Rotation Numbers for Generalised Stars

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Abstract. Let G be a p-vertex graph which is rooted at x. Informally, the rotation number $h(G,x)$ is the smallest number of edges in a p-vertex graph F such that, for every vertex y of F, there is a copy of G in F with x at y. In this article, we consider rotation numbers for the generalised star graph consisting of k paths of length n, all of which have a common endvertex, rooted at a vertex adjacent to the centre. In particular, if $k = 3$ we determine the rotation numbers to within 1, 2 or 3 depending on the residue of n modulo 6.

1. Introduction

A *rooted graph* is an ordered pair (G,x) where G is a simple undirected graph and x is a vertex of G called the *root*. Let (G,x) and (F,y) be rooted graphs. We say, informally, that (G,x) is a rooted subgraph of (F,y) if there is a copy of G in F with x at y. Formally, (G,x) is a *rooted subgraph* of (F,y) if there exists a 1-1 function $f : V(G) \to V(F)$ that satisfies

(1) $[u,v] \in E(G) \Rightarrow [f(u), f(v)] \in E(F)$
(2) $f(x) = y$

This property is denoted by $(G,x) \prec (F,y)$. If $(G,x) \prec (F,y)$ for every $y \in V(F)$, we say (G,x) *is a homogeneous rooted subgraph of* F and write $(G,x) \prec F$. The *rotation number* $h(G,x)$ of the p-vertex rooted graph (G,x) is defined to be the smallest number $q(F)$ of edges in a p-vertex graph F such that $(G,x) \prec F$. The graph F is called an *extremal* (G,x) *graph*.

These concepts were first defined by Cockayne and Lorimer [4]. Rotation numbers and extremal graphs have been calculated for complete bipartite graphs [1], [4], unions of two circuits [2], [5], lollipop graphs [7], unions of two complete graphs [6] and the union of two stars [3].

Rotation numbers of certain rooted trees are related to the minimum number of edges in minimal broadcast graphs , which model networks which allow optimal information dissemination from an arbitrary node (see [4]).

In this paper we begin the study of rotation numbers for the generalised star $S(k,n)$ which consists of k paths of length n, all of which have one endvertex in common (Figure 1). The root is the vertex a_1 adjacent to the common endvertex.

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$S(k,n)$:

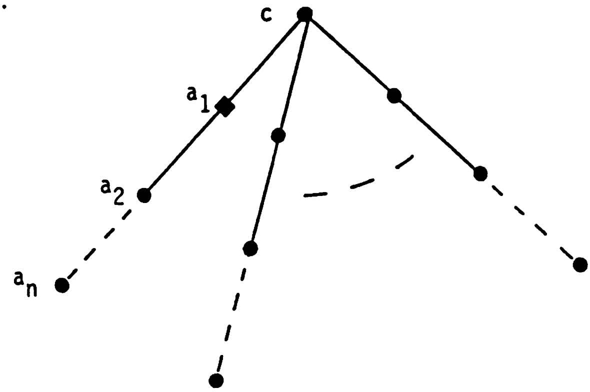

Figure 1. The generalised star $S(k,n)$ with root a_1.

In Section 2.1 we obtain a lower bound for $h(S(k,n), a_1)$ which is attained for $k = 3$ and $n = 1, 3, 7$. The analysis of Section 2.2 demonstrates for $k = 3$, $n \equiv 1$ or $5 \mod 6$ and $n \not\equiv 1, 7$, that the lower bound is not exact. Finally in Section 3 we establish an upper bound for the case $k = 3$.

Our results establish the rotation numbers for $k = 3$ to within 1,2 or 3 depending on the residue of $n \mod 6$. The exact determination remains an interesting open problem.

2. Lower Bounds

2.1 A lower bound for $h(S(k,n), a_1)$. We begin by eastablishing a lower bound for $h(S(k,n), a_1)$ for $k \geq 3$ and $n \geq 1$. For $k = 3$, this lower bound is attained for several small values of n. We exhibit graphs to illustrate this for the cases $n = 1, 3$ and 7 only.

For each $k \geq 3$ and $m \geq 1$, let $\mathcal{H}(k,n)$ be the class of all graphs H of order $kn+1$ such that $\delta(H) \geq 2$ and each vertex of H is adjacent to a vertex of degree at least k. Since the complete graph K_{kn+1} satisfies these requirements, $\mathcal{H}(k,n)$ is non-empty. Let

$$q = \min_{H \in \mathcal{H}(k,n)} \{|E(H)|\}$$

and let

$$\mathcal{G}(k,n) = \{G \in \mathcal{H}(k,n) \,|\, |E(G)| = q\}.$$

Theorem 1. *If $\mathcal{G}$ is any graph in $\mathcal{G}(k,n)$, then*

$$|E(G)| = q \geq \lceil \frac{3kn - 2n + 3}{2} \rceil.$$

Proof: Let G be a graph in $\mathcal{G}(k,n)$ and let X (Y respectively) be the set of all vertices of G of degree greater (less) than k. Let

