Polygonal Path Constructions for Tuscan-k Squares

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Introduction and summary.

An *Italian square* is an $n \times n$ array in which each of the symbols $1, 2, \ldots, n$ appears exactly once in each row.

By definition a *Tuscan-k square* is an Italian square with the further property that for any two symbols a and b, and for each m from 1 to k, there is at most one row in which b is the mth symbol to the right of a.

Tuscan-1 squares are known to exist [7] for all n except $n = 3$ and $n = 5$, although no simple construction is known for odd $n > 5$.

By definition a *Circular Tuscan-k array* is an $n \times (n+1)$ array A in which each of the symbols $\star, 1, 2, \ldots, n$ appears exactly once in each row, and in which the Tuscan-k square property holds when the rows are taken to be circular. In matrix notation the rows are indexed 1 to n while the columns are indexed 0 to n $(\text{mod } n+1)$. For each m from 1 to k, $(A(i,j), A(i,j+m)) \neq (A(r,t), A(r,t+m))$ unless $i = r$ and $j = t$.

A Tuscan-$(n-1)$ square is called a *Florentine square*, and a circular Tuscan-n array is called a *Circular Florentine array*. A Florentine square is known to exist whenever $n+1$ is prime, and not known otherwise. All those known can be made into circular Florentine arrays by adjoining a column full of asterisks on the left. When a Florentine square is Latin we call it a *Vatican square*.

The "Add Zero" transformation [1] starts by adjoining a zeroth column full of asterisks on the left of a Tuscan-k square. The rows are then cycled to bring another symbol into the zero column, which is then deleted. The symbol $\star$ is then replaced with the deleted symbol. A Tuscan-k square can be made into a circular Tuscan-k square iff all of its Add Zero transforms are Tuscan-k squares.

Our main effort has been to explore the Tuscan possibilities with the polygonal path construction described in section C. An exhaustive (computer) search of polygonal path Tuscan-1 squares up to $n = 14$ was carried out by Peter Pacini, who is a sophomore at the University of Southern California. Sample results are exhibited in the figures of section C.

Twelve questions.

For which integers $n > 1$ do these exist?

1. Tuscan-2 square.
2. Polygonal Path Tuscan-2 square.
3. Symmetric Polygonal Path Tuscan-2 square.
4. Circular Tuscan-2 array.
5. Polygonal Path Circular Tuscan-2 array.
6. Symmetric Polygonal Path Circular Tuscan-2 array.

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7. Florentine square.
8. Polygonal Path Florentine square.
9. Symmetric Polygonal Path Florentine square.
10. Circular Florentine array.
11. Polygonal Path Circular Florentine array.
12. Symmetric Polygonal Path Circular Florentine array.

Our current state of knowledge is as follows.

1. Tuscan-2 squares.
   None exist for odd $n \leq 7$.
   None known for odd $n > 1$.
   Six in standard form for $n = 8$, none of them circular, none of them Latin.
   There are no known examples of Tuscan-3 squares when $n+1$ is composite. We do not have proof that $n \times n$ Tuscan-2 squares exist for infinitely many composite values of $n+1$.
2. Polygonal Path Tuscan-2 squares.
   Enumerated by exhaustive search to $n = 18$. See E12.
   All that exist from $n = 4$ to $n = 14$ are in Figures 4A, 6A, 10A, 10B, 14B, 14C.
3. Symmetric Polygonal Path Tuscan-2 squares.
   (3.) exist iff (6.) exist, by Lemma E12.
   Enumerated by exhaustive search to $n = 28$.
4. Circular Tuscan-2 arrays.
   Existence implies a pair of orthogonal $(n+1) \times (n+1)$ Latin squares by Theorem A.
   Existence guarantees non-Latin Tuscan-2 squares.
5. Polygonal Path Circular Tuscan-2 arrays.
   See Lemma E1.
   Some of these have asymmetric paths.
6. Symmetric Polygonal Path Circular Tuscan-2 arrays.
   (6.) iff (3.).
   We have examples by exploratory search up to $n = 52$.
7. Florentine squares.
   None are known when $n+1$ is composite.
   None exist for odd $n \leq 9$.
   No Vatican squares exist for odd $n \leq 11$.
8. Polygonal Path Florentine squares.
   Equivalent to a singly periodic Costas array, by Theorem D.
   For $n \leq 20$ the only ones that exist are those in (12.) where $n+1$ is prime.
9. Symmetric Polygonal Path Florentine squares.
   Existence implies that an "isotope" of C_n exists with 0 on the main diagonal, $\frac{n}{2}$ on the cross main diagonal, and each of the remaining symbols

forming a pattern of n non-attacking queens. In the terminology of [2] an isotope of a Latin square is anything obtainable from the Latin square by permuting rows, columns, and symbols.

10. Circular Florentine arrays.
    Non-existence is known for all odd $n > 1$, by Theorem B.
    Non-existence is known for composite odd $n+1$ when the Bruck-Ryser theorem rules out a projective plane of the same odd order, by Theorem A.
11. Polygonal Path Circular Florentine arrays.
    (11.) exist iff (12.) exist, by Lemma E1.
    Non-existence is known for several infinite classes, covered in E1-E12.
    Fewer than 33 values of $n \leq 1000$ remain unsettled.
12. Symmetric Polygonal Path Circular Florentine arrays.
    These exist whenever $n+1$ is prime.

A. A necessary condition for circular Tuscan-k arrays.

Theorem A. *Any circular $n \times (n+1)$ Tuscan-k array A induces a set $\{A_1, A_2, \ldots, A_k\}$ of pairwise orthogonal $(n+1) \times (n+1)$ Latin squares.*

Proof: Let the rows of A be numbered $1, 2, \ldots, n$. Each row of A has a permutation of the symbols $0, 1, 2, \ldots, n$. The columns of A are numbered $0, 1, 2, \ldots, n$ modulo $n+1$ so that, for example, we consider the symbol in row r, column 1 of A to be two steps to the right of the symbol in row r, column n of A. Using "$A_t(i,j)$" to denote the symbol in row i, column j of A_t, we define $A_1, A_2, \ldots, A_k$ as follows. For $t = 1$,

$$A_1(i,j) = \begin{cases} 0 & \text{if } i = j \\ r & \text{if the symbol } j \text{ occurs one step to the right} \\ & \text{of the symbol } i \text{ in row } r \text{ of } A. \end{cases}$$

For each t from 2 to k,

$$A_t(i,j) = \begin{cases} j & \text{if } i = j \\ h & \text{if the symbol } h \text{ is } t \text{ steps to the right of the symbol } i \text{ in the row} \\ & \text{of } A \text{ in which the symbol } j \text{ is one step to the right of } i. \end{cases}$$

When $i = j$ it is clear that $A_t(i,j)$ is uniquely determined. When $i = j$ the Tuscan-1 property of A tells us that j is one step to the right of i in one and only one row of A. Thus, for each ordered pair (i,j) of symbols from A, and each t from 1 to k, $A_t(i,j)$ is a uniquely determined symbol from A.

Recalling that each symbol occurs exactly once in each row of A, it is clear that if $A_1(i,x) = A_1(i,y)$, or if $A_1(x,j) = A_1(y,i)$, then $x = y$. Thus, we need only the Tuscan-1 property of A to see that A_1 is a Latin square.

For each t from 2 to k the Tuscan-k property of A tells us that h is t steps to the right of i in one and only one row of A, unless $h = i$. Likewise h is $t-1$

steps to the right of j in one and only one row of A, unless $h = j$. Consider the ordered triples (i, j, h) such that $A_t(i, j) = h$. If $i = j$ or $i = h$ or $j = h$, we must have $i = j = h$. If $i \neq j$ or $i \neq h$ or $j \neq h$, then $i \neq j \neq h \neq i$ and any two of the symbols uniquely determine the remaining one. Consequently, A_t is a Latin square.

To show that for $1 \leq s < t \leq k$, A_s and A_t are orthogonal we need to see that the set

$$\{(A_s(i,j), A_t(i,j)): \quad 0 \leq i \leq n, \ 0 \leq j \leq n\}$$

is equal to the set

$$\{(i,j): \quad 0 \leq i \leq n, \ 0 \leq j \leq n\}.$$

This is easy for $s = 1$ because on the positions where A_1 is constant, A_t has a permutation of the symbols $0, 1, 2, \ldots, n$.

For $s > 1$ first notice that on the positions where $i = j$ we have all the ordered pairs with $A_t(i,j) = A_s(i,j) = j$. Where $i \neq j$ notice that the symbol $A_t(i,j)$ is always $t - s$ steps to the right of $A_s(i,j)$ in some row of A, so we have all the ordered pairs with $A_t(i,j) \neq A_s(i,j)$. ∎

B. Subproblem for circular Florentine arrays.

The question whether any $n \times n$ Florentine squares exist for odd $n > 1$ is still open. We shall prove that the answer is negative for $n \times (n+1)$ circular Florentine arrays.

Let $f(n+1)$ be the maximum number of symbols which can be placed in an $n \times (n+1)$ row-circular array so that each symbol appears once in each of the n rows, and for each ordered pair (a, b) of distinct symbols b is t steps to the right of a exactly once for each t from 1 to n. Most of what we know about f is summarized by I and II.

I. $f(n+1) \geq p$, where p is the smallest prime divisor of $n+1$.

II. $f(n+1) = 2$, when 2 divides $n+1$.

Proof of I: Number the columns with the least residues $0, 1, \ldots, n \pmod{n+1}$, and number the rows $1, 2, \ldots, n$. Use the symbols $0, 1, \ldots, p-1$. The construction is: In row r put the symbol a in the column whose number is congruent to ra modulo $n+1$. For distinct symbols a and b we can assume $0 \leq a < b < p$, so that $0 < b - a < p$ and therefore, since p is the smallest prime divisor of $n+1$, $b - a$ must be coprime to $n+1$. If b is the same number of steps to the right of a in rows r and s, then $rb - ra \equiv sb - sa \pmod{n+1}$. But then $r(b-a) \equiv s(b-a)$ and we can cancel $(b - a)$ leaving $r \equiv s \pmod{n+1}$. Therefore, $r = s$. ∎

Comment: When $n+1 = p$ the above construction gives the same $n \times n$ Vatican square as the multiplicative group of integers modulo p.

Proof of II: Supposing 2 divides $n+1$, we know from I that $f(n+1) \geq 2$.

To show that $f(n+1) < 3$ suppose a, b, and c are three distinct symbols in the $n \times (n+1)$ circular array. Letting a_r, b_r, and c_r denote the column numbers locating a, b, and c in row r, each of the values $(b_r - a_r)$, $(c_r - b_r)$, and $(c_r - a_r)$ is supposed to run through $1, 2, \ldots, n$ in some order as r takes the row numbers from 1 to n. But $(b_r - a_r) + (c_r - b_r) \equiv (c_r - a_r) \pmod{n+1}$. Thus, $f(n+1) \geq 3$ would require that $\sum_{r=1}^{n} r + \sum_{r=1}^{n} r \equiv \sum_{r=1}^{n} r \pmod{n+1}$, or $0 \equiv \frac{n+1}{2} \pmod{n+1}$, which does not happen when 2 divides $n+1$. ∎

Theorem B. *An $n \times (n+1)$ circular Florentine array does not exist for odd $n > 1$.*

Proof: An immediate consequence of II. ∎

Comment:

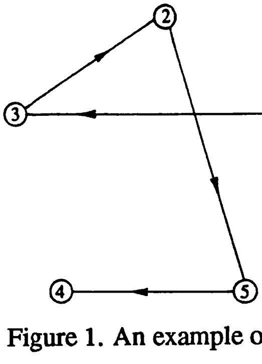
$$f(15) \geq 5.$$
$$f(9) = 3.$$

C. Description of the polygonal path construction.

Number the vertices of a regular n-gon with $1, 2, \ldots, n$ in counterclockwise order.

A path which starts at one vertex, ends at another, and proceeds along directed chords to visit the n vertices once each, is what we will call a *polygonal path.* If the picture of one polygonal path can be rotated to make it coincide with another, we consider them the same. Thus, $(n-1)!$ will be the number of different polygonal paths.

Rotation of any polygonal path constructs a Latin square as follows. Put the starting vertex of the path on the vertex of the n-gon which is numbered 1. Let the order in which the path visits the numbers 1 to n determine their order in the first row of the Latin square. Then rotating the polygonal path successively to start on $2, 3, \ldots, n$ determines the second row, third row, and so on to the nth row.



1	3	2	5	4
2	4	3	1	5
3	5	4	2	1
4	1	5	3	2
5	2	1	4	3

Figure 1. An example of polygonal path construction with $n = 5$.

Observe that each column of the Latin square has the numbers 1 to n in the same cyclic order. This happens because each column, reading down, records the successive numbers covered by one vertex of the path. Consequently, every polygonal path construction can be read as an addition table for C_n, the cyclic group of integers mod n. Figure 2 illustrates C_6 in its familiar diagonal stripe form.

Using matrix notation for a polygonal path construction A, let $A(i,j)$ denote the number where the ith row meets the jth column. The observation made above is equivalent to the fact that

$$1 + A(i,j) \equiv A(i+1,j) \pmod{n},$$

with the understanding that the row indices are taken mod n also.

Another equivalent fact is emphasized in the following congruence:

$$A(i,j+c) - A(i,j) \equiv A(i+b,j+c) - A(i+b,j) \pmod{n}$$

where $1 \leq j < j+c \leq n$ while the i's and b's are taken mod n.

Figure 3 shows an example of the "Zig Zag" construction for Tuscan-1 squares, discussed in [1]. The reason why it works is that the directed chords of the polygonal path are all different – one cannot be rotated to coincide with another.

Lemma C1. *A polygonal path generates a Tuscan-1 square iff all the directed chords of the path are different. Equivalently, in each row i of the Latin square A constructed by the polygonal path all the differences $A(i,j+1) - A(i,j)$ must be distinct modulo n.*

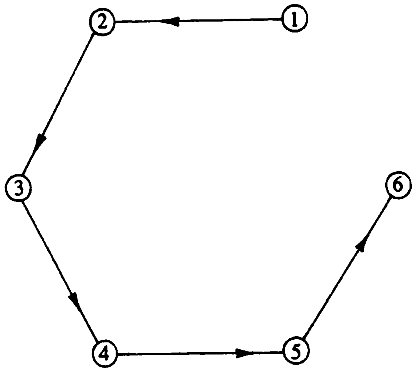
Proof: Direct observation. ∎

Lemma C2. *For odd $n > 1$ the polygonal path construction cannot generate a Tuscan-1 square.*

Proof: If $n = 2l+1$, then to be all different as Lemma A1 says, the $2l$ directed chords would have to be $+1, -1, +2, -2, \ldots, +l, -l$. But then they would sum to zero, the path would have to end where it began, and so would not be a path. ∎

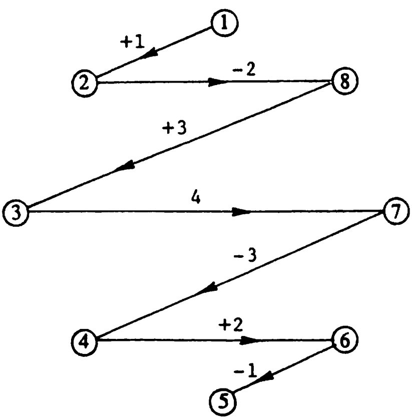
Remark: Using Lemma A2, together with our earlier observation that $(n-1)!$ is the number of polygonal path $n \times n$ Latin squares, and that all of them are "isotopes" of C_n, we can deduce that an odd Tuscan-1 square can never be obtained by row-column-symbol permutations of C_n.

The idea is as follows. Since row and/or symbol permutations do not affect the property of being Tuscan-1, we can without loss of generality reverse those permutations, and put the supposed odd Tuscan-1 square back into a form which can be reached by column permutations only from, say, the original diagonal stripe form of C_n. The last step will be to circulate rows so that the first column reading



+	6	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	4	5	6	1	2
4	4	5	6	1	2	3
5	5	6	1	2	3	4
6	6	1	2	3	4	5

Figure 2. Addition table for C_6.



1	2	8	3	7	4	6	5
2	3	1	4	8	5	7	6
3	4	2	5	1	6	8	7
4	5	3	6	2	7	1	8
5	6	4	7	3	8	2	1
6	7	5	8	4	1	3	2
7	8	6	1	5	2	4	3
8	1	7	2	6	3	5	4

Figure 3. Zig Zag Tuscan-1 square for $n = 8$.

down has the numbers 1 to n in natural order. In this form the square must be one of the $(n-1)!$ possible squares generated by the polygonal path construction. But not one of them has the Tuscan-1 property.

Tuscan squares by the polygonal path construction.

Nodes	$\phi(N)$	Automorphism equivalence classes	Symmetric polygonal path	Asymmetric polygonal path	Percent symmetric	Squares in standard form	Polygonal paths per square
4	2	1	1	0	100.0	1	2
6	2	2	2	0	100.0	2	2
8	4	4	2	2	50.0	6	4
10	4	42	12	30	28.6	72	4
12	4	504	44	460	8.7	964	4
14	6	7492	96	7396	1.3	14,888	6

Non-Latin Tuscan squares derived from the polygonal path constructions by the add-zero transformation

	Symmetric polygonal path		Asymmetric polygonal path	
Nodes	Total non-Latins	Non-Latins per path	Total non-Latins	Non-Latins per path
4	0	0	0	0
6	6	6	0	0
8	16	8	32	16
10	110	10	600	20
12	516	12	11,040	24
14	1344	14	207,088	28

Figures 4A, 6A, 10A, 12A, 10B, 14B, and 14C represent all polygonal path Tuscan-2 squares that exist from $N = 4$ to $N = 14$.
Figure 8A shows an example of an asymmetric polygonal path Tuscan-1 having $18 = 2(N+1)$ squares in its superequivalence class.
Figure 10C shows a symmetric polygonal path Tuscan-1 having $11 = N + 1$ squares in its superequivalence class.
Figure 14A exhibits both standard form and path form for a symmetric polygonal path Tuscan-1 square with $N = 14$.

