $\# V(G) > 2$ for each crystallization of a non-trivial n-manifold M. ∎

Now we give three immediate applications of Proposition 5 to minimum crystallizations, the Heegaard genus of a closed 3-manifold and the Tsukui conjecture.

(I): By definition, a *minimum crystallization* of an n-manifold M is a crystallization which has the smallest possible number of vertices among all the crystallizations representing the same manifold M (see [13], [14]). Some properties of minimum crystallizations were studied by S. Lins and A. Mandel in the above quoted papers. As a direct consequence of Proposition 5, the following proposition is easily proved:

Proposition 6. *Each minimum crystallization of a closed non-trivial handle free n-manifold is regular.*

(II): A *Heegaard splitting* of an orientable closed 3-manifold M^3 is a closed (connected) orientable surface F imbedded in M and dividing M into two handlebodies. The genus of the Heegaard splitting is the genus of F and the *Heegaard genus* of M, written $h(M)$, is the smallest integer h such that M has a Heegaard splitting of genus h. The proof of Proposition 5 directly implies the following

Proposition 7. *Let M^3 be a closed orientable non-trivial handle free 3-manifold with Heegaard genus $h(M)$. Then there exists a regular crystallization (G, c) of M with the property $\# G_{\{0,1\}} = \# G_{\{2,3\}} = h(M) + 1$.*

(III): Now we construct a regular crystallization of the 3-sphere S^3 which represents a counter-example to the cited conjecture of Y. Tsukui. In Figure 3 we show a Heegaard diagram of S^3 obtained from the one given in [16] by using suitable Singer-Reidemeister moves on Heegaard diagrams.

Let (G, c) be the 4-coloured (non contracted) graph representing S^3 and directly constructed from the above Heegaard diagram by using the procedure described in [6] (see Figure 4).

An application of the graphical moves involved in the proof of Proposition 5 takes (G, c) into a regular crystallization of S^3 (see Figure 5 and Figure 6).

Thus the Conjecture 5.4 of [20] is *false*.

Our counterexample shows that in the 3-dimensional case the assumption "non-trivial" in Proposition 5 is not necessary. We do not know whether the same is true for arbitrary dimension. Thus it seems natural to state the following

Conjecture 1. *The n-sphere S^n admits a regular crystallization.*

5. Pseudo-handles in graphs.

In this section we give a further application of the concept of regular crystallization to combinatorial handles in graphs. The purpose is to find simpler crystallizations of a manifold by using a sequence of graphical moves which do not change the manifold. Thus the study of the topological structure of a manifold can be performed by simply considering these simplified crystallizations.

Given an $(n+1)$-coloured graph (G, c), let Q be a subgraph of G induced by $n+1$ edges $e_0, e_1, \ldots, e_n$ coloured $0, 1, \ldots, n$, respectively.

Definition 2: With the above notation, Q is said to be a *(combinatorial) pseudo-handle* in (G, c) if there exist two distinct colours $i, j \in \Delta_n$ such that the subgraphs $Q - e_i$, $Q - e_j$ are handles in $G_{\hat{i}}, G_{\hat{j}}$, respectively.

Obviously handles in graphs are pseudo-handles but the converse is generally false, because an $\{i, j\}$-coloured cycle containing e_i and e_j might not exist in G. If the pseudo-handle Q is not a handle, then $K(Q) \subset K(G)$ is a combinatorial $(n-1)$-disk which has $n+1$ $(n-1)$-simplexes. Further $K(Q)$ and $K(G)$ have the same vertex set. The boundary $\partial K(Q)$ is the simplest contracted triangulation of the $(n-2)$-sphere, obtained by pairwise identifying the $(n-3)$-faces of two $(n-2)$-simplexes.

By using Proposition 5, we can prove the following

Proposition 8. *Let M^n be a closed non-trivial handle free n-manifold. Then there exists a regular crystallization of M without pseudo-handles.*