$$m = \sum_{x \in X} (\deg_G(x) - k).$$

If $Y = \emptyset$, let u and v be adjacent vertices of G and let $G_1 = G - [u,v]$. Since $k \geq 3$, every vertex in G_1 is adjacent to a vertex distinct from u and v, and hence to a vertex of degree at least k. Also, $\delta(G_1) \geq 2$ and so G_1 is in $\mathcal{H}(k,n)$ but has fewer edges than G. This is impossible; therefore $Y \neq \emptyset$. Let

$$\ell = \max_{y \in Y} \{\deg_G(y)\}.$$

We assume that among the graphs in $\mathcal{G}(k,n)$, G is one for which m is minimum and prove that $m < k - \ell$. Suppose to the contrary that $m \geq k - \ell$ and let v be a vertex of G of degree ℓ. For every x in X, let $f(x)$ be a (fixed) neighbour of x of degree at least k. Every x in X also has at least $\deg_G(x) - k$ neighbours distinct from $f(x)$ nonadjacent to v. Let

$$F = \cup_{i=1}^{m} \{[x_i, y_i]\}$$

be edges such that $x_i \in X, y_i \neq f(x_i)$ and y_i is nonadjacent to v, and such that $G - F$ has no vertices of degree greater then k. Note that each x in X occurs $\deg_G(x) - k$ times as an endvertex of an edge in F. Let

$$G_2 = (G - \cup_{i=1}^{k-\ell}\{[x_i, y_i]\}) + \cup_{i=1}^{k-\ell}\{[y_i, v]\}.$$

Then $\deg_{G_2}(v) = k, \deg_{G_2}(y_i) = \deg_G(y_i)$ and $k \leq \deg_{G_2}(x_i) \leq \deg_G(x_i)$. Therefore every vertex in G_2 is adjacent to a vertex of degree at least k, and since $|E(G_2)| = q$, it follows that $G_2 \in \mathcal{G}(k,n)$. However, if X_2 is the set of all vertices of G_2 of degree greater than k, then $X_2 \subseteq X$ and, since $k - \ell \geq 1$,

$$\sum_{x \in X_2} (\deg_{G_2}(x) - k) < m,$$

contradicting the choice of G. Hence $m < k - \ell$.

Let Z be the set of vertices of G of degree at least k and suppose $|Z| < n$. Then $\delta\langle Z \rangle \geq 1$, where $\langle Z \rangle$ denotes the subgraph (of G) induced by Z. Hence at most $(k-1)(n-1) + m$ edges join vertices of Z to vertices of Y. But

$$\begin{aligned}(k-1)(n-1) + m &\leq (k-1)(n-1) + k - \ell - 1 \\ &= (k-1)n - \ell \\ &\leq (k-1)n - 2 \qquad \text{since } \ell \geq 2,\end{aligned}$$

and

$$|Y| \geq (k-1)n+2.$$

Therefore not every vertex in Y is adjacent to a vertex in Z, contradicting the fact that $G \in \mathcal{H}(k,n)$. Consequently $|Z| \geq n$. In fact, it is evident from the above inequalities that $|Z| \geq n+1$ if $m = 0$. It follows that

$$|E(G)| \geq \begin{cases} \frac{1}{2}(k(n+1) + 2(k-1)n) & \text{if } m = 0 \\ \frac{1}{2}(kn+m+2(k-1)n+2) & \text{if } m \geq 1 \end{cases}$$
$$\geq \frac{1}{2}(3kn-2n+3) \qquad \text{since } k \geq 3. \ \blacksquare$$

Corollary. *For any* $k \geq 3$ *and* $n \geq 1$,

$$h(S(k,n),a_1) \geq \lceil \frac{3kn+3-2n}{2} \rceil.$$

Proof: Let F be an extremal $(S(k,n),a_1)$ graph. Then $F \in \mathcal{H}(k,n)$ and therefore by Theorem 1,

$$|E(F)| \geq \lceil \frac{3kn+3-2n}{2} \rceil. \ \blacksquare$$

Graphs for which this lower bound is attained for $k = 3$ are depicted in Figure 2.

2.2 The lower bound for $h(S(3,n),a_1)$ for $n \equiv 1$ or $5 \mod 6$.

In the rest of the paper, we shall only be concerned with the case $k = 3$ and will establish that there are infinitely many values of n for which the lower bound of Section 2.1 may not be attained. More specifically, the following sequence of lemmas will show that for $n \equiv 1$ or $5 \mod 6$ the rotation number is strictly greater than the lower bound of $\frac{(7n+3)}{2}$ unless $n = 1$ or 7 (in which cases the bound is attained by the graph of Figure 2). Throughout this section F will denote a supposed $(3n+1)$-vertex graph, where $n \equiv 1$ or $5 \mod 6$, $n \neq 1, 7$, which has $\frac{(7n+3)}{2}$ edges and satisfies $(S(3,n),a_1) \prec F$.

For any copy of $S(3,n)$ in F, the vertex of F corresponding to the vertex c of $S(3,n)$ will be called the *centre* of the copy and the three paths of length n in the copy will be termed *roads*. The endvertices of the three roads (other than the centre) will be called *terminals*.

In what follows the impossibility of various situations will be shown by proving the existence of at least four terminals. The details will sometimes be omitted for brevity and the argument merely referred to as 4TA (meaning a four terminal argument).