Lemma C3. *When $n = 2L$, any polygonal path which generates an $n \times n$ Tuscan-1 square must have length congruent to L modulo n.*

Proof: The length of the path modulo n is given by the sum of its chords. To be all different the chords must be (in some order) $+1, -1, +2, -2, \ldots, +(L-1), -(L-1)$, and L. Since $L \equiv -L \pmod{2L}$ it occurs only once, and so we have the sum $\equiv L \pmod{n}$. ∎

The Vatican squares for $N = 4$ to $N = 12$
all of them constructed from symmetric polygonal paths

Figure 4A
The only polygonal path
Tuscan for $N = 4$

STANDARD FORM

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

PATH FORM

1	4	2	3
2	1	3	4
3	2	4	1
4	3	1	2

Figure 6A
The only polygonal path
Tucsan-2 for $N = 6$

STANDARD FORM

1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

PATH FORM

1	5	6	3	2	4
2	6	1	4	3	5
3	1	2	5	4	6
4	2	3	6	5	1
5	3	4	1	6	2
6	4	5	2	1	3

Figure 10A
One of the two polygonal path Tuscan-2 squares for $N = 10$

STANDARD FORM

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	7	9
3	6	9	1	4	7	10	2	5	8
4	8	1	5	9	2	6	10	3	7
5	10	4	9	3	8	2	7	1	6
6	1	7	2	8	3	9	4	10	5
7	3	10	6	2	9	5	1	8	4
8	5	2	10	7	4	1	9	6	3
9	7	5	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

PATH FORM

1	2	9	3	5	10	8	4	7	6
2	3	10	4	6	1	9	5	8	7
3	4	1	5	7	2	10	6	9	8
4	5	2	6	8	3	1	7	10	9
5	6	3	7	9	4	2	8	1	10
6	7	4	8	10	5	3	9	2	1
7	8	5	9	1	6	4	10	3	2
8	9	6	10	2	7	5	1	4	3
9	10	7	1	3	8	6	2	5	4
10	1	8	2	4	9	7	3	6	5

Figure 12A
The only polygonal path Tuscan-2 for $N = 12$

STANDARD FORM

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	1	3	5	7	9	11
3	6	9	12	2	5	8	11	1	4	7	10
4	8	12	3	7	11	2	6	10	1	5	9
5	10	2	7	12	4	9	1	6	11	3	8
6	12	5	11	4	10	3	9	2	8	1	7
7	1	8	2	9	3	10	4	11	5	12	6
8	3	11	6	1	9	4	12	7	2	10	5
9	5	1	10	6	2	11	7	3	12	8	4
10	7	4	1	11	8	5	2	12	9	6	3
11	9	7	5	3	1	12	10	8	6	4	2
12	11	10	9	8	7	6	5	4	3	2	1

PATH FORM

1	8	5	3	4	12	6	10	9	11	2	7
2	9	6	4	5	1	7	11	10	12	3	8
3	10	7	5	6	2	8	12	11	1	4	9
4	11	8	6	7	3	9	1	12	2	5	10
5	12	9	7	8	4	10	2	1	3	6	11
6	1	10	8	9	5	11	3	2	4	7	12
7	2	11	9	10	6	12	4	3	5	8	1
8	3	12	10	11	7	1	5	4	6	9	2
9	4	1	11	12	8	2	6	5	7	10	3
10	5	2	12	1	9	3	7	6	8	11	4
11	6	3	1	2	10	4	8	7	9	12	5
12	7	4	2	3	11	5	9	8	10	1	6

Figure 8A
The full superequivalence class for a Tuscan-1 square (N = 8) asymmetric polygonal path

1	2	3	4	5	6	7	8
2	7	1	8	3	5	4	6
3	1	5	7	6	8	2	4
4	8	7	5	2	1	6	3
5	3	6	2	8	4	1	7
6	5	8	1	4	7	3	2
7	4	2	6	1	3	8	5
8	6	4	3	7	2	5	1

1	2	3	4	5	6	7	8
2	8	5	1	7	3	6	4
3	5	4	6	1	8	2	7
4	1	6	8	3	7	5	2
5	7	1	3	2	4	8	6
6	3	8	7	4	2	1	5
7	6	2	5	8	1	4	3
8	4	7	2	6	5	3	1

1	2	3	4	5	6	7	8
2	7	4	8	6	3	1	5
3	6	2	1	8	5	4	7
4	6	5	7	1	3	8	2
5	2	8	3	7	6	4	1
6	8	4	2	5	1	7	3
7	2	4	3	5	8	1	6
8	7	5	3	2	6	1	4

1	2	3	4	5	6	7	8
2	7	1	8	3	5	4	6
3	8	7	6	4	2	1	5
4	8	1	6	3	7	5	2
5	8	6	1	3	2	4	7
6	2	8	5	7	3	1	4
7	4	3	6	8	2	5	1
8	4	1	7	2	6	5	3

1	2	3	4	5	6	7	8
2	1	3	7	5	8	4	6
3	2	8	6	5	1	4	7
4	8	5	2	7	6	3	1
5	7	2	6	4	1	8	3
6	2	5	3	8	1	7	4
7	1	6	8	2	4	3	5
8	7	3	6	1	5	4	2

1	2	3	4	5	6	7	8
2	6	4	7	3	5	8	1
3	6	8	2	1	7	5	4
4	3	1	8	7	6	2	5
5	7	1	3	2	4	8	6
6	3	8	5	1	4	2	7
7	2	8	4	6	1	5	3
8	3	7	4	1	6	5	2

1	2	3	4	5	6	7	8
2	1	3	6	8	4	7	5
3	8	5	1	7	4	6	2
4	8	2	7	6	5	3	1
5	2	6	4	1	8	3	7
6	1	5	4	2	8	7	3
7	2	4	3	5	8	1	6
8	6	3	2	5	7	1	4

1	2	3	4	5	6	7	8
2	5	7	3	6	4	8	1
3	8	7	5	2	1	4	6
4	3	1	7	6	2	8	5
5	8	6	1	3	2	4	7
6	3	5	1	8	2	7	4
7	2	6	8	4	1	5	3
8	3	7	1	6	5	4	2

1	2	3	4	5	6	7	8
2	1	3	7	5	8	4	6
3	1	6	8	5	2	7	4
4	2	8	6	1	7	3	5
5	4	1	8	7	2	6	3
6	4	3	8	2	5	7	1
7	6	5	1	4	8	3	2
8	1	5	3	6	2	4	7

1	2	3	4	5	6	7	8
2	6	4	7	3	5	8	1
3	7	2	1	8	6	5	4
4	2	5	1	3	6	8	7
5	7	4	1	6	3	8	2
6	2	4	8	3	1	7	5
7	6	1	5	2	8	4	3
8	5	3	2	7	1	4	6

1	2	3	4	5	6	7	8
2	1	3	6	8	4	7	5
3	5	8	2	7	6	4	1
4	2	8	6	1	5	7	3
5	4	3	1	8	7	2	6
6	5	1	7	4	8	3	2
7	1	6	3	8	5	2	4
8	1	4	6	2	5	3	7

1	2	3	4	5	6	7	8
2	5	7	3	6	4	8	1
3	5	1	4	2	6	8	7
4	6	2	8	3	1	7	5
5	2	7	4	1	3	8	6
6	3	7	2	1	8	5	4
7	6	1	5	8	4	3	2
8	2	4	7	1	6	5	3

1	2	3	4	5	6	7	8
2	5	7	6	8	3	1	4
3	5	8	7	4	2	1	6
4	1	7	3	2	8	6	5
5	2	7	1	8	4	6	3
6	4	8	5	1	3	7	2
7	5	4	3	8	2	6	1
8	1	5	3	6	2	4	7

1	2	3	4	5	6	7	8
2	6	1	8	5	3	7	4
3	8	1	4	6	5	7	2
4	2	5	1	3	6	8	7
5	8	2	4	7	6	3	1
6	2	1	7	5	4	8	3
7	3	5	2	8	4	1	6
8	6	4	3	2	7	1	5

1	2	3	4	5	6	7	8
2	7	5	8	4	1	6	3
3	1	5	7	6	8	2	4
4	7	3	2	1	8	6	5
5	4	2	8	7	1	3	6
6	4	8	3	5	1	7	2
7	4	3	8	5	2	6	1
8	1	4	6	2	5	3	7

1	2	3	4	5	6	7	8
2	5	8	4	3	1	7	6
3	5	1	4	2	6	8	7
4	6	5	7	1	3	8	2
5	2	8	1	6	4	7	3
6	1	8	5	3	7	2	4
7	5	4	8	3	6	2	1
8	6	3	2	7	4	1	5

1	2	3	4	5	6	7	8
2	5	7	6	8	3	1	4
3	5	4	6	1	8	2	7
4	8	7	5	3	2	1	6
5	8	6	2	4	1	7	3
6	4	3	7	2	8	5	1
7	4	2	6	3	8	1	5
8	4	7	1	3	6	5	2

1	2	3	4	5	6	7	8
2	5	4	1	8	7	3	6
3	8	1	4	6	5	7	2
4	8	6	3	1	5	2	7
5	8	3	7	6	4	2	1
6	2	8	4	7	5	1	3
7	1	6	8	2	4	3	5
8	5	3	2	6	1	7	4

Figure 14A

$N = 14$, not Tuscan-2, symmetric polygonal path

STANDARD FORM

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	6	8	14	11	3	5	10	12	4	1	7	9	13
3	8	4	9	2	10	1	14	5	13	6	11	7	12
4	14	9	5	8	13	3	12	2	7	10	6	1	11
5	11	2	8	12	1	9	6	14	3	7	13	4	10
6	3	10	13	1	8	11	4	7	14	2	5	12	9
7	5	1	3	9	11	13	2	4	6	12	14	10	8
8	10	14	12	6	4	2	13	11	9	3	1	5	7
9	12	5	2	14	7	4	11	8	1	13	10	3	6
10	4	13	7	3	14	6	9	1	12	8	2	11	5
11	1	6	10	7	2	12	3	13	8	5	9	14	4
12	7	11	6	13	5	14	1	10	2	9	4	8	3
13	9	7	1	4	12	10	5	3	11	14	8	6	2
14	13	12	11	10	9	8	7	6	5	4	3	2	1

PATH FORM

1	6	2	3	5	11	14	7	4	12	10	9	13	8
2	7	3	4	6	12	1	8	5	13	11	10	14	9
3	8	4	5	7	13	2	9	6	14	12	11	1	10
4	9	5	6	8	14	3	10	7	1	13	12	2	11
5	10	6	7	9	1	4	11	8	2	14	13	3	12
6	11	7	8	10	2	5	12	9	3	1	14	4	13
7	12	8	9	11	3	6	13	10	4	2	1	5	14
8	13	9	10	12	4	7	14	11	5	3	2	6	1
9	14	10	11	13	5	8	1	12	6	4	3	7	2
10	1	11	12	14	6	9	2	13	7	5	4	8	3
11	2	12	13	1	7	10	3	14	8	6	5	9	4
12	3	13	14	2	8	11	4	1	9	7	6	10	5
13	4	14	1	3	9	12	5	2	10	8	7	11	6
14	5	1	2	4	10	13	6	3	11	9	8	12	7

Figure 14B

Asymmetric path, one of the two polygonal path Tuscan-2 squares for $N = 14$

STANDARD FORM

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	7	10	13	9	8	5	11	3	14	1	6	12	4
3	10	12	11	4	7	14	5	13	6	9	2	1	8
4	13	11	7	6	3	12	10	8	1	14	9	5	2
5	9	4	6	10	1	3	2	14	13	7	11	8	12
6	8	7	3	1	4	11	13	2	5	12	14	10	9
7	5	14	12	3	11	9	1	10	4	2	8	6	13
8	11	5	10	2	13	1	12	7	9	6	4	14	3
9	3	13	8	14	2	10	7	4	12	5	1	11	6
10	14	6	1	13	5	4	9	12	8	3	7	2	11
11	1	9	14	7	12	2	6	5	3	8	13	4	10
12	6	2	9	11	14	8	4	1	7	13	10	3	5
13	12	1	5	8	10	6	14	11	2	4	3	9	7
14	4	8	2	12	9	13	3	6	11	10	5	7	1

PATH FORM

1	2	6	9	4	12	3	13	5	7	14	11	10	8
2	3	7	10	5	13	4	14	6	8	1	12	11	9
3	4	8	11	6	14	5	1	7	9	2	13	12	10
4	5	9	12	7	1	6	2	8	10	3	14	13	11
5	6	10	13	8	2	7	3	9	11	4	1	14	12
6	7	11	14	9	3	8	4	10	12	5	2	1	13
7	8	12	1	10	4	9	5	11	13	6	3	2	14
8	9	13	2	11	5	10	6	12	14	7	4	3	1
9	10	14	3	12	6	11	7	13	1	8	5	4	2
10	11	1	4	13	7	12	8	14	2	9	6	5	3
11	12	2	5	14	8	13	9	1	3	10	7	6	4
12	13	3	6	1	9	14	10	2	4	11	8	7	5
13	14	4	7	2	10	1	11	3	5	12	9	8	6
14	1	5	8	3	11	2	12	4	6	13	10	9	7

Figure 14C

Symmetric path, the only polygonal path Tuscan-2 ($N = 14$) preserved by the add-zero transform

STANDARD FORM

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	8	5	14	9	11	3	12	4	6	1	10	7	13
3	5	11	8	1	13	6	9	2	14	7	4	10	12
4	14	8	6	12	5	2	13	10	3	9	7	1	11
5	9	1	12	2	7	11	4	8	13	3	14	6	10
6	11	13	5	7	12	14	1	3	8	10	2	4	9
7	3	6	2	11	14	10	5	1	4	13	9	12	8
8	12	9	13	4	1	5	10	14	11	2	6	3	7
9	4	2	10	8	3	1	14	12	7	5	13	11	6
10	6	14	3	13	8	4	11	7	2	12	1	9	5
11	1	7	9	3	10	13	2	5	12	6	8	14	4
12	10	4	7	14	2	9	6	13	1	8	11	5	3
13	7	10	1	6	4	12	3	11	9	14	5	8	2
14	13	12	11	10	9	8	7	6	5	4	3	2	1

PATH FORM

1	11	10	12	6	9	14	7	2	13	5	3	4	8
2	12	11	13	7	10	1	8	3	14	6	4	5	9
3	13	12	14	8	11	2	9	4	1	7	5	6	10
4	14	13	1	9	12	3	10	5	2	8	6	7	11
5	1	14	2	10	13	4	11	6	3	9	7	8	12
6	2	1	3	11	14	5	12	7	4	10	8	9	13
7	3	2	4	12	1	6	13	8	5	11	9	10	14
8	4	3	5	13	2	7	14	9	6	12	10	11	1
9	5	4	6	14	3	8	1	10	7	13	11	12	2
10	6	5	7	1	4	9	2	11	8	14	12	13	3
11	7	6	8	2	5	10	3	12	9	1	13	14	4
12	8	7	9	3	6	11	4	13	10	2	14	1	5
13	9	8	10	4	7	12	5	14	11	3	1	2	6
14	10	9	11	5	8	13	6	1	12	4	2	3	7

Figure 10B
The full superequivalence class for the only polygonal path Tuscan-2 ($N = 10$) other than the Vatican, preserved by the add-zero transform: symmetric polygonal path