Proof: Let $(U^{(1)}, u^{(1)})$ be a regular crystallization of M. Let us suppose that $U^{(1)}$ has a pseudo-handle. Then there exists a subgraph $Q^{(1)}$ of $U^{(1)}$ induced by $n+1$ edges $e_0, e_1, \ldots, e_n$ coloured by $0, 1, \ldots, n$, respectively, such that $Q^{(1)} - e_i$, $Q^{(1)} - e_j$ are handles in $U^{(1)}_{\hat{i}}$, $U^{(1)}_{\hat{j}}$ for two distinct colours $i, j \in \Delta_n$. We may assume that $i = n-1$ and $j = n$ without loss of generality. Thus the subgraph $Q^{(1)} - e_n$ is a handle in the n-coloured graph $(U^{(1)}_{\hat{n}}, u^{(1)}_{\hat{n}})$, where $u^{(1)}_{\hat{n}}\colon E(U^{(1)}_{\hat{n}}) = (u^{(1)})^{-1}(\Delta_{n-1}) \longrightarrow \Delta_{n-1} \subset \Delta_n$ is the restriction of $u^{(1)}$ to $E(U^{(1)}_{\hat{n}})$. Moreover $U^{(1)}_{\hat{n}}$ is the unique Δ_{n-1}-residue of $U^{(1)}$ which corresponds to the vertex v_n of the contracted triangulation $K^{(1)} = K(U^{(1)})$ of M. By construction, the pseudo-complex $K(U^{(1)}_{\hat{n}})$ is abstractly isomorphic to the disjoined star $std(v_n, K^{(1)})$ so

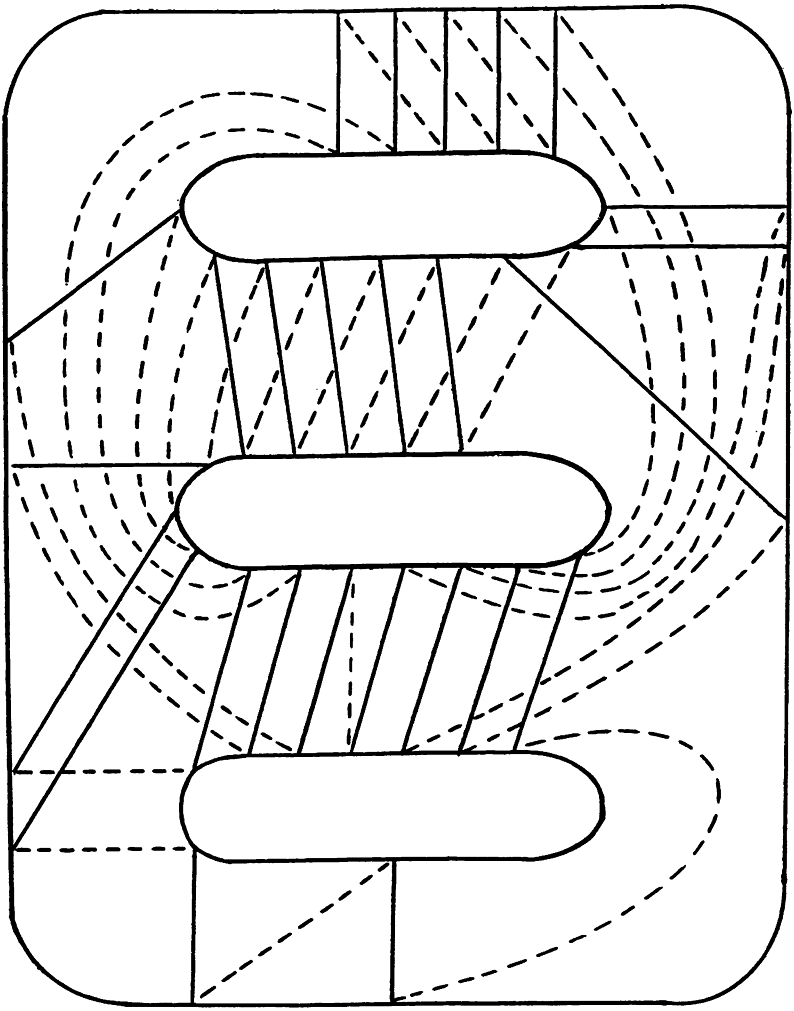

Figure 3
A Heegaard diagram of the 3-sphere S^3.