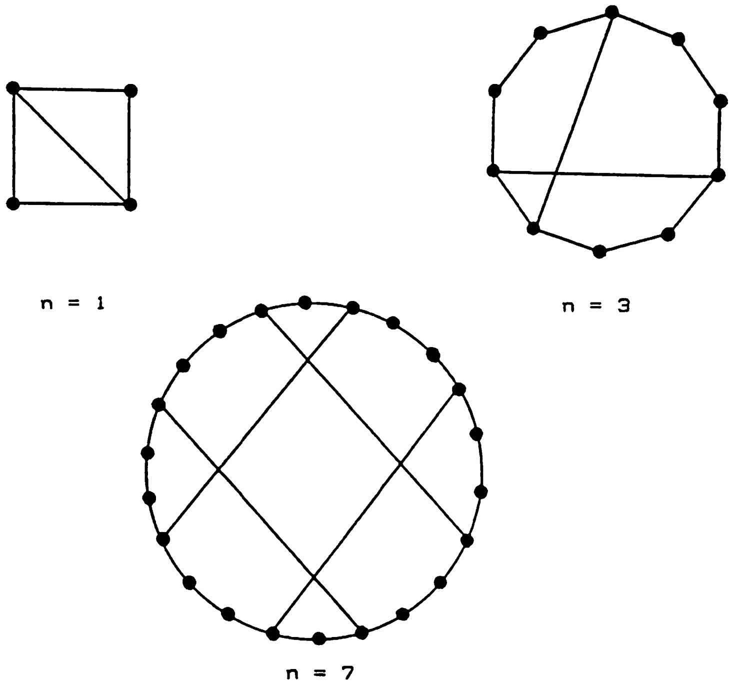

Figure 2. Graphs for which $h(S(3,n),a_1) = \lceil \frac{7n+3}{2} \rceil$.

Lemma 1. *F has exactly $n+1$ vertices of degree 3 and $2n$ vertices of degree 2.*

Proof: Since $(S(3,n),a_1) \prec F, \delta(F) \geq 2$ and every vertex is adjacent to a vertex of degree at least three. Suppose there are t, T and $(3n+1-t-T)$ vertices whose degrees are three, greater that three and two respectively. Further, let Σ be the degree sum of the T "large degree vertices". Adding degrees we have

$$3t + 2(3n+1-t-T) + \Sigma = 7n+3$$

and

$$3t + 2(3n+1-t-T) + 4T \leq 7n+3.$$

Therefore

$$t - 2T + \Sigma = n+1 \tag{1}$$

and

$$t + 2T \leq n+1. \tag{2}$$

Since each vertex of degree at least three is adjacent to another of this type, the number of edges from these vertices to the degree two vertices is at most $(\Sigma - T) + 2t$. The degree two vertices are dominated by the rest, hence

$$\Sigma - T + 2t \geq 3n+1-T-t.$$

Therefore

$$\Sigma \geq 3n + 1 - 3t. \tag{3}$$

From (1) and (3),

$$n + 1 - t + 2T \geq 3n + 1 - 3t$$

i.e.

$$t + T \geq n. \tag{4}$$

The only values of (t, T) satisfying (2) and (4) are $(n+1, 0)$, $(n, 0)$ and $(n-1, 1)$.

The second solution implies an odd number of vertices of odd degree. If $t = n - 1$ and $T = 1$, a simple degree count shows that the one large degree vertex has degree four. Let A be the set of n vertices of degree at least three. At most $3 + 2(n-1) = 2n + 1$ edges join vertices in A to the degree two vertices and these are all required, since there are exactly $2n + 1$ degree two vertices. Hence, $\langle A \rangle$ is regular of degree one, which is impossible, since n is odd. ∎

We denote by B, L the set of $n+1$ degree three vertices and the set of $2n$ degree two vertices respectively whose existence is asserted by Lemma 1. Elements of B and L will be called *b-vertices* and *ℓ-vertices* respectively. We note that each vertex of F is adjacent to at least one b-vertex. There are at most two vertices of degree two in $\langle B \rangle$ (otherwise there would be at most $(n-2)2 + 3 = 2n - 1$ edges available to dominate L) and the other vertices in $\langle B \rangle$ have degree one. Hence by parity there are 0 or 2 degree 2 vertices in $\langle B \rangle$ and, more specifically, $\langle B \rangle = mP_2, 2P_3 \cup mP_2$ or $P_4 \cup mP_2$ for some $m \geq 0$.

Lemma 2. *There is no graph F with $\langle B \rangle = 2P_3 \cup mP_2$.*

Proof: Suppose, to the contrary, that $\langle B \rangle = 2P_3 \cup mP_2$ for some $m \geq 0$. Let $U = \{u_1, \cdots, u_8\}$ and $V = \{v_1, \cdots, v_8\}$ be the vertex subsets of the two copies of P_3 in $\langle B \rangle$ and the degree two neighbours of these vertices (see Figure 3). The remainder of $\langle B \rangle$ is mP_2 where $m \geq 0$ and we denote by $W_i, i = 1 \cdots, m$ the vertex subset of the ith P_2 in $\langle B \rangle$ together with the four degree two neighbours of these vertices.