1	2	3	4	5	6	7	8	9	10
2	7	5	8	10	1	3	6	4	9
3	5	9	1	4	7	10	2	6	8
4	8	1	5	2	9	6	10	3	7
5	10	4	2	8	3	9	7	1	6
6	1	7	9	3	8	2	4	10	5
7	3	10	6	9	2	5	1	8	4
8	6	2	10	7	4	1	9	5	3
9	4	6	3	1	10	8	5	7	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	5	3	8	10	1	6	4	7	9
3	6	9	1	5	7	10	2	4	8
4	1	8	5	9	2	6	10	3	7
5	10	4	9	3	1	7	2	8	6
6	8	2	7	1	3	9	4	10	5
7	3	10	6	2	9	5	8	1	4
8	4	2	10	7	5	1	9	6	3
9	7	4	6	1	10	8	3	5	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	8	3	9	4	10	5	7	1	6
3	10	6	2	9	5	1	8	4	7
4	6	9	1	3	7	10	2	5	8
5	3	6	8	10	1	4	2	7	9
6	1	7	5	10	4	9	3	8	2
7	4	8	1	5	9	2	6	10	3
8	5	2	10	7	3	1	9	6	4
9	7	2	4	1	10	8	6	3	5
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	6	9	1	4	7	10	3	5	8
3	1	6	8	10	4	2	5	7	9
4	8	1	5	9	2	10	6	3	7
5	10	1	7	2	8	3	9	4	6
6	4	9	3	8	2	7	1	10	5
7	3	6	10	2	9	5	1	8	4
8	5	3	10	7	4	1	9	6	2
9	7	5	2	4	10	8	6	1	3
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	10	8	6	4	1	3	9	7	5
3	6	9	2	4	7	10	1	5	8
4	8	1	9	5	2	6	10	3	7
5	7	9	3	1	4	6	8	10	2
6	1	7	2	8	3	5	10	4	9
7	3	10	6	2	5	9	1	8	4
8	5	1	10	7	4	2	9	6	3
9	4	10	5	3	8	2	7	1	6
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	8	6	4	10	1	9	7	5	3
3	5	7	9	1	10	4	6	8	2
4	7	10	2	5	8	1	3	6	9
5	10	8	3	9	4	2	7	1	6
6	1	7	2	4	9	3	8	10	5
7	3	10	6	2	9	5	1	4	8
8	4	1	5	9	2	6	10	3	7
9	6	3	1	8	5	2	10	7	4
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	5	8	10	3	6	9	1	4	7
3	7	10	6	2	9	5	1	8	4
4	8	1	5	9	2	6	10	7	3
5	10	4	9	7	2	8	3	1	6
6	1	3	8	2	7	9	4	10	5
7	4	1	9	6	3	10	8	5	2
8	6	4	2	10	1	7	5	3	9
9	3	5	7	1	10	2	4	6	8
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	7	1	6	8	3	9	4	10	5
3	6	10	1	4	7	9	2	5	8
4	8	1	5	9	6	2	10	3	7
5	10	4	9	3	8	6	1	7	2
6	4	2	8	10	7	5	3	1	9
7	3	10	2	6	9	5	1	8	4
8	5	2	9	7	4	1	10	6	3
9	1	3	5	7	10	8	2	4	6
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	9	7	1	3	5	10	8
3	6	8	1	4	7	10	2	5	9
4	8	5	1	9	2	6	10	3	7
5	7	2	8	3	9	4	10	1	6
6	1	10	4	9	3	8	2	7	5
7	3	10	6	2	9	1	5	8	4
8	10	5	3	1	7	9	6	4	2
9	5	2	10	7	4	1	8	6	3
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	9	7	10	1	3	5	8	6
3	6	9	1	4	8	10	2	5	7
4	7	3	10	6	2	9	5	1	8
5	10	4	6	1	7	2	8	3	9
6	8	5	3	1	10	7	9	4	2
7	5	2	10	8	4	1	9	6	3
8	1	5	9	2	6	10	3	7	4
9	3	8	2	7	1	6	4	10	5
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	7	5	10	1	3	8	6	9
3	7	9	1	4	6	10	2	5	8
4	8	1	5	9	2	6	3	10	7
5	3	9	4	10	8	2	7	1	6
6	1	7	2	8	10	4	9	3	5
7	10	3	6	2	9	5	1	8	4
8	5	2	10	6	4	1	9	7	3
9	6	8	3	1	10	5	7	4	2
10	9	8	7	6	5	4	3	2	1

Figure 10C
The full superequivalence class for a Tuscan-1 square ($N = 10$), symmetric polygonal path

1	2	3	4	5	6	7	8	9	10
2	4	6	8	1	10	3	5	7	9
3	6	2	10	7	4	1	9	5	8
4	8	10	5	2	9	6	1	3	7
5	1	7	2	8	3	9	4	10	6
6	10	4	9	3	8	2	7	1	5
7	3	1	6	9	2	5	10	8	4
8	5	9	1	4	7	10	2	6	3
9	7	5	3	10	1	8	6	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	6	10	3	7	9	4	1	8	5
3	6	9	1	5	2	10	7	4	8
4	10	5	9	3	8	2	7	1	6
5	8	1	4	9	7	3	10	6	2
6	1	7	2	8	3	9	5	10	4
7	5	3	1	10	8	6	4	2	9
8	4	7	10	2	5	1	9	6	3
9	2	4	6	8	10	1	3	5	7
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	10	8	1	3	5	7	9
3	8	6	2	9	5	1	10	4	7
4	1	9	6	3	7	10	2	5	8
5	10	3	9	4	8	2	7	1	6
6	1	7	2	8	4	9	3	10	5
7	4	10	1	5	9	2	6	8	3
8	5	2	10	7	3	6	9	1	4
9	7	5	3	1	8	10	6	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	5	9	7
3	10	7	4	1	9	6	2	5	8
4	8	1	5	7	2	10	6	3	9
5	10	4	9	2	8	3	7	1	6
6	1	7	3	8	2	9	4	10	5
7	9	5	3	1	10	8	6	4	2
8	5	2	6	9	1	4	7	10	3
9	3	6	10	2	7	5	1	8	4
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	8	6	10	1	3	5	7	9
3	6	9	2	10	7	4	1	5	8
4	6	1	9	5	2	8	10	3	7
5	10	4	9	3	8	1	7	2	6
6	2	7	1	8	3	9	4	10	5
7	3	10	8	2	5	9	1	6	4
8	5	1	4	7	10	2	9	6	3
9	7	5	3	1	10	6	8	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	3	7	5	9
3	6	9	1	8	5	2	10	7	4
4	7	10	2	5	8	1	9	6	3
5	3	10	6	2	9	7	1	4	8
6	1	5	10	4	9	3	8	2	7
7	2	8	3	9	4	10	5	1	6
8	4	1	7	9	2	6	10	3	5
9	5	7	3	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	6	4	8	10	1	3	5	7	9
3	7	10	4	2	9	5	1	8	6
4	9	3	8	2	7	1	6	10	5
5	10	6	1	7	2	8	3	9	4
6	8	1	5	9	2	4	10	7	3
7	4	1	9	6	3	10	2	5	8
8	5	2	10	3	6	9	1	4	7
9	7	5	3	1	10	8	4	6	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	1	5	3	7	9
3	6	10	7	4	1	9	2	5	8
4	8	1	3	9	6	2	10	5	7
5	9	4	10	3	8	2	7	1	6
6	1	7	2	8	3	10	4	9	5
7	5	10	2	6	9	3	1	8	4
8	5	2	9	1	4	7	10	6	3
9	7	3	5	1	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	8	5	1	9	4	6	10	3	7
3	6	9	5	2	10	7	4	1	8
4	2	6	8	10	1	3	5	7	9
5	10	4	8	3	9	2	7	1	6
6	1	7	2	9	3	8	4	10	5
7	3	10	6	4	9	1	5	8	2
8	1	4	7	10	2	5	9	6	3
9	7	5	3	1	10	8	6	2	4
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	3	1	5	7	9
3	6	9	1	4	8	5	2	10	7
4	7	1	10	6	2	9	5	3	8
5	10	4	9	3	7	2	8	1	6
6	1	8	2	7	3	9	4	10	5
7	10	2	5	8	4	1	9	6	3
8	3	5	9	2	6	10	1	7	4
9	7	5	1	3	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

1	2	3	4	5	6	7	8	9	10
2	9	7	5	3	1	10	8	6	4
3	7	4	1	9	6	10	2	5	8
4	6	8	10	1	3	5	7	9	2
5	10	4	9	3	8	2	6	1	7
6	3	10	7	2	4	8	1	5	9
7	1	6	2	8	3	9	4	10	5
8	5	2	10	6	9	1	4	7	3
9	5	1	8	4	2	7	10	3	6
10	9	8	7	6	5	4	3	2	1

Figure 14D
The non-Latin 14 × 14 Tuscan-2 squares descended from the 14 × 15 circular Tuscan-2 array

*	14	2	13	5	1	10	11	4	3	8	12	6	9	7
*	1	3	14	6	2	11	12	5	4	9	13	7	10	8
*	2	4	1	7	3	12	13	6	5	10	14	8	11	9
*	3	5	2	8	4	13	14	7	6	11	1	9	12	10
*	4	6	3	9	5	14	1	8	7	12	2	10	13	11
*	5	7	4	10	6	1	2	9	8	13	3	11	14	12
*	6	8	5	11	7	2	3	10	9	14	4	12	1	13
*	7	9	6	12	8	3	4	11	10	1	5	13	2	14
*	8	10	7	13	9	4	5	12	11	2	6	14	3	1
*	9	11	8	14	10	5	6	13	12	3	7	1	4	2
*	10	12	9	1	11	6	7	14	13	4	8	2	5	3
*	11	13	10	2	12	7	8	1	14	5	9	3	6	4
*	12	14	11	3	13	8	9	2	1	6	10	4	7	5
*	13	1	12	4	14	9	10	3	2	7	11	5	8	6

14	2	13	5	1	10	11	4	3	8	12	6	9	7	*
14	6	2	11	12	5	4	9	13	7	10	8	*	1	3
14	8	11	9	*	2	4	1	7	3	12	13	6	5	10
14	7	6	11	1	9	12	10	*	3	5	2	8	4	13
14	1	8	7	12	2	10	13	11	*	4	6	3	9	5
14	12	*	5	7	4	10	6	1	2	9	8	13	3	11
14	4	12	1	13	*	6	8	5	11	7	2	3	10	9
14	*	7	9	6	12	8	3	4	11	10	1	5	13	2
14	3	1	*	8	10	7	13	9	4	5	12	11	2	6
14	10	5	6	13	12	3	7	1	4	2	*	9	11	8
14	13	4	8	2	5	3	*	10	12	9	1	11	6	7
14	5	9	3	6	4	*	11	13	10	2	12	7	8	1
14	11	3	13	8	9	2	1	6	10	4	7	5	*	12
14	9	10	3	2	7	11	5	8	6	*	13	1	12	4

7	*	14	2	13	5	1	10	11	4	3	8	12	6	9
7	10	8	*	1	3	14	6	2	11	12	5	4	9	13
7	3	12	13	6	5	10	14	8	11	9	*	2	4	1
7	6	11	1	9	12	10	*	3	5	2	8	4	13	14
7	12	2	10	13	11	*	4	6	3	9	5	14	1	8
7	4	10	6	1	2	9	8	13	3	11	14	12	*	5
7	2	3	10	9	14	4	12	1	13	*	6	8	5	11
7	9	6	12	8	3	4	11	10	1	5	13	2	14	*
7	13	9	4	5	12	11	2	6	14	3	1	*	8	10
7	1	4	2	*	9	11	8	14	10	5	6	13	12	3
7	14	13	4	8	2	5	3	*	10	12	9	1	11	6
7	8	1	14	5	9	3	6	4	*	11	13	10	2	12
7	5	*	12	14	11	3	13	8	9	2	1	6	10	4
7	11	5	8	6	*	13	1	12	4	14	9	10	3	2

Figure 14D (continued)

2	13	5	1	10	11	4	3	8	12	6	9	7	*	14
2	11	12	5	4	9	13	7	10	8	*	1	3	14	6
2	4	1	7	3	12	13	6	5	10	14	8	11	9	*
2	8	4	13	14	7	6	11	1	9	12	10	*	3	5
2	10	13	11	*	4	6	3	9	5	14	1	8	7	12
2	9	8	13	3	11	14	12	*	5	7	4	10	6	1
2	3	10	9	14	4	12	1	13	*	6	8	5	11	7
2	14	*	7	9	6	12	8	3	4	11	10	1	5	13
2	6	14	3	1	*	8	10	7	13	9	4	5	12	11
2	*	9	11	8	14	10	5	6	13	12	3	7	1	4
2	5	3	*	10	12	9	1	11	6	7	14	13	4	8
2	12	7	8	1	14	5	9	3	6	4	*	11	13	10
2	1	6	10	4	7	5	*	12	14	11	3	13	8	9
2	7	11	5	8	6	*	13	1	12	4	14	9	10	3

13	5	1	10	11	4	3	8	12	6	9	7	*	14	2
13	7	10	8	*	1	3	14	6	2	11	12	5	4	9
13	6	5	10	14	8	11	9	*	2	4	1	7	3	12
13	14	7	6	11	1	9	12	10	*	3	5	2	8	4
13	11	*	4	6	3	9	5	14	1	8	7	12	2	10
13	3	11	14	12	*	5	7	4	10	6	1	2	9	8
13	*	6	8	5	11	7	2	3	10	9	14	4	12	1
13	2	14	*	7	9	6	12	8	3	4	11	10	1	5
13	9	4	5	12	11	2	6	14	3	1	*	8	10	7
13	12	3	7	1	4	2	*	9	11	8	14	10	5	6
13	4	8	2	5	3	*	10	12	9	1	11	6	7	14
13	10	2	12	7	8	1	14	5	9	3	6	4	*	11
13	8	9	2	1	6	10	4	7	5	*	12	14	11	3
13	1	12	4	14	9	10	3	2	7	11	5	8	6	*

5	1	10	11	4	3	8	12	6	9	7	*	14	2	13
5	4	9	13	7	10	8	*	1	3	14	6	2	11	12
5	10	14	8	11	9	*	2	4	1	7	3	12	13	6
5	2	8	4	13	14	7	6	11	1	9	12	10	*	3
5	14	1	8	7	12	2	10	13	11	*	4	6	3	9
5	7	4	10	6	1	2	9	8	13	3	11	14	12	*
5	11	7	2	3	10	9	14	4	12	1	13	*	6	8
5	13	2	14	*	7	9	6	12	8	3	4	11	10	1
5	12	11	2	6	14	3	1	*	8	10	7	13	9	4
5	6	13	12	3	7	1	4	2	*	9	11	8	14	10
5	3	*	10	12	9	1	11	6	7	14	13	4	8	2
5	9	3	6	4	*	11	13	10	2	12	7	8	1	14
5	*	12	14	11	3	13	8	9	2	1	6	10	4	7
5	8	6	*	13	1	12	4	14	9	10	3	2	7	11

1	10	11	4	3	8	12	6	9	7	*	14	2	13	5
1	3	14	6	2	11	12	5	4	9	13	7	10	8	*
1	7	3	12	13	6	5	10	14	8	11	9	*	2	4
1	9	12	10	*	3	5	2	8	4	13	14	7	6	11
1	8	7	12	2	10	13	11	*	4	6	3	9	5	14
1	2	9	8	13	3	11	14	12	*	5	7	4	10	6
1	13	*	6	8	5	11	7	2	3	10	9	14	4	12
1	5	13	2	14	*	7	9	6	12	8	3	4	11	10
1	*	8	10	7	13	9	4	5	12	11	2	6	14	3
1	4	2	*	9	11	8	14	10	5	6	13	12	3	7
1	11	6	7	14	13	4	8	2	5	3	*	10	12	9
1	14	5	9	3	6	4	*	11	13	10	2	12	7	8
1	6	10	4	7	5	*	12	14	11	3	13	8	9	2
1	12	4	14	9	10	3	2	7	11	5	8	6	*	13

10	11	4	3	8	12	6	9	7	*	14	2	13	5	1
10	8	*	1	3	14	6	2	11	12	5	4	9	13	7
10	14	8	11	9	*	2	4	1	7	3	12	13	6	5
10	*	3	5	2	8	4	13	14	7	6	11	1	9	12
10	13	11	*	4	6	3	9	5	14	1	8	7	12	2
10	6	1	2	9	8	13	3	11	14	12	*	5	7	4
10	9	14	4	12	1	13	*	6	8	5	11	7	2	3
10	1	5	13	2	14	*	7	9	6	12	8	3	4	11
10	7	13	9	4	5	12	11	2	6	14	3	1	*	8
10	5	6	13	12	3	7	1	4	2	*	9	11	8	14
10	12	9	1	11	6	7	14	13	4	8	2	5	3	*
10	2	12	7	8	1	14	5	9	3	6	4	*	11	13
10	4	7	5	*	12	14	11	3	13	8	9	2	1	6
10	3	2	7	11	5	8	6	*	13	1	12	4	14	9

11	4	3	8	12	6	9	7	*	14	2	13	5	1	10
11	12	5	4	9	13	7	10	8	*	1	3	14	6	2
11	9	*	2	4	1	7	3	12	13	6	5	10	14	8
11	1	9	12	10	*	3	5	2	8	4	13	14	7	6
11	*	4	6	3	9	5	14	1	8	7	12	2	10	13
11	14	12	*	5	7	4	10	6	1	2	9	8	13	3
11	7	2	3	10	9	14	4	12	1	13	*	6	8	5
11	10	1	5	13	2	14	*	7	9	6	12	8	3	4
11	2	6	14	3	1	*	8	10	7	13	9	4	5	12
11	8	14	10	5	6	13	12	3	7	1	4	2	*	9
11	6	7	14	13	4	8	2	5	3	*	10	12	9	1
11	13	10	2	12	7	8	1	14	5	9	3	6	4	*
11	3	13	8	9	2	1	6	10	4	7	5	*	12	14
11	5	8	6	*	13	1	12	4	14	9	10	3	2	7

Figure 14D (continued)

9	7	*	14	2	13	5	1	10	11	4	3	8	12	6
9	13	7	10	8	*	1	3	14	6	2	11	12	5	4
9	*	2	4	1	7	3	12	13	6	5	10	14	8	11
9	12	10	*	3	5	2	8	4	13	14	7	6	11	1
9	5	14	1	8	7	12	2	10	13	11	*	4	6	3
9	8	13	3	11	14	12	*	5	7	4	10	6	1	2
9	14	4	12	1	13	*	6	8	5	11	7	2	3	10
9	6	12	8	3	4	11	10	1	5	13	2	14	*	7
9	4	5	12	11	2	6	14	3	1	*	8	10	7	13
9	11	8	14	10	5	6	13	12	3	7	1	4	2	*
9	1	11	6	7	14	13	4	8	2	5	3	*	10	12
9	3	6	4	*	11	13	10	2	12	7	8	1	14	5
9	2	1	6	10	4	7	5	*	12	14	11	3	13	8
9	10	3	2	7	11	5	8	6	*	13	1	12	4	14