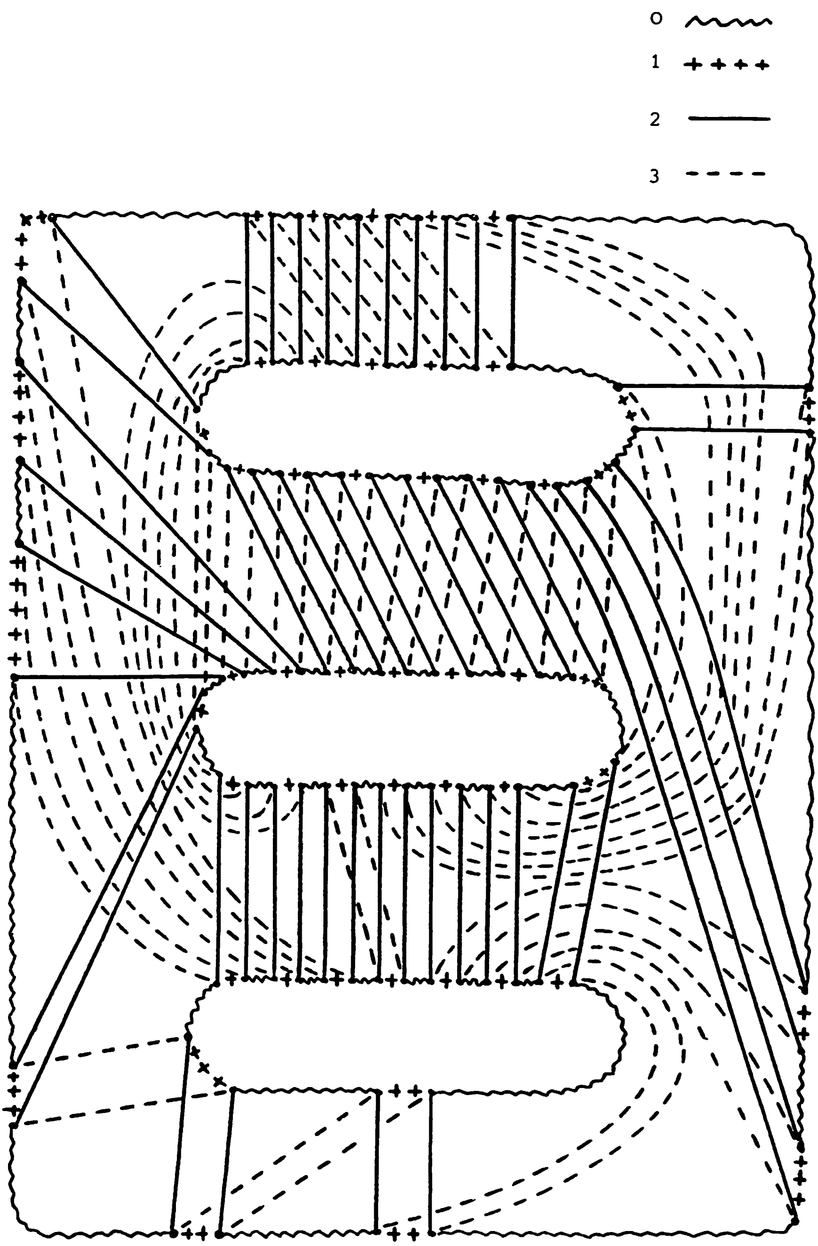

Figure 4

A non contracted 4-coloured graph representing the 3-sphere S^3.