The edges of F missing in Figure 3 join pairs of vertices which appear to have degree one in the figure. By considering various b-vertices as the centre of a copy of $S(3, n)$ in F, it is easy to see that each missing edge joins vertices from distinct sets in $\{U, V, W_1, \cdots, W_m\}$.

Consider the copy of $S(3, n)$ in F with centre at u_1. Without loss of generality, the initial vertices of the three roads may be taken as $R_1 : u_1, u_6$, $R_2 : u_1, u_2, u_4$ and $R_3 : u_1, u_3, u_7$. Therefore u_5 and u_8 are terminals.

No matter how these roads enter and leave V, there is at least one terminal in V. The four terminal argument (4TA) then asserts that any road which enters a set W_i, uses precisely three vertices of W_i—one vertex of degree three and its two neighbours of degree two (otherwise a fourth terminal is created in W_i). Moreover,

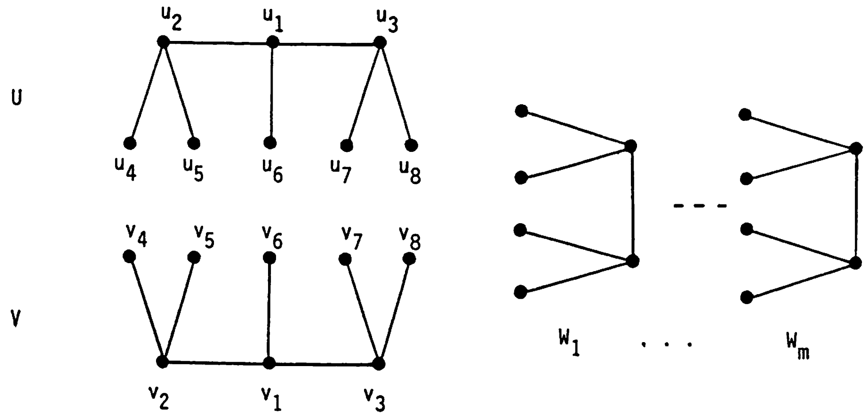

Figure 3. The case $\langle B \rangle = 2P_3 \cup mP_2$.

without losing generality, the third terminal is v_1 or v_4 (otherwise a fourth terminal would be created in V).

Suppose that the third terminal is v_1. Then by 4TA, some road terminates in v_6v_1. If $v_6v_1 \in R_1$, then $R_1 - \{u\}$ contains one vertex of U, two of V and $3q$ other vertices for some $q \geq 0$. Hence $n \equiv 0 \mod 3$, contrary to hypothesis. If v_6v_1 (without losing generality) is in R_2, then it is easy to show that the number of vertices on R_1 and R_3 have different residues $\mod 3$ which is impossible.

If the third terminal is v_4, then by 4TA, the road ending there either contains no other vertex of V, or it contains the subpath $Q_1 = v_5v_2v_1v_6$ or the subpath $Q_2 = v_7v_3v_8$. If R_1 terminates at v_4 and contains Q_1, then $n \equiv 0 \mod 3$. If R_1 terminates at V_4 and contains Q_2 or no vertex of V other than v_4, one may show that the number of vertices on R_2 andR_3 have difference residues $\mod 3$, a contradiction. If (say) R_2 terminates at v_4 and contains Q_2 or no vertex of V other than v_4, then $n \equiv 0 \mod 3$. Finally, if R_2 contains Q_1, then is once again easy to see that the number of vertices on R_1 and R_3 have different residues $\mod 3$. ∎

From Lemma 2 we know that $\langle B \rangle = mP_2$ or $P_4 \cup mP_2$. Form a graph F^* from F as follows: If $\langle B \rangle = mP_2$, delete all edges of $\langle B \rangle$ to form F^*. If $\langle B \rangle = P_4 \cup mP_2$, delete all edges of $\langle B \rangle$ except the single edge of $\langle B \rangle$ which connects the two vertices of degree two in $\langle B \rangle$. In each case the resulting graph F^* is regular of degree two.

Lemma 3. *Suppose that F^* has t_i cycles of length congruent to $i \mod 3$ for $i = 0, 1, 2$. Then*

(a) $t_1 \leq 1$
(b) $t_2 \leq 2$

(c) *either* t_1 *or* t_2 *is zero.*

Proof: All parts of this proof rely on the facts that the b-vertices of any cycle of F^* form a dominating set of the cycle and that the domination number of $\cup_{j=1}^{s} C_{k_j}$ where $\sum_{j=1}^{s} k_j = \lambda$, is at least $\lceil \frac{\lambda}{3} \rceil$.

(a) Suppose $t_1 \geq 2$ and F^* contains cycles of length $3q_1 + 1$ and $3q_2 + 1$. Then from the above, at least

$$\lceil \frac{3q_1 + 1}{3} \rceil + \lceil \frac{3q_2 + 1}{3} \rceil + \lceil \frac{3n + 1 - (3q_1 + 1) - (3q_2 + 1)}{3} \rceil$$
$$= (q_1 + 1) + (q_2 + 1) + (n - q_1 - q_2) = n + 2$$

vertices are required in any dominating set of F^* contrary to Lemma 1. The proofs of (b), (c) are omitted. ∎

Since the b-vertices dominate any cycle C of F^*, the vertex sequence of C may be represented by a sequence of α's, β's and γ's where α, β, γ means a b-vertex followed by two, one or zero ℓ-vertices respectively. This will be called an α, β, γ-*code* of the cycle. For example, an α, β, γ-*code* of the C_{19} of Figure 4 in which large and small dots represent b-vertices and ℓ-vertices respectively, is $\alpha\beta\alpha\alpha\gamma\alpha\beta\beta$.