6	9	7	*	14	2	13	5	1	10	11	4	3	8	12
6	2	11	12	5	4	9	13	7	10	8	*	1	3	14
6	5	10	14	8	11	9	*	2	4	1	7	3	12	13
6	11	1	9	12	10	*	3	5	2	8	4	13	14	7
6	3	9	5	14	1	8	7	12	2	10	13	11	*	4
6	1	2	9	8	13	3	11	14	12	*	5	7	4	10
6	8	5	11	7	2	3	10	9	14	4	12	1	13	*
6	12	8	3	4	11	10	1	5	13	2	14	*	7	9
6	14	3	1	*	8	10	7	13	9	4	5	12	11	2
6	13	12	3	7	1	4	2	*	9	11	8	14	10	5
6	7	14	13	4	8	2	5	3	*	10	12	9	1	11
6	4	*	11	13	10	2	12	7	8	1	14	5	9	3
6	10	4	7	5	*	12	14	11	3	13	8	9	2	1
6	*	13	1	12	4	14	9	10	3	2	7	11	5	8

12	6	9	7	*	14	2	13	5	1	10	11	4	3	8
12	5	4	9	13	7	10	8	*	1	3	14	6	2	11
12	13	6	5	10	14	8	11	9	*	2	4	1	7	3
12	10	*	3	5	2	8	4	13	14	7	6	11	1	9
12	2	10	13	11	*	4	6	3	9	5	14	1	8	7
12	*	5	7	4	10	6	1	2	9	8	13	3	11	14
12	1	13	*	6	8	5	11	7	2	3	10	9	14	4
12	8	3	4	11	10	1	5	13	2	14	*	7	9	6
12	11	2	6	14	3	1	*	8	10	7	13	9	4	5
12	3	7	1	4	2	*	9	11	8	14	10	5	6	13
12	9	1	11	6	7	14	13	4	8	2	5	3	*	10
12	7	8	1	14	5	9	3	6	4	*	11	13	10	2
12	14	11	3	13	8	9	2	1	6	10	4	7	5	*
12	4	14	9	10	3	2	7	11	5	8	6	*	13	1

8	12	6	9	7	*	14	2	13	5	1	10	11	4	3
8	*	1	3	14	6	2	11	12	5	4	9	13	7	10
8	11	9	*	2	4	1	7	3	12	13	6	5	10	14
8	4	13	14	7	6	11	1	9	12	10	*	3	5	2
8	7	12	2	10	13	11	*	4	6	3	9	5	14	1
8	13	3	11	14	12	*	5	7	4	10	6	1	2	9
8	5	11	7	2	3	10	9	14	4	12	1	13	*	6
8	3	4	11	10	1	5	13	2	14	*	7	9	6	12
8	10	7	13	9	4	5	12	11	2	6	14	3	1	*
8	14	10	5	6	13	12	3	7	1	4	2	*	9	11
8	2	5	3	*	10	12	9	1	11	6	7	14	13	4
8	1	14	5	9	3	6	4	*	11	13	10	2	12	7
8	9	2	1	6	10	4	7	5	*	12	14	11	3	13
8	6	*	13	1	12	4	14	9	10	3	2	7	11	5

3	8	12	6	9	7	*	14	2	13	5	1	10	11	4
3	14	6	2	11	12	5	4	9	13	7	10	8	*	1
3	12	13	6	5	10	14	8	11	9	*	2	4	1	7
3	5	2	8	4	13	14	7	6	11	1	9	12	10	*
3	9	5	14	1	8	7	12	2	10	13	11	*	4	6
3	11	14	12	*	5	7	4	10	6	1	2	9	8	13
3	10	9	14	4	12	1	13	*	6	8	5	11	7	2
3	4	11	10	1	5	13	2	14	*	7	9	6	12	8
3	1	*	8	10	7	13	9	4	5	12	11	2	6	14
3	7	1	4	2	*	9	11	8	14	10	5	6	13	12
3	*	10	12	9	1	11	6	7	14	13	4	8	2	5
3	6	4	*	11	13	10	2	12	7	8	1	14	5	9
3	13	8	9	2	1	6	10	4	7	5	*	12	14	11
3	2	7	11	5	8	6	*	13	1	12	4	14	9	10

4	3	8	12	6	9	7	*	14	2	13	5	1	10	11
4	9	13	7	10	8	*	1	3	14	6	2	11	12	5
4	1	7	3	12	13	6	5	10	14	8	11	9	*	2
4	13	14	7	6	11	1	9	12	10	*	3	5	2	8
4	6	3	9	5	14	1	8	7	12	2	10	13	11	*
4	10	6	1	2	9	8	13	3	11	14	12	*	5	7
4	12	1	13	*	6	8	5	11	7	2	3	10	9	14
4	11	10	1	5	13	2	14	*	7	9	6	12	8	3
4	5	12	11	2	6	14	3	1	*	8	10	7	13	9
4	2	*	9	11	8	14	10	5	6	13	12	3	7	1
4	8	2	5	3	*	10	12	9	1	11	6	7	14	13
4	*	11	13	10	2	12	7	8	1	14	5	9	3	6
4	7	5	*	12	14	11	3	13	8	9	2	1	6	10
4	14	9	10	3	2	7	11	5	8	6	*	13	1	12

D. A necessary and sufficient condition.

Theorem D. *An $n \times n$ polygonal path Vatican square exists iff an $n \times \infty$ singly periodic Costas array exists.*

Proof: See [6] for definition and discussion of Costas arrays.

Both are equivalent to the existence of a sequence $X_1, X_2, \ldots, X_n$ which is a permutation of the integers $1, 2, \ldots, n$ satisfying property D: $X_{s+k} - X_s \not\equiv X_{t+k} - X_t \pmod{n}$ for every s, t, k such that $1 \leq s < t < t+k \leq n$.

A polygonal path $X_1, X_2, \ldots, X_n$ produces a Vatican square iff it satisfies property D.

A singly periodic Costas array exists iff an $n \times \infty$ matrix can be described as follows. For $1 \leq i \leq n$, $-\infty < j < +\infty$ put a dot at position (i, j) iff $j \equiv X_i \pmod{n}$, where $X_1, X_2, \ldots, X_n$ is a permutation of $1, 2, \ldots, n$ which satisfies property D.

Theorem D then follows directly from the definitions. ∎

E. Polygonal path circular Tuscan-k arrays.

Lemma E1. *If a polygonal path $X_1, X_2, \ldots, X_n$ produces an $n \times (n+1)$ circular Tuscan-k array, then for each i from 1 to k,*

$$X_{n+1-i} \equiv \frac{n}{2} + X_i \pmod{n}.$$

Proof: Letting the symbol $\star = X_0$ we will confine our attention to $X_0, X_1, X_2, \ldots, X_n$ which we may assume is the top row of the $n \times (n+1)$ circular Tuscan-k array. For fixed i, with $1 \leq i \leq k$, look at all the cycles (one or more depending on divisors of $n+1$) of the form $X_a, X_{a+i}, X_{a+2i}, \ldots$, where the subscripts are taken modulo $n+1$. Every symbol gets hit exactly once by some one of the cycles, and since the array is circular Tuscan-i, each of the differences $1, 2, \ldots, n-1$ modulo n must occur exactly once in only one cycle as $X_{a+(r+1)i} - X_{a+ri}$. Notice that the sum of all these differences is congruent to $\frac{n}{2}$ modulo n, and that every cycle not hitting the asterisk X_0 has the sum of its differences congruent to 0 modulo n. The one cycle which does hit X_0 will in fact contain $\ldots, X_{n+1-i}, X_0, X_i, \ldots$ successively, which accounts for the two pairs of symbols, successive in a cycle, to which we assign no difference value modulo n. The remaining differences therefore sum to $\frac{n}{2}$, so we can evaluate

$$(X_{n+1-i} - X_{n+1-2i}) + (X_{n+1-2i} - X_{n+1-3i}) + \ldots + (X_{2i} - X_i) = X_{n+1-i} - X_i \equiv \frac{n}{2} \pmod{n}.$$

∎

Corollary E1.1. *If the polygonal path* $X_1, X_2, \dots, X_n$ *produces an* $n \times (n+1)$ *circular Vatican array, then the path is symmetric.*

Corollary E1.2. *If the polygonal path* $X_1, X_2, \dots, X_n$ *produces an* $n \times (n+1)$ *circular Vatican array, then for* $1 \leq i \leq n$, *we have the following.*

(a) *For* $n+1 \equiv 1 \pmod 8$, X_i *is even iff* X_{n+1-i} *is even.*
(b) *For* $n+1 \equiv 3 \pmod 8$, X_i *is even iff* X_{n+1-i} *is odd.*
(c) *For* $n+1 \equiv 5 \pmod 8$, X_i *is even iff* X_{n+1-i} *is even.*
(d) *For* $n+1 \equiv 7 \pmod 8$, X_i *is even iff* X_{n+1-i} *is odd.*

Lemma E2. *If a polygonal path* $X_1, X_2, \dots, X_n$ *produces an* $n \times (n+1)$ *circular Florentine array, then for* $1 \leq i \leq n$ *we have the following.*

(a) *For* $n+1 \equiv 1 \pmod 8$, X_i *is even iff* X_{2i} *is even.*
(b) *For* $n+1 \equiv 3 \pmod 8$, X_i *is even iff* X_{2i} *is odd.*
(c) *For* $n+1 \equiv 5 \pmod 8$, X_i *is even iff* X_{2i} *is odd.*
(d) *For* $n+1 \equiv 7 \pmod 8$, X_i *is even iff* X_{2i} *is even.*

Proof: Again with $\star = X_0$ we consider $X_0, X_1, X_2, \dots, X_n$ taking it to be the top row of the $n \times (n+1)$ circular Florentine array. For fixed i, with $1 \leq i \leq n$, look at all the cycles (one or more) of the form $X_a, X_{a+2i}, X_{a+4i}, X_{a+6i}, \dots$, where the subscripts are taken modulo $n+1$. Every symbol (including $\star = X_0$) gets hit exactly once by some one of the cycles. And since the array is circular Florentine each of the values $1, 2, \dots, n-1$ modulo n occurs exactly once in only one cycle as a successive difference $X_{a+(r+1)2i} - X_{a+r2i}$. As if the asterisk were at infinity, it makes no difference value modulo n with any of the symbols $X_1, X_2, \dots, X_n$, so no value is assigned to a successive difference when $a + (r+1)2i = 0$ or when $a + 2ri = 0$. The sum of all the differences in a cycle which does not hit the asterisk is congruent to 0 modulo n, and since by Corollary E1.1 such cycles come in symmetrically opposite pairs, we infer that the number of odd differences in those cycles is congruent to 0 modulo 4. The cycle which contains the asterisk is its own symmetric opposite, so the cycle looks like this:

$$X_{2i}, X_{4i}, \dots, X_{n+1-i}, X_i, \dots, X_{n+1-4i}, X_{n+1-2i}, X_0.$$

The symmetry guaranteed by Lemma E1 tells us that the number of odd differences between X_{2i} and X_{n+1-i} will be equal to the number of odd differences between X_i and X_{n+1-2i}. Since $X_i - X_{n+1-i}$ is congruent to $\frac{n}{2}$ modulo n, this difference will be odd if $\frac{n}{2}$ is odd, and even if $\frac{n}{2}$ is even.

We distinguish four cases as follows, letting $D(\star)$ denote the number of odd differences in the cycle which contains the asterisk.

(a) $n+1 \equiv 1 \pmod 8$. The number of odd differences is congruent to 0 (mod 4), and $X_i - X_{n+1-i}$ is even. Half of $D(\star)$ is even, therefore

$X_{n+1-i} - X_{2i}$ is even, and $X_i - X_{2i}$ is even. Hence, X_i is even iff X_{2i} is even.

(b) $n+1 \equiv 3 \pmod 8$. The number of odd differences is congruent to 1 (mod 4), and $X_i - X_{n+1-i}$ is odd. Half of $(D(\star) - 1)$ is even, therefore $X_{n+1-i} - X_{2i}$ is even, and $X_i - X_{2i}$ is odd. Hence, X_i is even iff X_{2i} is odd.

(c) For $n+1 \equiv 5 \pmod 8$. The number of odd differences is congruent to 2 (mod 4), and $X_i - X_{n+1-i}$ is even. Half of $D(\star)$ is odd, therefore $X_{n+1-i} - X_{2i}$ is odd, and $X_i - X_{2i}$ is odd. Hence, X_i is even iff X_{2i} is odd.

(d) For $n+1 \equiv 7 \pmod 8$. The number of odd differences is congruent to 3 (mod 4), and $X_i - X_{n+1-i}$ is odd. Half of $(D(\star) - 1)$ is odd, therefore $X_{n+1-i} - X_{2i}$ is odd, and $X_i - X_{2i}$ is even. Hence, X_i is even iff X_{2i} is even.

∎

Lemma E3. *If a polygonal path circular Florentine array exists with first row $X_0, X_1, X_2, \ldots, X_n$, and $n+1 = md$ where $m > 1$, and $d > 1$, then the mod 2 values of $X_{\frac{m-1}{2}d} - X_d$ and $X_{\frac{m+1}{2}d} - X_d$ must agree with the table shown below.*

	$X_{\frac{m-1}{2}d} - X_d$	$X_{\frac{m+1}{2}d} - X_d$
$n+1 \equiv 1 \pmod 8$	*even*	*even*
$n+1 \equiv 3 \pmod 8$	*even*	*odd*
$n+1 \equiv 5 \pmod 8$	*odd*	*odd*
$n+1 \equiv 7 \pmod 8$	*odd*	*even.*

Proof: Consider the decomposition of the top row into cycles of the form $[X_j, X_{j+d}, X_{j+2d}, X_{j+3d}, \ldots]$. Excluding $X_d - X_0$ and $X_0 - X_{md-d}$, the other differences of the form $X_{j+(t+1)d} - X_{j+td}$ must be distinct, and so must take the values $1, 2, 3, \ldots, n-1$. The number of such differences with odd value is congruent to $\frac{n}{2}$ modulo 4. Since the cycles with $j \neq 0$ come in symmetric pairs, with differences summing to zero, the number of odd differences in such cycles is congruent to 0 modulo 4. Letting $D(\star)$ denote the number of odd differences in the one cycle which includes $\star = X_0$, we see that $D(\star) \equiv \frac{n}{2} \pmod 4$.

Let x be the number of odd differences, of the form $X_{(t+1)d} - X_{td}$, in the sequence $X_d, X_{2d}, X_{3d}, \ldots, X_{\frac{m-1}{2}d}$. By Lemma E1, x will also be the number of such odd differences in the sequence $X_{md-d}, X_{md-2d}, \ldots, X_{\frac{m+1}{2}d}$. Lemma E1 tells us in particular that $X_{\frac{m+1}{2}d} - X_{\frac{m-1}{2}d} \equiv \frac{n}{2} \pmod n$, so let $y = 0$ if $\frac{n}{2}$ is even, $y = 1$ if $\frac{n}{2}$ is odd. Putting these facts together we have $D(\star) = x + y + x \equiv \frac{n}{2} \pmod 4$. When n is given mod 8 we can solve for x and y mod 2. Thus, the entries in the table are determined with the observation that $X_{\frac{m-1}{2}d} - X_d \equiv x \pmod 2$, and $X_{\frac{m+1}{2}d} - X_d \equiv x + y \pmod 2$. ∎

Comment: A novel proof that 2 is not a primitive root for any prime congruent to 1 or 7 modulo 8 is a consequence of Lemma E2 combined with the fact that a polygonal path exists for an $n \times (n+1)$ circular Vatican array whenever $n+1$ is prime. Indeed if 2 is a primitive root of the prime $n+1$, it means that powers of two, $1, 2, 4, 8, \ldots$, taken modulo $n+1$, will run through all the subscripts $1, 2, 3, \ldots, n$. But then suppose $n+1 \equiv 1 \pmod{8}$, or $n+1 \equiv 7 \pmod{8}$. By Lemma E2, if X_1 is even, then X_2 is even, and X_4 is even, etc., and these implications make all of $X_1, X_2, \ldots, X_n$ even. Likewise if X_1 is odd, then X_2 is odd, and then X_4 is odd, etc., making all of $X_1, X_2, \ldots, X_n$ odd. These contradictions conclude the proof.

Lemma E4.

1. *If p is prime and $p \equiv 7 \pmod{8}$, then there exists $r > 0$ such that $2^r \equiv \frac{p+1}{2} \pmod{p}$, and the least such r is EVEN.*
2. *If p is prime and $p \equiv 5 \pmod{8}$, then there exists $r > 0$ such that $2^4 \equiv \frac{p-1}{2} \pmod{p}$, and any such r is ODD.*
3. *If p is prime and $p \equiv 3 \pmod{8}$, then there exists $r > 0$ such that $2^r \equiv \frac{p-1}{2} \pmod{p}$, and any such r is EVEN.*

Proof: Let p be an odd prime. According to the theory of quadratic reciprocity [4], [5], the congruence $x^2 \equiv 2 \pmod{p}$ has a solution if $p \equiv \pm 1 \pmod{8}$, and has no solution if $p \equiv \pm 3 \pmod{8}$. Another basic fact is that $-1 \equiv x^2 \pmod{p}$ has a solution if $p \equiv 1 \pmod{4}$, and has no solution if $p \equiv -1 \pmod{4}$.