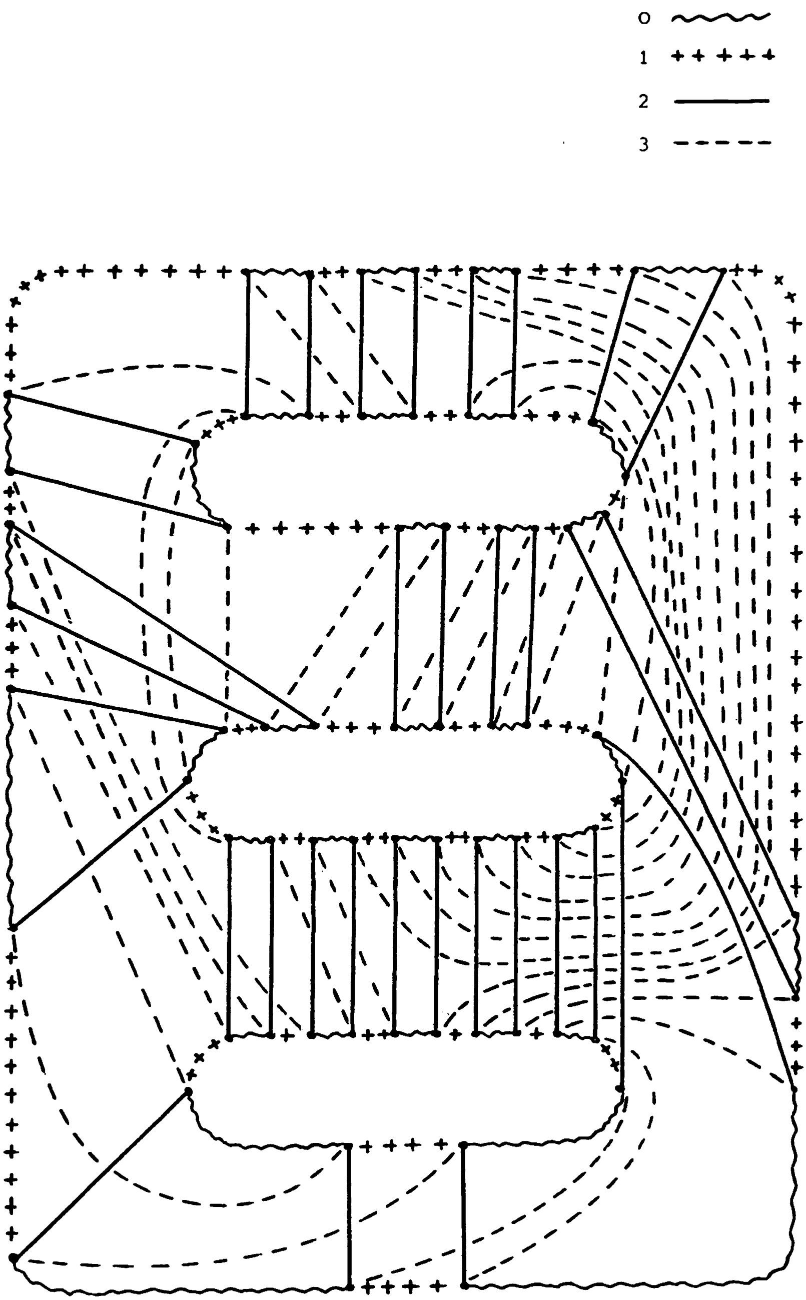

Figure 5
A (non regular) crystallization of the 3-sphere S^3.