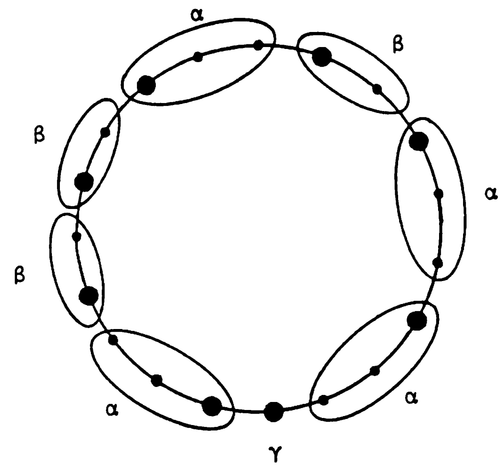

Figure 4. Example of α, β, γ-code

Domination considerations similar to those used in the proof of Lemma 3, also establish the following lemma. We omit the proof.

Lemma 4. *Let J be an α, β, γ-code of C, a cycle of F^*, with $|V(C)| \equiv j$ mod 3. Then*

(a) $j = 0$ *implies that J has only α's.*
(b) $j = 1$ *implies that J has α's and precisely one γ or α's and precisely two β's.*
(c) $j = 2$ *implies that J has α's and precisely one β.* ∎

An edge of $F - F^*$ is called a *chord* if it is adjacent to two vertices of the same cycle of F^*, otherwise it is called a *link*. We shall make repeated use of the following proposition whose proof is obvious and omitted.

Proposition 1. *Suppose that a road uses a chord $[y, z]$ of the cycle C or a link $[y, z]$ to or from C with $z \in V(C)$ and that z is not the centre of the copy of $S(3, n)$. If the neighbours of z on C are ℓ-vertices, then one of these is a terminal of the copy.* ∎

Various situations will be eliminated by this proposition and 4TA.

Lemma 5. *F^* has at most two cycles.*

Proof: Suppose the contrary. Then by Lemma 3, F^* has a cycle C_1 of length congruent to 0 mod 3. Lemma 3 and the fact that there are $3n + 1$ vertices, imply that F^* has a second cycle C_2 whose length is congruent to 0 or 2 mod 3. Consider a copy of $S(3, n)$ in F whose centre is in C_1. Note that such a copy exists since no ℓ-vertex x of C_1 is adjacent to a b-vertex not in C_1, and there is a copy of $S(3, n)$ in F with a_1 at x. Suppose that all three roads leave C_1. By Proposition 1 and Lemma 4, there are at least two terminals in C_1. Therefore two of the roads must eventually re-enter C_1 and the same two results provide a contradiction using 4TA.

Hence at least one road stays in C_1 which therefore has at least $n + 4$ vertices. Similarly C_2 has at least $n + 3$ vertices and so F^* has a cycle C_3 with at most $n - 6$ vertices. Consider any copy of $S(3, n)$ in F with centre in C_3. Two roads must enter each of C_1 and C_2 and therefore Proposition 1 and 4TA give the required contradiction. ∎

Lemma 6. *F^* is hamiltonian.*

Proof: By the above lemmas, if F^* is not hamiltonian, then F^* has exactly two cycles C_1, C_2 where (case 1) these have lengths congruent to 0 and 1 mod 3 respectively or (case 2) both lengths are congruent to 2 mod 3.

In each case consider a copy of $S(3, n)$ in F with centre in C_1. An identical argument to that used in Lemma 5, obtains a contradiction if three roads leave C_1. If two roads leave by b-vertices neither of which is the centre, there are two terminals in C_1 (Proposition 1). Therefore one of the roads must re-enter C_1 which means that the third terminal is in C_1. However, the two roads enter C_2 and one must eventually leave C_2 to return to C_1. At least one of the three entry or exit vertices of these roads in C_2, has ℓ-vertex neighbours in either of the two cases (Lemma 4). Hence, there is a fourth terminal in C_2—a contradiction. If exactly one road leaves C_1, then C_1 has at least $2n+1$ vertices and so C_2 has at most n vertices. Consider a copy of $S(3,n)$ in F with centre in C_2. All three roads must leave C_2. Proposition 1 asserts the existence of at least one terminal in C_2 and three in C_1 which is impossible.

Hence in each case, for every centre in C_1, two roads leave C_1, one of which leaves directly from the centre. Hence C_1 has no chord and has at least $n+4$ vertices.

Case 1: C_1 and C_2 have lengths congruent to 0 and 1 $\mod 3$ respectively.