1. Let $p - 1 = 2k$. Since $p \equiv 7 \pmod{8}$, k is odd. Certainly $2^{2k} \equiv 1 \pmod{p}$. Suppose s is the smallest exponent satisfying $0 < s \leq 2k$ and $2^s \equiv 1 \pmod{p}$. If $s = 2t$ we have, working in $GF(p)$, $2^{2t} = 1$, $(2^t - 1)(2^t + 1) = 0$, and therefore $2^t = 1$ or $2^t = -1$. But since 2 is a square and -1 is not a square we see that $2^t \neq -1$. Thus $0 < t < s$ and $2^t \equiv 1 \pmod{p}$ contrary to the supposition that s was smallest. So s must be odd, and notice that $s > 1$ because $2 \not\equiv 1 \pmod{p}$. Letting $r = s - 1$, adding $0 \equiv p \pmod{p}$, and dividing both sides by 2, we obtain $2^r \equiv \frac{p+1}{2} \pmod{p}$ and see that r is EVEN.
2. When $p = 5 \pmod{8}$, let $p - 1 = 4k$ where k is odd. Since $2^{4k} \equiv 1 \pmod{p}$, in $GF(p)$ we have $(2^{2k} - 1)(2^{2k} + 1) = 0$. Since 2 is not a square while 1 is a square, and -1 is a square, we see that $2^k - 1 \neq 0$, $2^k + 1 \neq 0$, and therefore $2^{2k} + 1 = 0$. Thus t exists such that $2^t \equiv -1 \pmod{p}$, and incidentally any such t cannot be odd. Letting $r = t - 1$, adding $0 \equiv p \pmod{p}$, and dividing both sides by 2, we obtain $2^r \equiv \frac{p-1}{2} \pmod{p}$, and see that r must be ODD.
3. When $p \equiv 3 \pmod{8}$, let $p - 1 = 2k$ where k is odd. Since $2^{2k} \equiv 1 \pmod{p}$, in $GF(p)$ we have $(2^k - 1)(2^k + 1) = 0$. Since 2 is not a square, 2^k is not a square, 1 is a square, $2^k - 1 \neq 0$, and therefore

$2^k = -1$. Thus t exists such that $2^t \equiv -1 \pmod{p}$, and incidentally since -1 is not a square, any such t cannot be even. Letting $r = t - 1$, adding $0 \equiv p \pmod{p}$ to both sides, and dividing both sides by 2, we obtain $2^r \equiv \frac{p-1}{2} \pmod{p}$, and see that r must be EVEN.

∎

Lemma E5. *Suppose the polygonal path $X_1, X_2, \ldots, X_n$ produces an $n \times (n+1)$ circular Florentine array, with $n+1 = md$, and $m > 1$.*

1. *In case $n+1 \equiv \pm 1 \pmod{8}$:*
   If $r > 0$ exists such that $2^r \equiv \frac{m-1}{2} \pmod{m}$, then $X_{\frac{m-1}{2}d} - X_d$ is EVEN.
2. *In case $n+1 \equiv \pm 3 \pmod{8}$:*
   If a least $r > 0$ exists such that $2^r \equiv \frac{m-1}{2} \pmod{m}$, then $X_{\frac{m-1}{2}d} - X_d \equiv r \pmod{2}$.
3. *In case $n+1 \equiv \pm 3 \pmod{8}$:*
   If $t > 0$ is the least such that $2^t \equiv \frac{m+1}{2} \pmod{m}$, then $X_{\frac{m+1}{2}d} - X_d \equiv t \pmod{2}$.

Proof: Follows immediately from Lemma E2. ∎

Theorem E6.

	$m \equiv 1 \pmod{8}$	$m \equiv 3 \pmod{8}$	$m \equiv 5 \pmod{8}$	$m \equiv 7 \pmod{8}$
$n+1 \equiv 1 \pmod{8}$				
$n+1 \equiv 3 \pmod{8}$	The least $r > 0$ exists: r is odd and $2^r \equiv \frac{m-1}{2} \pmod{m}$, or r is even and $2^4 \equiv \frac{m+1}{2} \pmod{m}$.		m is prime	m is prime
$n+1 \equiv 5 \pmod{8}$	The least $r > 0$ exists: r is even and $2^r \equiv \frac{m-1}{2} \pmod{m}$, or r is even and $2^r \equiv \frac{m+1}{2} \pmod{m}$.	m is prime		m is prime
$n+1 \equiv 7 \pmod{8}$	The least $r > 0$ exists: $2^4 \equiv \frac{m-1}{2} \pmod{m}$.	m is prime	m is prime	

If $n+1 = md$, $m > 1$, $d > 1$, and the above table has an entry which is true, then an $n \times (n+1)$ polygonal path circular Florentine array does not exist.

Proof: For brevity's sake let $X(-1) \equiv X_{\frac{m-1}{2}d} - X_d \pmod 2$, and $X(+1) \equiv X_{\frac{m+1}{2}d} - X_d \pmod 2$.

In each of the nine cases, assuming a polygonal path circular Florentine array does exist, there will be a contradiction between the ODD/EVEN value of $X(-1)$ or $X(+1)$ required by Lemma E3, and that required by Lemma E5 and Lemma E4, as follows. In each case we assume the table entry is true.

1. $n+1 \equiv 3$ and $m \equiv 1 \pmod 8$
   E5.2 says $X(-1)$ is ODD, or E5.3 says $X(+1)$ is EVEN. But E3 says $X(-1)$ is EVEN and $X(+1)$ is ODD.
2. $n+1 \equiv 3$ and $m \equiv 5 \pmod 8$
   E4.2 and E5.2 say $X(-1)$ is ODD. But E3 says $X(-1)$ is EVEN.
3. $n+1 \equiv 3$ and $m \equiv 7 \pmod 8$
   E4.1 and E5.3 say $X(+1)$ is EVEN. But E3 says $X(+1)$ is ODD.
4. $n+1 \equiv 5$ and $m \equiv 1 \pmod 8$
   E5.2 says $X(-1)$ is EVEN, or E5.3 says $X(+1)$ is EVEN. But E3 says $X(-1)$ is ODD and $X(+1)$ is ODD.
5. $n+1 \equiv 5$ and $m \equiv 3 \pmod 8$
   E4.3 and E5.2 say $X(-1)$ is EVEN. But E3 says $X(-1)$ is ODD.
6. $n+1 \equiv 5$ and $m \equiv 7 \pmod 8$
   E4.1 and E5.3 say $X(+1)$ is EVEN. But E3 says $X(+1)$ is ODD.
7. $n+1 \equiv 7$ and $m \equiv 1 \pmod 8$
   E5.1 says $X(-1)$ is EVEN. But E3 says $X(-1)$ is ODD.
8. $n+1 \equiv 7$ and $m \equiv 3 \pmod 8$
   E4.3 and E5.1 say $X(-1)$ is EVEN. But E3 says $X(-1)$ is ODD.
9. $n+1 \equiv 7$ and $m \equiv 5 \pmod 8$
   E4.2 and E5.1 say $X(-1)$ is EVEN. But E3 says $X(-1)$ is ODD.

∎

The partition $E(n)$. *When $n+1 \equiv 1 \pmod 8$ we can define an equivalence relation R on the set of integers $\{1,2,\dots,n\}$ by letting xRy iff $y \equiv \pm 2^r x \pmod{n+1}$. Let $E(n)$ denote the partition of $\{1,2,\dots,n\}$ induced by R.*

Two examples.

Example 1: When $n+1 = 25$, $E(n) = \{[5,10,20,15],[1,2,4,8,16,7,14,3,6,12,24,23,21,17,9,18,11,22,19,13]\}$. Since $25 \equiv 1 \pmod 8$, Lemma E2 tells us that if there were a polygonal path construction for a 24×25 circular Florentine array, then among the symbols $1,2,\dots,24$ twenty would be odd or twenty would be even.

Example 2: When $n+1 = 49$, $E(n) = \{[7,14,28,42,35,21],[1,2,4,8,16,32,15,30,11,22,44,39,29,9,18,36,23,46,43,37,25,48,47,45,41,33,17,34,19,38,27,5,10,20,40,31,13,26,3,6,12,24]\}$. Since $49 \equiv 1$ (mod

8), Lemma E2 together with Corollary E1.2 would tell us that among the symbols $1, 2, \ldots, 48$ forty-two would be odd or forty-two would be even.

Theorem E7. *No polygonal path $n \times (n+1)$ circular Florentine array can exist when $n+1 \equiv 1 \pmod 8$ and we cannot contain exactly half of the numbers from 1 to n in some of the parts which belong to $E(n)$.*

Proof: Follows from E2 together with Corollary E1.2. ▮

Theorem E8. *If $n+1 = 3m$, where m is not divisible by 3, then the polygonal path construction cannot produce an $n \times (n+1)$ circular Florentine array.*

Proof: Suppose $n+1 = 3m$, and a polygonal path $X_1, X_2, \ldots, X_n$ produces an $n \times (n+1)$ circular Florentine array. Let $F(i,j)$ denote the symbol in row i, column j from 0 to n. Let the symbol $\star = X_0$, and let $F(1,j) = X_j$. $F(i,0) = X_0$, so the asterisk fills the leftmost column. The rest of the array, generated by rotating the polygonal path, is described by $F(i,j) \equiv i-1+X_j \pmod n$ where $j \neq 0$, and the residue (mod n) is from the symbols $1, 2, \ldots, n$.

Since they must occur at different directed distances from the asterisk, the symbols n, $\frac{n}{2}$, and 1 must occur once each in the columns numbered 1 to n.

Lemma E1 tells us that if $F(i,j) = n$, then $F(i, n+1-j) = \frac{n}{2}$. Indeed, calculating (mod n) we have $n = F(i,j) \equiv i-1+X_j \equiv i-1+\frac{n}{2}+X_{n+1-j}$ and therefore $\frac{n}{2} \equiv i-1+X_{n+1-j} \equiv F(i, n+1-j)$.

Now we shall use the assumption that $n+1 = 3m$, defining nine variables: $Y_{000}, Y_{001}, Y_{002}, Y_{120}, Y_{121}, Y_{122}, Y_{210}, Y_{211}, Y_{212}$. Let Y_{abc} stand for the number of rows in which $F(i,r) = n$, $F(i,s) = \frac{n}{2}$, $F(i,t) = 1$, and $r \equiv a \pmod 3$, $s \equiv b \pmod 3$, $t \equiv c \pmod 3$. Since $r+s \equiv 0 \pmod{n+1}$, and $n+1 \equiv 0 \pmod 3$, we only need the rows where $a+b \equiv 0 \pmod 3$, and these nine variables cover all possible cases.

Since in the column numbers $1, 2, \ldots, 3m-1$ there are m numbers which are congruent to $1 \pmod 3$, m which are congruent to $2 \pmod 3$, and $m-1$ which are congruent to $0 \pmod 3$, we can form the following nine linear equations.

(Equation 1) $Y_{000} + Y_{120} + Y_{210} = m-1$

(Equation 2) $Y_{001} + Y_{121} + Y_{211} = m$

(Equation 3) $Y_{002} + Y_{122} + Y_{212} = m$

(Equation 4) $Y_{120} + Y_{121} + Y_{122} = m$

(Equation 5) $Y_{210} + Y_{211} + Y_{212} = m$

(Equation 6) $Y_{001} + Y_{122} + Y_{210} = m$

(Equation 7) $Y_{002} + Y_{120} + Y_{211} = m$

(Equation 8) $Y_{001} + Y_{120} + Y_{212} = m$

(Equation 9) $Y_{002} + Y_{121} + Y_{210} = m.$

(Equation 1) is true since the number of times the symbol 1 should be in columns congruent to $0 \pmod 3$ is $m-1$. (Equation 2) and (Equation 3) are true because the symbol 1 should be in columns congruent to $1 \pmod 3$ m times, and in columns congruent to $2 \pmod 3$ m times. (Equation 4) and (Equation 5) are true because m is the number of times the symbol n should be in columns congruent to $1 \pmod 3$, and respectively congruent to $2 \pmod 3$. Considering the rows in which $F(i,r) = n$ and $F(i,t) = 1$, (Equation 6) is true because it should occur m times that $t-r \equiv 1 \pmod 3$, and (Equation 7) is true because it should occur m times that $t-r \equiv 2 \pmod 3$ (Equation 8) and (Equation 9) are true because among rows in which $F(i,s) = \frac{n}{2}$ and $F(i,t) = 1$, there should be m rows where $t-s \equiv 1 \pmod 3$ and m rows where $t-s \equiv 2 \pmod 3$.

It can be verified by standard methods that the nine linear equations with nine variables are linearly independent. Their unique solution is $Y_{000} = (m-3)/3$, and $Y_{001} = Y_{002} = Y_{120} = Y_{121} = Y_{122} = Y_{210} = Y_{211} = Y_{212} = m/3$. Hence, if $n+1 = 3m$ and m is not divisible by 3, there cannot exist an $n \times (n+1)$ circular Florentine array by the polygonal path construction. ∎

Remark E9: Quoting from [3], the Bruck-Ryser theorem says: "If $w \equiv 1$ or $2 \pmod 4$, and the square-free part of w contains at least one prime factor $p \equiv 3 \pmod 4$, then there does not exist a projective plane of order w."

It is well known [3] that a projective plane of order $w = n+1$ exists iff a set of n pairwise orthogonal $(n+1) \times (n+1)$ Latin squares exists.

Thus by virtue of Theorem A we know that *no* $n \times (n+1)$ circular Florentine array can exist, whenever the Bruck-Ryser theorem says no projective plane of order $n+1$ can exist. The first few odd numbers ruled out are: 21, 33, 57, 69, 77, 93, 105.

E10. Non-existence of some polygonal path circular Florentine arrays.

Using the lemmas, theorems, and remark, E1-E9, we have found that for $n+1 \leq 1001$ there are only 33 composite numbers for which we cannot rule out the existence of a polygonal path $X_1, X_2, \ldots, X_n$ which produces an $n \times (n+1)$ circular Florentine array. Those numbers are 85, 99, 125, 171, 205, 221, 243, 289, 325, 343, 387, 425, 485, 493, 511, 531, 533, 565, 585, 603, 623, 629, 685, 697, 725, 747, 845, 891, 901, 925, 959, 963, 965.

E12. Circular Tuscan-2 arrays and Tuscan-3 squares.

Although we have proved that for many composite integers there is no polygonal path which forms a circular Florentine array, symmetric polygonal paths do form other Tuscan-k arrays.

Lemma E12. *If a polygonal path $X_1, X_2, \ldots, X_n$ is symmetric, that is, for $1 \leq i \leq n$, $\frac{n}{2} + X_i \equiv X_{n+1-i} \pmod n$, and forms a Tuscan-2 square, then the polygonal path $X_0, X_1, X_2, \ldots, X_n$ (with $\star = X_0$) forms an $n \times (n+1)$ circular Tuscan-2 array.*

Proof: For every $1 \le j \le n-2$ we have $X_{j+2} - X_j \equiv X_{n-j-1} + \frac{n}{2} - (X_{n-j+1} + \frac{n}{2}) \equiv X_{n-j-1} - X_{n-j+1} \pmod{n}$. Hence, $X_{j+2} - X_j \equiv n - (X_{n-j+1} - X_{n-j-1}) \pmod{n}$, and since the polygonal path forms a Tuscan-2 square it follows that $X_{j+2} - X_j \not\equiv \frac{n}{2} \pmod{n}$. Now, $X_n - X_1 \equiv \frac{n}{2} \pmod{n}$ by symmetry, and therefore the polygonal path $X_0, X_1, X_2, \ldots, X_n$ forms an $n \times (n+1)$ circular Tuscan-2 array. ∎

Let $X_1, X_2, \ldots, X_n$ be a polygonal path which forms a Tuscan-k square, and let m be an integer relatively prime to n. Then it can be easily verified that the polygonal path defined by $Y_i \equiv mX_i \pmod{n}$, $1 \le i \le n$, also forms a Tuscan-k square. If $X_0, X_1, X_2, \ldots, X_n$ forms a circular Tuscan-k array, then $Y_0, Y_1, Y_2, \ldots, Y_n$ forms a circular Tuscan-k array. $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ will be called equivalent.

We have found examples of symmetric polygonal paths which form Tuscan-2 squares for all even n, $10 \le n \le 50$. These examples are listed in Figure E. An exhaustive computer search has found that for $n \le 20$ all the Tuscan-3 squares which are formed by polygonal paths are the known Vatican squares. For $n \le 28$ an exhaustive search of all the symmetric polygonal paths which form Tuscan-2 squares has found that only for $n = 22$ is there a Tuscan-3 which is not Vatican. There is only one (using the equivalence described above), and it does not form a circular Tuscan-3 array, as can be seen in Figure 22A. Incidentally this example shows that Lemma E12 cannot be extended to cover symmetric polygonal paths which form Tuscan-3 squares.

Three other inequivalent symmetric polygonal paths which form Tuscan-3 squares, but not Vatican, and not circular Tuscan-3, were found for $n = 30$. However, the search was not exhaustive. The three paths are shown in Figures 30A, 30B, and 30C.

Let $T_2(n)$ denote the number of inequivalent polygonal paths of order n which form Tuscan-2 squares. Let $TS_2(n)$ denote the number of inequivalent symmetric polygonal paths which form Tuscan-2 squares. The following table summarizes the results of exhaustive search.

n	$T_2(n)$	$TS_2(n)$
10	2	2
12	1	1
14	3	1
16	14	6
18	135	9
20	?	19
22	?	48
24	?	178
26	?	453
28	?	1751

Open problems.

1. Find an $n \times n$ Tuscan-3 square for composite $n + 1$, or prove that none exist.
2. Find an $n \times n$ Tuscan-2 square for odd $n > 1$, or prove that none exist.
3. Prove that no $n \times n$ Florentine squares exist for odd $n > 1$, or find one.
4. Do any polygonal path $n \times n$ Florentine squares exist for composite $n + 1$?