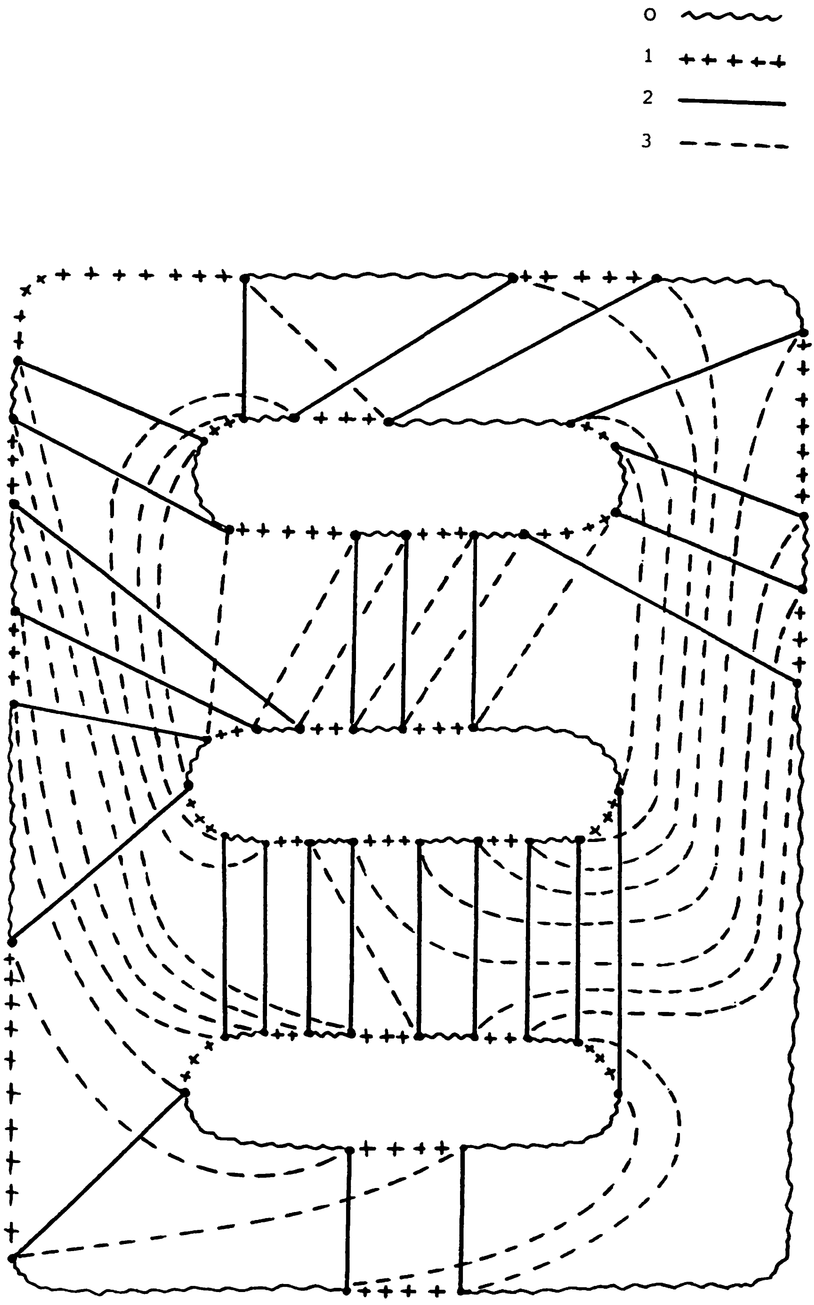

Figure 6
A regular crystallization (that is, strong frame) of the 3-sphere S^3.

that $U_{\hat{n}}^{(1)}$ represents the $(n-1)$-sphere $S^{n-1} \simeq \partial|std(v_n, K^{(1)})|$. By Proposition 1, the graph $(\widetilde{U}_{\hat{n}}^{(1)}, \widetilde{u}_{\hat{n}}^{(1)})$ obtained from $(U_{\hat{n}}^{(1)}, u_{\hat{n}}^{(1)})$ by cancelling the handle $Q^{(1)}$ has two connected components (U_1, u_1), (U_2, u_2), both representing S^{n-1}. If $V(\widetilde{U}_{\hat{n}}^{(1)}) = V(U_{\hat{n}}^{(1)}) \cup \{X, Y\} = V(U^{(1)}) \cup \{X, Y\}$, then we take $X \in V(U_1)$ and $Y \in V(U_2)$. Let now $(U^{(2)}, u^{(2)})$ be the $(n+1)$-coloured graph obtained from $(\widetilde{U}_{\hat{n}}^{(1)}, \widetilde{u}_{\hat{n}}^{(1)})$ by the following rules:

(1) $V(U^{(2)}) = V(\widetilde{U}_{\hat{n}}^{(1)}), E(U_{\hat{n}}^{(2)}) = E(\widetilde{U}_{\hat{n}}^{(1)})$;

(2) two arbitrary vertices $u, v \in V(U^{(2)}) - \{X, Y\}$ are joined in $U^{(2)}$ by an n-coloured edge if and only if u, v were joined in $U^{(1)}$ by such an edge;

(3) put an n-coloured edge f_n between X and Y.

Since the subgraph $U_{\hat{n}}^{(2)} = \widetilde{U}_{\hat{n}}^{(1)}$ has the components U_1, U_2, the n-coloured edge f_n induces a dipole of type 1 between the vertices X, Y in $U^{(2)}$. By cancelling such a dipole from $U^{(2)}$ we get the initial crystallization $(U^{(1)}, u^{(1)})$. Thus $(U^{(2)}, u^{(2)})$ is an $(n+1)$-coloured graph (non contracted) representing M. Since the graph $(U^{(1)}, u^{(1)})$ has no handles, there is no $\{n-1, n\}$-coloured cycle in $U^{(2)}$ containing f_n and e_n. However the $\{n-1, n\}$-coloured cycle of $U^{(2)}$, which contains e_n, must join U_1 and U_2. Thus there exists an n-coloured edge g_n joining U_1 and U_2 which is different from e_n and f_n. By cancelling the dipole of type 1 induced by g_n, we get a new crystallization $(U^{(3)}, u^{(3)})$ of M such that $\# V(U^{(3)}) = \# V(U^{(1)})$. Now the graph $U^{(3)}$ is not regular because the edges e_n, f_n belong to the same $\{n, i\}$-coloured cycle of $U^{(3)}$ for every $i \in \Delta_{n-2}$. Then we can apply Proposition 5 to obtain a crystallization $(U^{(4)}, u^{(4)})$ of M with $\# V(U^{(4)}) \leq \# V(U^{(1)}) - 2$. If $U^{(4)}$ does not satisfy the statement, we repeat our constructions by replacing $U^{(1)}$ by $U^{(4)}$. Going on like this, we can obtain a final regular crystallization (G, c) of M without pseudo-handles. In fact the sequence of moves must be finite because $\# V(G) > 2$ for each crystallization of a non-trivial n-manifold M. ∎

As a corollary of Proposition 8, we have the following

Proposition 9. *Let M^3 be a closed non-trivial handle free 3-manifold. Then there exists a strong frame without pseudo-handles which represents M.*

The regular crystallization of the 3-sphere, shown in Figure 6, has no pseudo-handles. Thus it seems natural to state the following

Conjecture 2. *The n-sphere admits a regular crystallization without pseudo-handles.*

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