Since C_1 has no chord and for each centre in C_1 one road remains in C_1, it follows that $n \equiv 2 \mod 3$. Since F is connected, C_2 has a b-vertex, say z, incident with a link. Consider the copy of $S(3,n)$ with centre z. The size of C_1 implies that at least two roads enter C_1 and no road which enters C_1 may leave again (4TA). Since C_1 has no chord, each road entering C_1 uses $3q$ vertices of C_1 for some q. But the road from z which uses the link, has all vertices but z in C_1. Hence $n \equiv 0 \mod 3$—a contradiction.

Case 2: C_1 and C_2 both have lengths congruent to $2 \mod 3$.

In this case, the above implies that neither C_1 nor C_2 has a chord. Therefore the numbers of b-vertices in C_1 and C_2 are equal. Hence both C_1 and C_2 have $\frac{(3n+1)}{2}$ vertices. By Lemma 4, $C_1(C_2)$ has exactly one ℓ-vertex $w_1(w_2)$ which is adjacent to two b-vertices. Consider the copy of $S(3,n)$ whose centre z_1 is a b-vertex of C_1 whose distance on C_1 from w_1 is maximum.

One road, say R_1, uses the link $[z_1, z_2]$ and has all its n non-centre vertices in C_2. A second road, say R_2, leaving C_1 at y_1 by the link $[y_1, y_2]$, has $k_1 = \frac{n-1}{2}$ non-centre vertices in C_1 and hence the remaining $k_2 = \frac{n+1}{2}$ vertices in C_2. Finally the third road R_3 remains in C_1 and has terminal adjacent to y_1. Note that the portion of $R_1(R_2)$ in C_2 forms a path from $z_2(y_2)$ to a neighbour of $y_2(z_2)$.

By choice of centre, $w_1 \in R_3$ and so $n \equiv 1 \mod 3$. Whether or not w_2 is included, there is no path of n vertices in C_2 (disjoint from R_2) from z_2 to a neighbour of y_2. Hence R_1 cannot exist and the lemma is proved. ∎

The above lemma shows that if the lower bound is achieved by F, then F^* is isomorphic to C_{3n+1} and all edges of $F - F^*$ are chords of F^*. By Lemma 4(b) the $\alpha\beta\gamma$-code of F^* consists of either one γ and the rest α's or two β's and the rest α's.

Lemma 7. *For any copy of $S(3,n)$ in F, the three roads use exactly one chord of F^*.*

Proof: By Proposition 1, the roads of each copy use at most three chords. If the roads of a copy use exactly three chords, then, again by Proposition 1, the $\alpha\beta\gamma$-

code of F^* consists of one γ and the rest α's. In this case the roads either use exactly one chord each or one road uses all three chords, for otherwise the lengths of the roads have different residues $\mod 3$. But this is only possible if $n \equiv 0 \mod 3$, which contradicts the fact that $n \equiv 1$ or $n \equiv 5 \mod 6$.

Hence suppose the roads of a copy Q of $S(3, n)$ use precisely two chords. Let the vertex sequence of F^* be $(u_0, \cdots, u_{3n})$ with u_0 the centre of Q. We consider two cases.

Case 1: No copy of $S(3, n)$ in F has chords on two distinct roads.

Say the road R_1 of Q uses two chords of F^*. Roads R_2 and R_3 use only edges of F^* and therefore without loss of generality R_1 (heavy lines in Figure 5) may be taken to have vertex sequence $(u_0, u_m, u_{m-1}, \cdots, u_{n+1}, u_{2n}, u_{2n-1}, \cdots, u_{m+1})$ for some integer m with $n+1 < m < 2n$. Now consider the copy of $S(3, n)$ with centre u_{2n}. One of its roads, say T_1, uses the chord $[u_{2n}, u_{n+1}]$. Since there are only $n-2$ vertices between u_{2n} and u_{n+1}, a second road T_2 has vertex sequence $(u_{2n}, u_{2n+1}, \cdots, u_{3n})$ and the third road T_3, which includes the edge $[u_{2n}, u_{2n-1}]$, must use a chord $[u_i, u_j]$ where $i \in \{n+2, \cdots, 2n-1\}$ and $j \in \{0, \cdots, n\}$. This is contrary to assumption.

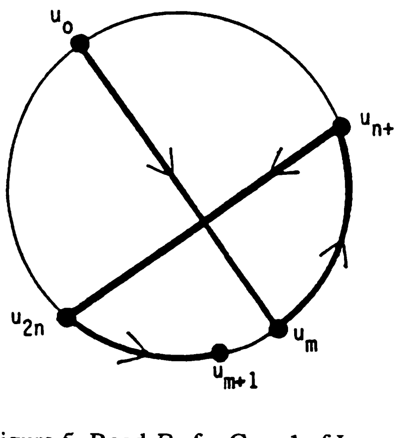

Figure 5. Road R_1 for Case 1 of Lemma 7

Case 2: There is a copy of $S(3, n)$ in F with chords on distinct roads.