Figure 16A
Symmetric polygonal path 16 $\times$ 17 circular Tuscan-2 array
add-asterisk transform example of a non-Latin Tuscan-2 square

*	16	2	1	12	5	15	3	6	14	11	7	13	4	9	10	8
*	1	3	2	13	6	16	4	7	15	12	8	14	5	10	11	9
*	2	4	3	14	7	1	5	8	16	13	9	15	6	11	12	10
*	3	5	4	15	8	2	6	9	1	14	10	16	7	12	13	11
*	4	6	5	16	9	3	7	10	2	15	11	1	8	13	14	12
*	5	7	6	1	10	4	8	11	3	16	12	2	9	14	15	13
*	6	8	7	2	11	5	9	12	4	1	13	3	10	15	16	14
*	7	9	8	3	12	6	10	13	5	2	14	4	11	16	1	15
*	8	10	9	4	13	7	11	14	6	3	15	5	12	1	2	16
*	9	11	10	5	14	8	12	15	7	4	16	6	13	2	3	1
*	10	12	11	6	15	9	13	16	8	5	1	7	14	3	4	2
*	11	13	12	7	16	10	14	1	9	6	2	8	15	4	5	3
*	12	14	13	8	1	11	15	2	10	7	3	9	16	5	6	4
*	13	15	14	9	2	12	16	3	11	8	4	10	1	6	7	5
*	14	16	15	10	3	13	1	4	12	9	5	11	2	7	8	6
*	15	1	16	11	4	14	2	5	13	10	6	12	3	8	9	7

16	2	1	12	5	15	3	6	14	11	7	13	4	9	10	8	*
16	4	7	15	12	8	14	5	10	11	9	*	1	3	2	13	6
16	13	9	15	6	11	12	10	*	2	4	3	14	7	1	5	8
16	7	12	13	11	*	3	5	4	15	8	2	6	9	1	14	10
16	9	3	7	10	2	15	11	1	8	13	14	12	*	4	6	5
16	12	2	9	14	15	13	*	5	7	6	1	10	4	8	11	3
16	14	*	6	8	7	2	11	5	9	12	4	1	13	3	10	15
16	1	15	*	7	9	8	3	12	6	10	13	5	2	14	4	11
16	*	8	10	9	4	13	7	11	14	6	3	15	5	12	1	2
16	6	13	2	3	1	*	9	11	10	5	14	8	12	15	7	4
16	8	5	1	7	14	3	4	2	*	10	12	11	6	15	9	13
16	10	14	1	9	6	2	8	15	4	5	3	*	11	13	12	7
16	5	6	4	*	12	14	13	8	1	11	15	2	10	7	3	9
16	3	11	8	4	10	1	6	7	5	*	13	15	14	9	2	12
16	15	10	3	13	1	4	12	9	5	11	2	7	8	6	*	14
16	11	4	14	2	5	13	10	6	12	3	8	9	7	*	15	1

Figure 20B

Symmetric polygonal path 20 × 21 circular Tuscan-2 array add-asterisk example of a non-Latin Tuscan-2 square

*	20	1	4	13	19	6	8	12	7	15	5	17	2	18	16	9	3	14	11	10
*	1	2	5	14	20	7	9	13	8	16	6	18	3	19	17	10	4	15	12	11
*	2	3	6	15	1	8	10	14	9	17	7	19	4	20	18	11	5	16	13	12
*	3	4	7	16	2	9	11	15	10	18	8	20	5	1	19	12	6	17	14	13
*	4	5	8	17	3	10	12	16	11	19	9	1	6	2	20	13	7	18	15	14
*	5	6	9	18	4	11	13	17	12	20	10	2	7	3	1	14	8	19	16	15
*	6	7	10	19	5	12	14	18	13	1	11	3	8	4	2	15	9	20	17	16
*	7	8	11	20	6	13	15	19	14	2	12	4	9	5	3	16	10	1	18	17
*	8	9	12	1	7	14	16	20	15	3	13	5	10	6	4	17	11	2	19	18
*	9	10	13	2	8	15	17	1	16	4	14	6	11	7	5	18	12	3	20	19
*	10	11	14	3	9	16	18	2	17	5	15	7	12	8	6	19	13	4	1	20
*	11	12	15	4	10	17	19	3	18	6	16	8	13	9	7	20	14	5	2	1
*	12	13	16	5	11	18	20	4	19	7	17	9	14	10	8	1	15	6	3	2
*	13	14	17	6	12	19	1	5	20	8	18	10	15	11	9	2	16	7	4	3
*	14	15	18	7	13	20	2	6	1	9	19	11	16	12	10	3	17	8	5	4
*	15	16	19	8	14	1	3	7	2	10	20	12	17	13	11	4	18	9	6	5
*	16	17	20	9	15	2	4	8	3	11	1	13	18	14	12	5	19	10	7	6
*	17	18	1	10	16	3	5	9	4	12	2	14	19	15	13	6	20	11	8	7
*	18	19	2	11	17	4	6	10	5	13	3	15	20	16	14	7	1	12	9	8
*	19	20	3	12	18	5	7	11	6	14	4	16	1	17	15	8	2	13	10	9

20	1	4	13	19	6	8	12	7	15	5	17	2	18	16	9	3	14	11	10	*
20	7	9	13	8	16	6	18	3	19	17	10	4	15	12	11	*	1	2	5	14
20	18	11	5	16	13	12	*	2	3	6	15	1	8	10	14	9	17	7	19	4
20	5	1	19	12	6	17	14	13	*	3	4	7	16	2	9	11	15	10	18	8
20	13	7	18	15	14	*	4	5	8	17	3	10	12	16	11	19	9	1	6	2
20	10	2	7	3	1	14	8	19	16	15	*	5	6	9	18	4	11	13	17	12
20	17	16	*	6	7	10	19	5	12	14	18	13	1	11	3	8	4	2	15	9
20	6	13	15	19	14	2	12	4	9	5	3	16	10	1	18	17	*	7	8	11
20	15	3	13	5	10	6	4	17	11	2	19	18	*	8	9	12	1	7	14	16
20	19	*	9	10	13	2	8	15	17	1	16	4	14	6	11	7	5	18	12	3
20	*	10	11	14	3	9	16	18	2	17	5	15	7	12	8	6	19	13	4	1
20	14	5	2	1	*	11	12	15	4	10	17	19	3	18	6	16	8	13	9	7
20	4	19	7	17	9	14	10	8	1	15	6	3	2	*	12	13	16	5	11	18
20	8	18	10	15	11	9	2	16	7	4	3	*	13	14	17	6	12	19	1	5
20	2	6	1	9	19	11	16	12	10	3	17	8	5	4	*	14	15	18	7	13
20	12	17	13	11	4	18	9	6	5	*	15	16	19	8	14	1	3	7	2	10
20	9	15	2	4	8	3	11	1	13	18	14	12	5	19	10	7	6	*	16	17
20	11	8	7	*	17	18	1	10	16	3	5	9	4	12	2	14	19	15	13	6
20	16	14	7	1	12	9	8	*	18	19	2	11	17	4	6	10	5	13	3	15
20	3	12	18	5	7	11	6	14	4	16	1	17	15	8	2	13	10	9	*	19

Figure 22A
Tuscan-3, $N = 22$, symmetric polygonal path

1	3	13	9	16	8	7	10	15	6	22	11	17	4	21	18	19	5	20	2	14	12
2	4	14	10	17	9	8	11	16	7	1	12	18	5	22	19	20	6	21	3	15	13
3	5	15	11	18	10	9	12	17	8	2	13	19	6	1	20	21	7	22	4	16	14
4	6	16	12	19	11	10	13	18	9	3	14	20	7	2	21	22	8	1	5	17	15
5	7	17	13	20	12	11	14	19	10	4	15	21	8	3	22	1	9	2	6	18	16
6	8	18	14	21	13	12	15	20	11	5	16	22	9	4	1	2	10	3	7	19	17
7	9	19	15	22	14	13	16	21	12	6	17	1	10	5	2	3	11	4	8	20	18
8	10	20	16	1	15	14	17	22	13	7	18	2	11	6	3	4	12	5	9	21	19
9	11	21	17	2	16	15	18	1	14	8	19	3	12	7	4	5	13	6	10	22	20
10	12	22	18	3	17	16	19	2	15	9	20	4	13	8	5	6	14	7	11	1	21
11	13	1	19	4	18	17	20	3	16	10	21	5	14	9	6	7	15	8	12	2	22
12	14	2	20	5	19	18	21	4	17	11	22	6	15	10	7	8	16	9	13	3	1
13	15	3	21	6	20	19	22	5	18	12	1	7	16	11	8	9	17	10	14	4	2
14	16	4	22	7	21	20	1	6	19	13	2	8	17	12	9	10	18	11	15	5	3
15	17	5	1	8	22	21	2	7	20	14	3	9	18	13	10	11	19	12	16	6	4
16	18	6	2	9	1	22	3	8	21	15	4	10	19	14	11	12	20	13	17	7	5
17	19	7	3	10	2	1	4	9	22	16	5	11	20	15	12	13	21	14	18	8	6
18	20	8	4	11	3	2	5	10	1	17	6	12	21	16	13	14	22	15	19	9	7
19	21	9	5	12	4	3	6	11	2	18	7	13	22	17	14	15	1	16	20	10	8
20	22	10	6	13	5	4	7	12	3	19	8	14	1	18	15	16	2	17	21	11	9
21	1	11	7	14	6	5	8	13	4	20	9	15	2	19	16	17	3	18	22	12	10
22	2	12	8	15	7	6	9	14	5	21	10	16	3	20	17	18	4	19	1	13	11

This is the only non-Vatican Tuscan-3 for $N < 30$ with a symmetric polygonal path . It is not circular Tuscan-3, but is necessarily circular Tuscan-2 because the path is symmetric. Except for the Vatican Squares, which occur when $N + 1$ is prime, there are no polygonal path Tuscan-3 squares for $N < 22$.

Figure 30A
Tuscan-3, $N = 30$, symmetric polygonal path

1	3	4	20	12	22	13	8	25	14	2	9	6	30	26	11	15	21	24	17	29	10	23	28	7	27	5	19	18	16
2	4	5	21	13	23	14	9	26	15	3	10	7	1	27	12	16	22	25	18	30	11	24	29	8	28	6	20	19	17
3	5	6	22	14	24	15	10	27	16	4	11	8	2	28	13	17	23	26	19	1	12	25	30	9	29	7	21	20	18
4	6	7	23	15	25	16	11	28	17	5	12	9	3	29	14	18	24	27	20	2	13	26	1	10	30	8	22	21	19
5	7	8	24	16	26	17	12	29	18	6	13	10	4	30	15	19	25	28	21	3	14	27	2	11	1	9	23	22	20
6	8	9	25	17	27	18	13	30	19	7	14	11	5	1	16	20	26	29	22	4	15	28	3	12	2	10	24	23	21
7	9	10	26	18	28	19	14	1	20	8	15	12	6	2	17	21	27	30	23	5	16	29	4	13	3	11	25	24	22
8	10	11	27	19	29	20	15	2	21	9	16	13	7	3	18	22	28	1	24	6	17	30	5	14	4	12	26	25	23
9	11	12	28	20	30	21	16	3	22	10	17	14	8	4	19	23	29	2	25	7	18	1	6	15	5	13	27	26	24
10	12	13	29	21	1	22	17	4	23	11	18	15	9	5	20	24	30	3	26	8	19	2	7	16	6	14	28	27	25
11	13	14	30	22	2	23	18	5	24	12	19	16	10	6	21	25	1	4	27	9	20	3	8	17	7	15	29	28	26
12	14	15	1	23	3	24	19	6	25	13	20	17	11	7	22	26	2	5	28	10	21	4	9	18	8	16	30	29	27
13	15	16	2	24	4	25	20	7	26	14	21	18	12	8	23	27	3	6	29	11	22	5	10	19	9	17	1	30	28
14	16	17	3	25	5	26	21	8	27	15	22	19	13	9	24	28	4	7	30	12	23	6	11	20	10	18	2	1	29
15	17	18	4	26	6	27	22	9	28	16	23	20	14	10	25	29	5	8	1	13	24	7	12	21	11	19	3	2	30
16	18	19	5	27	7	28	23	10	29	17	24	21	15	11	26	30	6	9	2	14	25	8	13	22	12	20	4	3	1
17	19	20	6	28	8	29	24	11	30	18	25	22	16	12	27	1	7	10	3	15	26	9	14	23	13	21	5	4	2
18	20	21	7	29	9	30	25	12	1	19	26	23	17	13	28	2	8	11	4	16	27	10	15	24	14	22	6	5	3
19	21	22	8	30	10	1	26	13	2	20	27	24	18	14	29	3	9	12	5	17	28	11	16	25	15	23	7	6	4
20	22	23	9	1	11	2	27	14	3	21	28	25	19	15	30	4	10	13	6	18	29	12	17	26	16	24	8	7	5
21	23	24	10	2	12	3	28	15	4	22	29	26	20	16	1	5	11	14	7	19	30	13	18	27	17	25	9	8	6
22	24	25	11	3	13	4	29	16	5	23	30	27	21	17	2	6	12	15	8	20	1	14	19	28	18	26	10	9	7
23	25	26	12	4	14	5	30	17	6	24	1	28	22	18	3	7	13	16	9	21	2	15	20	29	19	27	11	10	8
24	26	27	13	5	15	6	1	18	7	25	2	29	23	19	4	8	14	17	10	22	3	16	21	30	20	28	12	11	9
25	27	28	14	6	16	7	2	19	8	26	3	30	24	20	5	9	15	18	11	23	4	17	22	1	21	29	13	12	10
26	28	29	15	7	17	8	3	20	9	27	4	1	25	21	6	10	16	19	12	24	5	18	23	2	22	30	14	13	11
27	29	30	16	8	18	9	4	21	10	28	5	2	26	22	7	11	17	20	13	25	6	19	24	3	23	1	15	14	12
28	30	1	17	9	19	10	5	22	11	29	6	3	27	23	8	12	18	21	14	26	7	20	25	4	24	2	16	15	13
29	1	2	18	10	20	11	6	23	12	30	7	4	28	24	9	13	19	22	15	27	8	21	26	5	25	3	17	16	14
30	2	3	19	11	21	12	7	24	13	1	8	5	29	25	10	14	20	23	16	28	9	22	27	6	26	4	18	17	15

Figure 30B
Tuscan-3, $N = 30$, symmetric polygonal path

1	3	6	5	27	22	28	10	17	26	9	29	15	19	8	23	4	30	14	24	11	2	25	13	7	12	20	21	18	16
2	4	7	6	28	23	29	11	18	27	10	30	16	20	9	24	5	1	15	25	12	3	26	14	8	13	21	22	19	17
3	5	8	7	29	24	30	12	19	28	11	1	17	21	10	25	6	2	16	26	13	4	27	15	9	14	22	23	20	18
4	6	9	8	30	25	1	13	20	29	12	2	18	22	11	26	7	3	17	27	14	5	28	16	10	15	23	24	21	19
5	7	10	9	1	26	2	14	21	30	13	3	19	23	12	27	8	4	18	28	15	6	29	17	11	16	24	25	22	20
6	8	11	10	2	27	3	15	22	1	14	4	20	24	13	28	9	5	19	29	16	7	30	18	12	17	25	26	23	21
7	9	12	11	3	28	4	16	23	2	15	5	21	25	14	29	10	6	20	30	17	8	1	19	13	18	26	27	24	22
8	10	13	12	4	29	5	17	24	3	16	6	22	26	15	30	11	7	21	1	18	9	2	20	14	19	27	28	25	23
9	11	14	13	5	30	6	18	25	4	17	7	23	27	16	1	12	8	22	2	19	10	3	21	15	20	28	29	26	24
10	12	15	14	6	1	7	19	26	5	18	8	24	28	17	2	13	9	23	3	20	11	4	22	16	21	29	30	27	25
11	13	16	15	7	2	8	20	27	6	19	9	25	29	18	3	14	10	24	4	21	12	5	23	17	22	30	1	28	26
12	14	17	16	8	3	9	21	28	7	20	10	26	30	19	4	15	11	25	5	22	13	6	24	18	23	1	2	29	27
13	15	18	17	9	4	10	22	29	8	21	11	27	1	20	5	16	12	26	6	23	14	7	25	19	24	2	3	30	28
14	16	19	18	10	5	11	23	30	9	22	12	28	2	21	6	17	13	27	7	24	15	8	26	20	25	3	4	1	29
15	17	20	19	11	6	12	24	1	10	23	13	29	3	22	7	18	14	28	8	25	16	9	27	21	26	4	5	2	30
16	18	21	20	12	7	13	25	2	11	24	14	30	4	23	8	19	15	29	9	26	17	10	28	22	27	5	6	3	1
17	19	22	21	13	8	14	26	3	12	25	15	1	5	24	9	20	16	30	10	27	18	11	29	23	28	6	7	4	2
18	20	23	22	14	9	15	27	4	13	26	16	2	6	25	10	21	17	1	11	28	19	12	30	24	29	7	8	5	3
19	21	24	23	15	10	16	28	5	14	27	17	3	7	26	11	22	18	2	12	29	20	13	1	25	30	8	9	6	4
20	22	25	24	16	11	17	29	6	15	28	18	4	8	27	12	23	19	3	13	30	21	14	2	26	1	9	10	7	5
21	23	26	25	17	12	18	30	7	16	29	19	5	9	28	13	24	20	4	14	1	22	15	3	27	2	10	11	8	6
22	24	27	26	18	13	19	1	8	17	30	20	6	10	29	14	25	21	5	15	2	23	16	4	28	3	11	12	9	7
23	25	28	27	19	14	20	2	9	18	1	21	7	11	30	15	26	22	6	16	3	24	17	5	29	4	12	13	10	8
24	26	29	28	20	15	21	3	10	19	2	22	8	12	1	16	27	23	7	17	4	25	18	6	30	5	13	14	11	9
25	27	30	29	21	16	22	4	11	20	3	23	9	13	2	17	28	24	8	18	5	26	19	7	1	6	14	15	12	10
26	28	1	30	22	17	23	5	12	21	4	24	10	14	3	18	29	25	9	19	6	27	20	8	2	7	15	16	13	11
27	29	2	1	23	18	24	6	13	22	5	25	11	15	4	19	30	26	10	20	7	28	21	9	3	8	16	17	14	12
28	30	3	2	24	19	25	7	14	23	6	26	12	16	5	20	1	27	11	21	8	29	22	10	4	9	17	18	15	13
29	1	4	3	25	20	26	8	15	24	7	27	13	17	6	21	2	28	12	22	9	30	23	11	5	10	18	19	16	14
30	2	5	4	26	21	27	9	16	25	8	28	14	18	7	22	3	29	13	23	10	1	24	12	6	11	19	20	17	15