Without losing generality we may assume that Q is such a copy and that the two chords are used by the roads R_1 and R_2. Let R_3 have vertex sequence $(u_0, u_1, \cdots, u_n)$ and say R_1 contains the edge $[u_0, u_{3n}]$. Then R_2 has vertex sequence either $(u_1, u_m, u_{m+1}, \cdots, u_{m+n-1})$ where $m \in \{n+1, \cdots, 2n\}$, or $(u_0, u_\ell, u_{\ell-1}, \cdots, u_{\ell-n+1})$ where $\ell \in \{2n, \cdots, 3n-2\}$. In the latter case, $u\ell+1$ is necessarily a b-vertex of F (and hence by Lemma 4(b), $\ell \neq 3n-2$) and R_1 uses a chord of F^* incident with $u\ell+1$. By considering the road R_1 (R_2 respectively) if $\ell = 2n$ ($\ell \in \{2n+1, \cdots, 3n-3\}$ and R_1 uses the chord $[u_{\ell+1}, u_{\ell-n}]$), we see that $n \equiv 0 \mod 3$, a contradiction. Hence $\ell \in \{2n+1, \cdots, 3n-3\}$ and R_1 uses the chord $[u_{\ell+1}, u_{n+1}]$. But then the copy of $S(3, n)$ with centre u_ℓ has a road T containing the edge $[u_\ell, u_{\ell+1}]$ and a chord of F^* incident with u_{3n}, which is impossible by Lemma 4(b).

Similarly, if R_2 has vertex sequence $(u_0, u_m, u_{m+1}, \cdots, U_{m+n-1})$ where $m \in \{n+2, \cdots, 2n\}$ and R_1 uses the chord $[u_{m+n}, u_{m-1}]$, we obtain $n \equiv 0 \mod 3$ by considering the length of R_2. Hence in this case, either

(a) $m \in \{n+3, \cdots, 2n\}$ and R_1 uses the chord $[u_{m+n}, u_{n+1}]$, or

(b) $m = n+1$ and R_1 uses a chord $[u_i, u_{2n+1}]$ where $i \in \{2n+3, \cdots, 3n\}$.

The only possible $\alpha\beta\gamma$-code in either case is two β's and the rest α's, where R_1 and R_2 contain one β-sequence each (for otherwise the roads have lengths congruent to different residues modulo 3). If (a) occurs and the β-sequence on R_1 is contained in $(u_{m+n}, \cdots, u_{3n}, u_0)$ ($(u_{n+1}, u_{n+2}, \cdots, u_{m-1})$ respectively), then the copy of $S(3, n)$ with centre u_{m+n} (u_m respectively) has two roads containing one chord each and the third road must contain a β-sequence, which is impossible. If (b) occurs, a similar contradiction is obtained using a copy of $S(3, n)$ centred at u_{2n+1}. This completes the proof. ∎

We now require a further definition.

Let the graph G have $3n+1$ vertices and a Hamilton cycle C. Define vertex u of G to be (n, C)*-separated* if there exists a vertex v adjacent to u such that there is a path in C from u to v of length $n+1$.

The connection between (n, C)-separation and copies of $(S(3, n), a_1)$ is explained by the following simple fact.

Proposition 1. *If each vertex w of the $(3n+1)$-vertex graph G is adjacent to a vertex u which is (n, C_u)-separated for some Hamilton cycle C_u, then* $(S(3, n), a_1) \prec G$.

Proof: Immediate from the definition. ∎

Note that the $\alpha\beta\gamma$-code of F^* as described in Lemma 4 implies that F^* is the unique Hamilton cycle of F. It follows from Lemma 7 that if F achieves the lower bound, then each b-vertex of F is (n, F^*)-separated. We finally prove that 1 and 7 are the only values of $n \equiv 1$ or $5 \mod 6$ for which this is possible.

Lemma 8. *Each b-vertex of F is (n, F^*)-separated if and only if $n = 1$ or 7.*

Proof: By Lemma 4(b), F^* has $\alpha\beta\gamma$-code consisting of α's and either exactly two β's or exactly one γ. Suppose that each b-vertex of F is (n, F^*)-separated and for $b \in B$, let $f(b)$ denote the other endvertex of the chord of F^* incident with b.

Case 1: There are two β's.

The shorter path in F^* from each b to $f(b)$ contains exactly one of the β-sequences or else there would be two chords cutting off numbers of vertices of

F^* with different residues mod 3. This implies $n \equiv 1 \mod 3$. Hence, if F^* has vertex sequence $(0, \cdots, 3n)$, then without loss of generality we may take the β-sequences at $0, 1$ and $\lambda, \lambda + 1$ where $0, \lambda$ are b-vertices and $n+1 \leq \lambda \leq \frac{3n+1}{2}$. The possibilities for $f(3n-2)$ are $2n-3$ or $n-2$. Suppose $f(3n-2) = 2n-3$. Then since the shorter $(2n-3, 3n-2)$-path on F^* contains a β, we have $2n-3 \leq \frac{3n+1}{2}$, i.e. $n \leq 7$. Hence for $n > 7$, $f(3n-2) = n-2$. The candidates for $f(2n-1)$ are $3n$ and $n-2$. The former is impossible since the path from $2n-1$ to $3n$ contains no β. Therefore, $f(2n-1) = f(3n-2) = n-2$, a contradiction, which shows that Case 1 is impossible for $n > 7$. It is also impossible if $n = 5$ since $5 \not\equiv 1 \mod 3$.

Case 2: There is exactly one γ.

It is not possible for the shorter path in F^* from each b to $f(b)$ to include the γ if $n > 3$. Hence for each chord the path must exclude the γ. Therefore if $3n$ and 0 are the adjacent b-vertices, then $f(0) = f(2n+2) = n+1$, which is impossible.

Cases 1 and 2 have eliminated all values of $n \equiv 1$ or $5 \mod 6$ except 1 and 7. Figure 2 shows that for these values each b-vertex of F has the (n, F^*)-separation property. ∎

The above sequence of lemmas and the graphs of Figure 2 have proved:

Theorem 2.

(a) *If* $n \equiv 1$ *or* $5 \mod 6$ *and* $n \notin \{1, 7\}$, *then* $h(S(3, n), a_1) > \frac{(7n+3)}{2}$.
(b) *If* $n \in \{1, 3, 7\}$, *then* $h(S(3, n), a_1) = \frac{(7n+3)}{2}$. ∎

3. Upper bound for the rotation number of $(S(3, n), a_1)$

One further observation is required to establish the upper bound of two or three edges more than the lower bound. Let $[v_1, v_2]$ be an edge of the Hamilton cycle C of the graph G of order p and let $A = \{u_0, u_1, \cdots, u_{k+1}\}$ be a set (in order) of sequential vertices of C distinct from v_1 and v_2. Suppose G contains the edges $[v_1, u_1]$, $[v_2, u_k]$ and $[u_0, u_{k+1}]$. Then the cycle C^* obtained from C by deleting the edges $[u_0, u_1]$, $[u_k, u_{k+1}]$ and $[v_1, v_2]$ and adding the edges $[v_1, u_1]$, $[v_2, u_k]$ and $[u_0, u_{k+1}]$ is also a Hamilton cycle of G (see Figure 6).

Suppose that two vertices of $V(G) - A$ are joined by a path of length t in C which includes the edge $[v_1, v_2]$. Then it is clear that these vertices are joined by a path of length $t + k$ and a path of length $p - t - k$ in C^*.

Theorem 3. *For any* $n \geq 2$,

$$h(S(3, n)a_1) \leq \begin{cases} \lceil \frac{7n+3}{2} \rceil + 3 & \text{if } n \equiv 3 \mod 6 \\ \lceil \frac{7n+3}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

Proof: Technical details depend on the residue class of n modulo 6. We give the proof for $n = 6q + 1$, the other cases being similar, except that in the case where

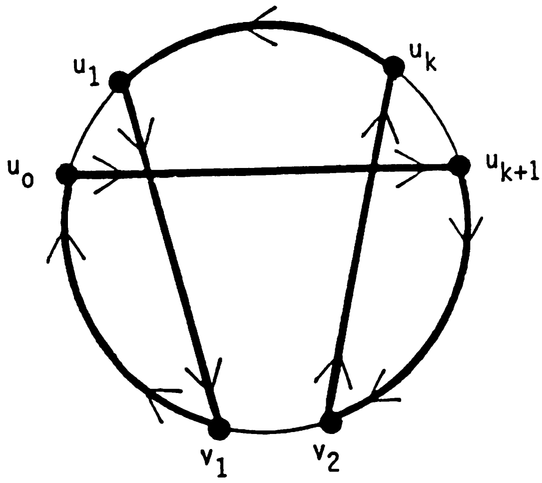

Figure 6. Two Hamilton cycles of G

$n = 6q+3$, this method yields an upper bound of three more than the lower bound. Let C be a cycle of order $p = 3n+1$ with vertex sequence $(0, 1, \cdots, 3n)$. Various edges will be added to C forming a graph F with the desired properties.

Firstly, add the sets $\{[3i, 3i+n+1] : i = 0, \cdots, q-1\}$ and $\{[3i+2, 3i+n+3] : i = 3q, \cdots, 4q-1\}$. So far each vertex of

$$X = \{p-(3q+1), p-3q, \cdots, 0, \cdots, 3q-3, 3q-2\} \cup \{6q+1, 6q+2, \cdots, 12q\}$$

is adjacent to an (n, C)-separated vertex and it remains to consider the set $Y = Y_1 \cup Y_2$, where $Y_1 = \{3q-1, 3q, \cdots, 6q\}$ and $Y_2 = \{12q+1, 12q+2, \cdots, 15q+2\}$.

Now add the set of edges $E = \{[3q+3j, 12q+2+3j] : j = 0, \cdots, q\}$. Observe that each vertex of Y is adjacent to an endvertex of E and the endvertices of any edge in E are joined by a path of length $9q+2$ in C which includes the edge $[0, 1]$. Apply the observation preceding this proof with $v_1 = 0, v_2 = 1, u_0 = n = 6q+1$ and $u_{k+1} = 9q+2$. Two extra edges, namely the edges $[1, 9q+1]$ and $[6q+1, 9q+2]$, are added to form a new Hamilton cycle C^* and all endvertices of E are (n, C^*)-separated. It follows by Proposition 2 that $(S(3,n)a_1) \prec F$ and

$$\begin{aligned} h(S(3,n)a_1) &\leq 3n+1+2q+q+1+2 \\ &= 3n+1+\frac{1}{2}(n+1)+2 \\ &= \lceil \frac{7n+3}{2} \rceil + 2. \ \blacksquare \end{aligned}$$

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