Figure 30C
Tuscan-3, $N = 30$, symmetric polygonal path

1	7	9	12	11	25	4	21	28	2	14	3	23	15	20	5	30	8	18	29	17	13	6	19	10	26	27	24	22	16
2	8	10	13	12	26	5	22	29	3	15	4	24	16	21	6	1	9	19	30	18	14	7	20	11	27	28	25	23	17
3	9	11	14	13	27	6	23	30	4	16	5	25	17	22	7	2	10	20	1	19	15	8	21	12	28	29	26	24	18
4	10	12	15	14	28	7	24	1	5	17	6	26	18	23	8	3	11	21	2	20	16	9	22	13	29	30	27	25	19
5	11	13	16	15	29	8	25	2	6	18	7	27	19	24	9	4	12	22	3	21	17	10	23	14	30	1	28	26	20
6	12	14	17	16	30	9	26	3	7	19	8	28	20	25	10	5	13	23	4	22	18	11	24	15	1	2	29	27	21
7	13	15	18	17	1	10	27	4	8	20	9	29	21	26	11	6	14	24	5	23	19	12	25	16	2	3	30	28	22
8	14	16	19	18	2	11	28	5	9	21	10	30	22	27	12	7	15	25	6	24	20	13	26	17	3	4	1	29	23
9	15	17	20	19	3	12	29	6	10	22	11	1	23	28	13	8	16	26	7	25	21	14	27	18	4	5	2	30	24
10	16	18	21	20	4	13	30	7	11	23	12	2	24	29	14	9	17	27	8	26	22	15	28	19	5	6	3	1	25
11	17	19	22	21	5	14	1	8	12	24	13	3	25	30	15	10	18	28	9	27	23	16	29	20	6	7	4	2	26
12	18	20	23	22	6	15	2	9	13	25	14	4	26	1	16	11	19	29	10	28	24	17	30	21	7	8	5	3	27
13	19	21	24	23	7	16	3	10	14	26	15	5	27	2	17	12	20	30	11	29	25	18	1	22	8	9	6	4	28
14	20	22	25	24	8	17	4	11	15	27	16	6	28	3	18	13	21	1	12	30	26	19	2	23	9	10	7	5	29
15	21	23	26	25	9	18	5	12	16	28	17	7	29	4	19	14	22	2	13	1	27	20	3	24	10	11	8	6	30
16	22	24	27	26	10	19	6	13	17	29	18	8	30	5	20	15	23	3	14	2	28	21	4	25	11	12	9	7	1
17	23	25	28	27	11	20	7	14	18	30	19	9	1	6	21	16	24	4	15	3	29	22	5	26	12	13	10	8	2
18	24	26	29	28	12	21	8	15	19	1	20	10	2	7	22	17	25	5	16	4	30	23	6	27	13	14	11	9	3
19	25	27	30	29	13	22	9	16	20	2	21	11	3	8	23	18	26	6	17	5	1	24	7	28	14	15	12	10	4
20	26	28	1	30	14	23	10	17	21	3	22	12	4	9	24	19	27	7	18	6	2	25	8	29	15	16	13	11	5
21	27	29	2	1	15	24	11	18	22	4	23	13	5	10	25	20	28	8	19	7	3	26	9	30	16	17	14	12	6
22	28	30	3	2	16	25	12	19	23	5	24	14	6	11	26	21	29	9	20	8	4	27	10	1	17	18	15	13	7
23	29	1	4	3	17	26	13	20	24	6	25	15	7	12	27	22	30	10	21	9	5	28	11	2	18	19	16	14	8
24	30	2	5	4	18	27	14	21	25	7	26	16	8	13	28	23	1	11	22	10	6	29	12	3	19	20	17	15	9
25	1	3	6	5	19	28	15	22	26	8	27	17	9	14	29	24	2	12	23	11	7	30	13	4	20	21	18	16	10
26	2	4	7	6	20	29	16	23	27	9	28	18	10	15	30	25	3	13	24	12	8	1	14	5	21	22	19	17	11
27	3	5	8	7	21	30	17	24	28	10	29	19	11	16	1	26	4	14	25	13	9	2	15	6	22	23	20	18	12
28	4	6	9	8	22	1	18	25	29	11	30	20	12	17	2	27	5	15	26	14	10	3	16	7	23	24	21	19	13
29	5	7	10	9	23	2	19	26	30	12	1	21	13	18	3	28	6	16	27	15	11	4	17	8	24	25	22	20	14
30	6	8	11	10	24	3	20	27	1	13	2	22	14	19	4	29	7	17	28	16	12	5	18	9	25	26	23	21	15

Figure 20A

Tuscan-2, $N = 20$, Asymmetric polygonal path
not symmetric in two segments at each end;
therefore not circular Tuscan-2

1	2	4	3	16	19	15	5	9	20	7	13	18	10	8	17	12	6	14	11
2	3	5	4	17	20	16	6	10	1	8	14	19	11	9	18	13	7	15	12
3	4	6	5	18	1	17	7	11	2	9	15	20	12	10	19	14	8	16	13
4	5	7	6	19	2	18	8	12	3	10	16	1	13	11	20	15	9	17	14
5	6	8	7	20	3	19	9	13	4	11	17	2	14	12	1	16	10	18	15
6	7	9	8	1	4	20	10	14	5	12	18	3	15	13	2	17	11	19	16
7	8	10	9	2	5	1	11	15	6	13	19	4	16	14	3	18	12	20	17
8	9	11	10	3	6	2	12	16	7	14	20	5	17	15	4	19	13	1	18
9	10	12	11	4	7	3	13	17	8	15	1	6	18	16	5	20	14	2	19
10	11	13	12	5	8	4	14	18	9	16	2	7	19	17	6	1	15	3	20
11	12	14	13	6	9	5	15	19	10	17	3	8	20	18	7	2	16	4	1
12	13	15	14	7	10	6	16	20	11	18	4	9	1	19	8	3	17	5	2
13	14	16	15	8	11	7	17	1	12	19	5	10	2	20	9	4	18	6	3
14	15	17	16	9	12	8	18	2	13	20	6	11	3	1	10	5	19	7	4
15	16	18	17	10	13	9	19	3	14	1	7	12	4	2	11	6	20	8	5
16	17	19	18	11	14	10	20	4	15	2	8	13	5	3	12	7	1	9	6
17	18	20	19	12	15	11	1	5	16	3	9	14	6	4	13	8	2	10	7
18	19	1	20	13	16	12	2	6	17	4	10	15	7	5	14	9	3	11	8
19	20	2	1	14	17	13	3	7	18	5	11	16	8	6	15	10	4	12	9
20	1	3	2	15	18	14	4	8	19	6	12	17	9	7	16	11	5	13	10

Figure 24A
Tuscan-2, $N = 24$, symmetric polygonal path
necessarily circular Tuscan-2

1	3	2	7	17	24	11	8	16	10	6	21	9	18	22	4	20	23	12	5	19	14	15	13
2	4	3	8	18	1	12	9	17	11	7	22	10	19	23	5	21	24	13	6	20	15	16	14
3	5	4	9	19	2	13	10	18	12	8	23	11	20	24	6	22	1	14	7	21	16	17	15
4	6	5	10	20	3	14	11	19	13	9	24	12	21	1	7	23	2	15	8	22	17	18	16
5	7	6	11	21	4	15	12	20	14	10	1	13	22	2	8	24	3	16	9	23	18	19	17
6	8	7	12	22	5	16	13	21	15	11	2	14	23	3	9	1	4	17	10	24	19	20	18
7	9	8	13	23	6	17	14	22	16	12	3	15	24	4	10	2	5	18	11	1	20	21	19
8	10	9	14	24	7	18	15	23	17	13	4	16	1	5	11	3	6	19	12	2	21	22	20
9	11	10	15	1	8	19	16	24	18	14	5	17	2	6	12	4	7	20	13	3	22	23	21
10	12	11	16	2	9	20	17	1	19	15	6	18	3	7	13	5	8	21	14	4	23	24	22
11	13	12	17	3	10	21	18	2	20	16	7	19	4	8	14	6	9	22	15	5	24	1	23
12	14	13	18	4	11	22	19	3	21	17	8	20	5	9	15	7	10	23	16	6	1	2	24
13	15	14	19	5	12	23	20	4	22	18	9	21	6	10	16	8	11	24	17	7	2	3	1
14	16	15	20	6	13	24	21	5	23	19	10	22	7	11	17	9	12	1	18	8	3	4	2
15	17	16	21	7	14	1	22	6	24	20	11	23	8	12	18	10	13	2	19	9	4	5	3
16	18	17	22	8	15	2	23	7	1	21	12	24	9	13	19	11	14	3	20	10	5	6	4
17	19	18	23	9	16	3	24	8	2	22	13	1	10	14	20	12	15	4	21	11	6	7	5
18	20	19	24	10	17	4	1	9	3	23	14	2	11	15	21	13	16	5	22	12	7	8	6
19	21	20	1	11	18	5	2	10	4	24	15	3	12	16	22	14	17	6	23	13	8	9	7
20	22	21	2	12	19	6	3	11	5	1	16	4	13	17	23	15	18	7	24	14	9	10	8
21	23	22	3	13	20	7	4	12	6	2	17	5	14	18	24	16	19	8	1	15	10	11	9
22	24	23	4	14	21	8	5	13	7	3	18	6	15	19	1	17	20	9	2	16	11	12	10
23	1	24	5	15	22	9	6	14	8	4	19	7	16	20	2	18	21	10	3	17	12	13	11
24	2	1	6	16	23	10	7	15	9	5	20	8	17	21	3	19	22	11	4	18	13	14	12

Figure 26A
Tuscan-2, $N = 26$, symmetric polygonal path
necessarily circular Tuscan-2

1	2	4	7	3	23	12	5	19	11	21	26	9	22	13	8	24	6	18	25	10	16	20	17	15	14
2	3	5	8	4	24	13	6	20	12	22	1	10	23	14	9	25	7	19	26	11	17	21	18	16	15
3	4	6	9	5	25	14	7	21	13	23	2	11	24	15	10	26	8	20	1	12	18	22	19	17	16
4	5	7	10	6	26	15	8	22	14	24	3	12	25	16	11	1	9	21	2	13	19	23	20	18	17
5	6	8	11	7	1	16	9	23	15	25	4	13	26	17	12	2	10	22	3	14	20	24	21	19	18
6	7	9	12	8	2	17	10	24	16	26	5	14	1	18	13	3	11	23	4	15	21	25	22	20	19
7	8	10	13	9	3	18	11	25	17	1	6	15	2	19	14	4	12	24	5	16	22	26	23	21	20
8	9	11	14	10	4	19	12	26	18	2	7	16	3	20	15	5	13	25	6	17	23	1	24	22	21
9	10	12	15	11	5	20	13	1	19	3	8	17	4	21	16	6	14	26	7	18	24	2	25	23	22
10	11	13	16	12	6	21	14	2	20	4	9	18	5	22	17	7	15	1	8	19	25	3	26	24	23
11	12	14	17	13	7	22	15	3	21	5	10	19	6	23	18	8	16	2	9	20	26	4	1	25	24
12	13	15	18	14	8	23	16	4	22	6	11	20	7	24	19	9	17	3	10	21	1	5	2	26	25
13	14	16	19	15	9	24	17	5	23	7	12	21	8	25	20	10	18	4	11	22	2	6	3	1	26
14	15	17	20	16	10	25	18	6	24	8	13	22	9	26	21	11	19	5	12	23	3	7	4	2	1
15	16	18	21	17	11	26	19	7	25	9	14	23	10	1	22	12	20	6	13	24	4	8	5	3	2
16	17	19	22	18	12	1	20	8	26	10	15	24	11	2	23	13	21	7	14	25	5	9	6	4	3
17	18	20	23	19	13	2	21	9	1	11	16	25	12	3	24	14	22	8	15	26	6	10	7	5	4
18	19	21	24	20	14	3	22	10	2	12	17	26	13	4	25	15	23	9	16	1	7	11	8	6	5
19	20	22	25	21	15	4	23	11	3	13	18	1	14	5	26	16	24	10	17	2	8	12	9	7	6
20	21	23	26	22	16	5	24	12	4	14	19	2	15	6	1	17	25	11	18	3	9	13	10	8	7
21	22	24	1	23	17	6	25	13	5	15	20	3	16	7	2	18	26	12	19	4	10	14	11	9	8
22	23	25	2	24	18	7	26	14	6	16	21	4	17	8	3	19	1	13	20	5	11	15	12	10	9
23	24	26	3	25	19	8	1	15	7	17	22	5	18	9	4	20	2	14	21	6	12	16	13	11	10
24	25	1	4	26	20	9	2	16	8	18	23	6	19	10	5	21	3	15	22	7	13	17	14	12	11
25	26	2	5	1	21	10	3	17	9	19	24	7	20	11	6	22	4	16	23	8	14	18	15	13	12
26	1	3	6	2	22	11	4	18	10	20	25	8	21	12	7	23	5	17	24	9	15	19	16	14	13

Figure 32A

Tuscan-2, $N = 32$, symmetric polygonal path

necessarily circular Tuscan-2

1	3	2	5	10	27	31	8	28	4	22	16	29	7	14	25	9	30	23	13	32	6	20	12	24	15	11	26	21	18	19	17
2	4	3	6	11	28	32	9	29	5	23	17	30	8	15	26	10	31	24	14	1	7	21	13	25	16	12	27	22	19	20	18
3	5	4	7	12	29	1	10	30	6	24	18	31	9	16	27	11	32	25	15	2	8	22	14	26	17	13	28	23	20	21	19
4	6	5	8	13	30	2	11	31	7	25	19	32	10	17	28	12	1	26	16	3	9	23	15	27	18	14	29	24	21	22	20
5	7	6	9	14	31	3	12	32	8	26	20	1	11	18	29	13	2	27	17	4	10	24	16	28	19	15	30	25	22	23	21
6	8	7	10	15	32	4	13	1	9	27	21	2	12	19	30	14	3	28	18	5	11	25	17	29	20	16	31	26	23	24	22
7	9	8	11	16	1	5	14	2	10	28	22	3	13	20	31	15	4	29	19	6	12	26	18	30	21	17	32	27	24	25	23
8	10	9	12	17	2	6	15	3	11	29	23	4	14	21	32	16	5	30	20	7	13	27	19	31	22	18	1	28	25	26	24
9	11	10	13	18	3	7	16	4	12	30	24	5	15	22	1	17	6	31	21	8	14	28	20	32	23	19	2	29	26	27	25
10	12	11	14	19	4	8	17	5	13	31	25	6	16	23	2	18	7	32	22	9	15	29	21	1	24	20	3	30	27	28	26
11	13	12	15	20	5	9	18	6	14	32	26	7	17	24	3	19	8	1	23	10	16	30	22	2	25	21	4	31	28	29	27
12	14	13	16	21	6	10	19	7	15	1	27	8	18	25	4	20	9	2	24	11	17	31	23	3	26	22	5	32	29	30	28
13	15	14	17	22	7	11	20	8	16	2	28	9	19	26	5	21	10	3	25	12	18	32	24	4	27	23	6	1	30	31	29
14	16	15	18	23	8	12	21	9	17	3	29	10	20	27	6	22	11	4	26	13	19	1	25	5	28	24	7	2	31	32	30
15	17	16	19	24	9	13	22	10	18	4	30	11	21	28	7	23	12	5	27	14	20	2	26	6	29	25	8	3	32	1	31
16	18	17	20	25	10	14	23	11	19	5	31	12	22	29	8	24	13	6	28	15	21	3	27	7	30	26	9	4	1	2	32
17	19	18	21	26	11	15	24	12	20	6	32	13	23	30	9	25	14	7	29	16	22	4	28	8	31	27	10	5	2	3	1
18	20	19	22	27	12	16	25	13	21	7	1	14	24	31	10	26	15	8	30	17	23	5	29	9	32	28	11	6	3	4	2
19	21	20	23	28	13	17	26	14	22	8	2	15	25	32	11	27	16	9	31	18	24	6	30	10	1	29	12	7	4	5	3
20	22	21	24	29	14	18	27	15	23	9	3	16	26	1	12	28	17	10	32	19	25	7	31	11	2	30	13	8	5	6	4
21	23	22	25	30	15	19	28	16	24	10	4	17	27	2	13	29	18	11	1	20	26	8	32	12	3	31	14	9	6	7	5
22	24	23	26	31	16	20	29	17	25	11	5	18	28	3	14	30	19	12	2	21	27	9	1	13	4	32	15	10	7	8	6
23	25	24	27	32	17	21	30	18	26	12	6	19	29	4	15	31	20	13	3	22	28	10	2	14	5	1	16	11	8	9	7
24	26	25	28	1	18	22	31	19	27	13	7	20	30	5	16	32	21	14	4	23	29	11	3	15	6	2	17	12	9	10	8
25	27	26	29	2	19	23	32	20	28	14	8	21	31	6	17	1	22	15	5	24	30	12	4	16	7	3	18	13	10	11	9
26	28	27	30	3	20	24	1	21	29	15	9	22	32	7	18	2	23	16	6	25	31	13	5	17	8	4	19	14	11	12	10
27	29	28	31	4	21	25	2	22	30	16	10	23	1	8	19	3	24	17	7	26	32	14	6	18	9	5	20	15	12	13	11
28	30	29	32	5	22	26	3	23	31	17	11	24	2	9	20	4	25	18	8	27	1	15	7	19	10	6	21	16	13	14	12
29	31	30	1	6	23	27	4	24	32	18	12	25	3	10	21	5	26	19	9	28	2	16	8	20	11	7	22	17	14	15	13
30	32	31	2	7	24	28	5	25	1	19	13	26	4	11	22	6	27	20	10	29	3	17	9	21	12	8	23	18	15	16	14
31	1	32	3	8	25	29	6	26	2	20	14	27	5	12	23	7	28	21	11	30	4	18	10	22	13	9	24	19	16	17	15
32	2	1	4	9	26	30	7	27	3	21	15	28	6	13	24	8	29	22	12	31	5	19	11	23	14	10	25	20	17	18	16

Figure 34A
Tuscan-2, $N = 34$, symmetric polygonal path necessarily circular Tuscan-2

1	2	4	7	3	13	22	33	26	6	14	32	27	8	29	17	11	28	34	12	25	10	15	31	23	9	16	5	30	20	24	21	19	18
2	3	5	8	4	14	23	34	27	7	15	33	28	9	30	18	12	29	1	13	26	11	16	32	24	10	17	6	31	21	25	22	20	19
3	4	6	9	5	15	24	1	28	8	16	34	29	10	31	19	13	30	2	14	27	12	17	33	25	11	18	7	32	22	26	23	21	20
4	5	7	10	6	16	25	2	29	9	17	1	30	11	32	20	14	31	3	15	28	13	18	34	26	12	19	8	33	23	27	24	22	21
5	6	8	11	7	17	26	3	30	10	18	2	31	12	33	21	15	32	4	16	29	14	19	1	27	13	20	9	34	24	28	25	23	22
6	7	9	12	8	18	27	4	31	11	19	3	32	13	34	22	16	33	5	17	30	15	20	2	28	14	21	10	1	25	29	26	24	23
7	8	10	13	9	19	28	5	32	12	20	4	33	14	1	23	17	34	6	18	31	16	21	3	29	15	22	11	2	26	30	27	25	24
8	9	11	14	10	20	29	6	33	13	21	5	34	15	2	24	18	1	7	19	32	17	22	4	30	16	23	12	3	27	31	28	26	25
9	10	12	15	11	21	30	7	34	14	22	6	1	16	3	25	19	2	8	20	33	18	23	5	31	17	24	13	4	28	32	29	27	26
10	11	13	16	12	22	31	8	1	15	23	7	2	17	4	26	20	3	9	21	34	19	24	6	32	18	25	14	5	29	33	30	28	27
11	12	14	17	13	23	32	9	2	16	24	8	3	18	5	27	21	4	10	22	1	20	25	7	33	19	26	15	6	30	34	31	29	28
12	13	15	18	14	24	33	10	3	17	25	9	4	19	6	28	22	5	11	23	2	21	26	8	34	20	27	16	7	31	1	32	30	29
13	14	16	19	15	25	34	11	4	18	26	10	5	20	7	29	23	6	12	24	3	22	27	9	1	21	28	17	8	32	2	33	31	30
14	15	17	20	16	26	1	12	5	19	27	11	6	21	8	30	24	7	13	25	4	23	28	10	2	22	29	18	9	33	3	34	32	31
15	16	18	21	17	27	2	13	6	20	28	12	7	22	9	31	25	8	14	26	5	24	29	11	3	23	30	19	10	34	4	1	33	32
16	17	19	22	18	28	3	14	7	21	29	13	8	23	10	32	26	9	15	27	6	25	30	12	4	24	31	20	11	1	5	2	34	33
17	18	20	23	19	29	4	15	8	22	30	14	9	24	11	33	27	10	16	28	7	26	31	13	5	25	32	21	12	2	6	3	1	34
18	19	21	24	20	30	5	16	9	23	31	15	10	25	12	34	28	11	17	29	8	27	32	14	6	26	33	22	13	3	7	4	2	1
19	20	22	25	21	31	6	17	10	24	32	16	11	26	13	1	29	12	18	30	9	28	33	15	7	27	34	23	14	4	8	5	3	2
20	21	23	26	22	32	7	18	11	25	33	17	12	27	14	2	30	13	19	31	10	29	34	16	8	28	1	24	15	5	9	6	4	3
21	22	24	27	23	33	8	19	12	26	34	18	13	28	15	3	31	14	20	32	11	30	1	17	9	29	2	25	16	6	10	7	5	4
22	23	25	28	24	34	9	20	13	27	1	19	14	29	16	4	32	15	21	33	12	31	2	18	10	30	3	26	17	7	11	8	6	5
23	24	26	29	25	1	10	21	14	28	2	20	15	30	17	5	33	16	22	34	13	32	3	19	11	31	4	27	18	8	12	9	7	6
24	25	27	30	26	2	11	22	15	29	3	21	16	31	18	6	34	17	23	1	14	33	4	20	12	32	5	28	19	9	13	10	8	7
25	26	28	31	27	3	12	23	16	30	4	22	17	32	19	7	1	18	24	2	15	34	5	21	13	33	6	29	20	10	14	11	9	8
26	27	29	32	28	4	13	24	17	31	5	23	18	33	20	8	2	19	25	3	16	1	6	22	14	34	7	30	21	11	15	12	10	9
27	28	30	33	29	5	14	25	18	32	6	24	19	34	21	9	3	20	26	4	17	2	7	23	15	1	8	31	22	12	16	13	11	10
28	29	31	34	30	6	15	26	19	33	7	25	20	1	22	10	4	21	27	5	18	3	8	24	16	2	9	32	23	13	17	14	12	11
29	30	32	1	31	7	16	27	20	34	8	26	21	2	23	11	5	22	28	6	19	4	9	25	17	3	10	33	24	14	18	15	13	12
30	31	33	2	32	8	17	28	21	1	9	27	22	3	24	12	6	23	29	7	20	5	10	26	18	4	11	34	25	15	19	16	14	13
31	32	34	3	33	9	18	29	22	2	10	28	23	4	25	13	7	24	30	8	21	6	11	27	19	5	12	1	26	16	20	17	15	14
32	33	1	4	34	10	19	30	23	3	11	29	24	5	26	14	8	25	31	9	22	7	12	28	20	6	13	2	27	17	21	18	16	15
33	34	2	5	1	11	20	31	24	4	12	30	25	6	27	15	9	26	32	10	23	8	13	29	21	7	14	3	28	18	22	19	17	16
34	1	3	6	2	12	21	32	25	5	13	31	26	7	28	16	10	27	33	11	24	9	14	30	22	8	15	4	29	19	23	20	18	17

Figure 38A
Tuscan-2, $N = 38$, symmetric polygonal path
necessarily circular Tuscan-2

1	3	2	5	9	14	23	35	19	27	37	12	26	15	32	25	10	30	36	17	11	29	6	13	34	7	31	18	8	38	16	4	33	28	24	21	22	20
2	4	3	6	10	15	24	36	20	28	38	13	27	16	33	26	11	31	37	18	12	30	7	14	35	8	32	19	9	1	17	5	34	29	25	22	23	21
3	5	4	7	11	16	25	37	21	29	1	14	28	17	34	27	12	32	38	19	13	31	8	15	36	9	33	20	10	2	18	6	35	30	26	23	24	22
4	6	5	8	12	17	26	38	22	30	2	15	29	18	35	28	13	33	1	20	14	32	9	16	37	10	34	21	11	3	19	7	36	31	27	24	25	23
5	7	6	9	13	18	27	1	23	31	3	16	30	19	36	29	14	34	2	21	15	33	10	17	38	11	35	22	12	4	20	8	37	32	28	25	26	24
6	8	7	10	14	19	28	2	24	32	4	17	31	20	37	30	15	35	3	22	16	34	11	18	1	12	36	23	13	5	21	9	38	33	29	26	27	25
7	9	8	11	15	20	29	3	25	33	5	18	32	21	38	31	16	36	4	23	17	35	12	19	2	13	37	24	14	6	22	10	1	34	30	27	28	26
8	10	9	12	16	21	30	4	26	34	6	19	33	22	1	32	17	37	5	24	18	36	13	20	3	14	38	25	15	7	23	11	2	35	31	28	29	27
9	11	10	13	17	22	31	5	27	35	7	20	34	23	2	33	18	38	6	25	19	37	14	21	4	15	1	26	16	8	24	12	3	36	32	29	30	28
10	12	11	14	18	23	32	6	28	36	8	21	35	24	3	34	19	1	7	26	20	38	15	22	5	16	2	27	17	9	25	13	4	37	33	30	31	29
11	13	12	15	19	24	33	7	29	37	9	22	36	25	4	35	20	2	8	27	21	1	16	23	6	17	3	28	18	10	26	14	5	38	34	31	32	30
12	14	13	16	20	25	34	8	30	38	10	23	37	26	5	36	21	3	9	28	22	2	17	24	7	18	4	29	19	11	27	15	6	1	35	32	33	31
13	15	14	17	21	26	35	9	31	1	11	24	38	27	6	37	22	4	10	29	23	3	18	25	8	19	5	30	20	12	28	16	7	2	36	33	34	32
14	16	15	18	22	27	36	10	32	2	12	25	1	28	7	38	23	5	11	30	24	4	19	26	9	20	6	31	21	13	29	17	8	3	37	34	35	33
15	17	16	19	23	28	37	11	33	3	13	26	2	29	8	1	24	6	12	31	25	5	20	27	10	21	7	32	22	14	30	18	9	4	38	35	36	34
16	18	17	20	24	29	38	12	34	4	14	27	3	30	9	2	25	7	13	32	26	6	21	28	11	22	8	33	23	15	31	19	10	5	1	36	37	35
17	19	18	21	25	30	1	13	35	5	15	28	4	31	10	3	26	8	14	33	27	7	22	29	12	23	9	34	24	16	32	20	11	6	2	37	38	36
18	20	19	22	26	31	2	14	36	6	16	29	5	32	11	4	27	9	15	34	28	8	23	30	13	24	10	35	25	17	33	21	12	7	3	38	1	37
19	21	20	23	27	32	3	15	37	7	17	30	6	33	12	5	28	10	16	35	29	9	24	31	14	25	11	36	26	18	34	22	13	8	4	1	2	38
20	22	21	24	28	33	4	16	38	8	18	31	7	34	13	6	29	11	17	36	30	10	25	32	15	26	12	37	27	19	35	23	14	9	5	2	3	1
21	23	22	25	29	34	5	17	1	9	19	32	8	35	14	7	30	12	18	37	31	11	26	33	16	27	13	38	28	20	36	24	15	10	6	3	4	2
22	24	23	26	30	35	6	18	2	10	20	33	9	36	15	8	31	13	19	38	32	12	27	34	17	28	14	1	29	21	37	25	16	11	7	4	5	3
23	25	24	27	31	36	7	19	3	11	21	34	10	37	16	9	32	14	20	1	33	13	28	35	18	29	15	2	30	22	38	26	17	12	8	5	6	4
24	26	25	28	32	37	8	20	4	12	22	35	11	38	17	10	33	15	21	2	34	14	29	36	19	30	16	3	31	23	1	27	18	13	9	6	7	5
25	27	26	29	33	38	9	21	5	13	23	36	12	1	18	11	34	16	22	3	35	15	30	37	20	31	17	4	32	24	2	28	19	14	10	7	8	6
26	28	27	30	34	1	10	22	6	14	24	37	13	2	19	12	35	17	23	4	36	16	31	38	21	32	18	5	33	25	3	29	20	15	11	8	9	7
27	29	28	31	35	2	11	23	7	15	25	38	14	3	20	13	36	18	24	5	37	17	32	1	22	33	19	6	34	26	4	30	21	16	12	9	10	8
28	30	29	32	36	3	12	24	8	16	26	1	15	4	21	14	37	19	25	6	38	18	33	2	23	34	20	7	35	27	5	31	22	17	13	10	11	9
29	31	30	33	37	4	13	25	9	17	27	2	16	5	22	15	38	20	26	7	1	19	34	3	24	35	21	8	36	28	6	32	23	18	14	11	12	10
30	32	31	34	38	5	14	26	10	18	28	3	17	6	23	16	1	21	27	8	2	20	35	4	25	36	22	9	37	29	7	33	24	19	15	12	13	11
31	33	32	35	1	6	15	27	11	19	29	4	18	7	24	17	2	22	28	9	3	21	36	5	26	37	23	10	38	30	8	34	25	20	16	13	14	12
32	34	33	36	2	7	16	28	12	20	30	5	19	8	25	18	3	23	29	10	4	22	37	6	27	38	24	11	1	31	9	35	26	21	17	14	15	13
33	35	34	37	3	8	17	29	13	21	31	6	20	9	26	19	4	24	30	11	5	23	38	7	28	1	25	12	2	32	10	36	27	22	18	15	16	14
34	36	35	38	4	9	18	30	14	22	32	7	21	10	27	20	5	25	31	12	6	24	1	8	29	2	26	13	3	33	11	37	28	23	19	16	17	15
35	37	36	1	5	10	19	31	15	23	33	8	22	11	28	21	6	26	32	13	7	25	2	9	30	3	27	14	4	34	12	38	29	24	20	17	18	16
36	38	37	2	6	11	20	32	16	24	34	9	23	12	29	22	7	27	33	14	8	26	3	10	31	4	28	15	5	35	13	1	30	25	21	18	19	17
37	1	38	3	7	12	21	33	17	25	35	10	24	13	30	23	8	28	34	15	9	27	4	11	32	5	29	16	6	36	14	2	31	26	22	19	20	18
38	2	1	4	8	13	22	34	18	26	36	11	25	14	31	24	9	29	35	16	10	28	5	12	33	6	30	17	7	37	15	3	32	27	23	20	21	19

Figure 44A
Tuscan-2, $N = 44$, symmetric polygonal path
necessarily circular Tuscan-2

1	2	4	7	3	9	14	28	37	19	8	39	20	13	34	5	21	11	38	18	10	22	44	32	40	16	33	43	27	12	35	42	17	30	41	15	6	36	31	25	29	26	24	23
2	3	5	8	4	10	15	29	38	20	9	40	21	14	35	6	22	12	39	19	11	23	1	33	41	17	34	44	28	13	36	43	18	31	42	16	7	37	32	26	30	27	25	24
3	4	6	9	5	11	16	30	39	21	10	41	22	15	36	7	23	13	40	20	12	24	2	34	42	18	35	1	29	14	37	44	19	32	43	17	8	38	33	27	31	28	26	25
4	5	7	10	6	12	17	31	40	22	11	42	23	16	37	8	24	14	41	21	13	25	3	35	43	19	36	2	30	15	38	1	20	33	44	18	9	39	34	28	32	29	27	26
5	6	8	11	7	13	18	32	41	23	12	43	24	17	38	9	25	15	42	22	14	26	4	36	44	20	37	3	31	16	39	2	21	34	1	19	10	40	35	29	33	30	28	27
6	7	9	12	8	14	19	33	42	24	13	44	25	18	39	10	26	16	43	23	15	27	5	37	1	21	38	4	32	17	40	3	22	35	2	20	11	41	36	30	34	31	29	28
7	8	10	13	9	15	20	34	43	25	14	1	26	19	40	11	27	17	44	24	16	28	6	38	2	22	39	5	33	18	41	4	23	36	3	21	12	42	37	31	35	32	30	29
8	9	11	14	10	16	21	35	44	26	15	2	27	20	41	12	28	18	1	25	17	29	7	39	3	23	40	6	34	19	42	5	24	37	4	22	13	43	38	32	36	33	31	30
9	10	12	15	11	17	22	36	1	27	16	3	28	21	42	13	29	19	2	26	18	30	8	40	4	24	41	7	35	20	43	6	25	38	5	23	14	44	39	33	37	34	32	31
10	11	13	16	12	18	23	37	2	28	17	4	29	22	43	14	30	20	3	27	19	31	9	41	5	25	42	8	36	21	44	7	26	39	6	24	15	1	40	34	38	35	33	32
11	12	14	17	13	19	24	38	3	29	18	5	30	23	44	15	31	21	4	28	20	32	10	42	6	26	43	9	37	22	1	8	27	40	7	25	16	2	41	35	39	36	34	33
12	13	15	18	14	20	25	39	4	30	19	6	31	24	1	16	32	22	5	29	21	33	11	43	7	27	44	10	38	23	2	9	28	41	8	26	17	3	42	36	40	37	35	34
13	14	16	19	15	21	26	40	5	31	20	7	32	25	2	17	33	23	6	30	22	34	12	44	8	28	1	11	39	24	3	10	29	42	9	27	18	4	43	37	41	38	36	35
14	15	17	20	16	22	27	41	6	32	21	8	33	26	3	18	34	24	7	31	23	35	13	1	9	29	2	12	40	25	4	11	30	43	10	28	19	5	44	38	42	39	37	36
15	16	18	21	17	23	28	42	7	33	22	9	34	27	4	19	35	25	8	32	24	36	14	2	10	30	3	13	41	26	5	12	31	44	11	29	20	6	1	39	43	40	38	37
16	17	19	22	18	24	29	43	8	34	23	10	35	28	5	20	36	26	9	33	25	37	15	3	11	31	4	14	42	27	6	13	32	1	12	30	21	7	2	40	44	41	39	38
17	18	20	23	19	25	30	44	9	35	24	11	36	29	6	21	37	27	10	34	26	38	16	4	12	32	5	15	43	28	7	14	33	2	13	31	22	8	3	41	1	42	40	39
18	19	21	24	20	26	31	1	10	36	25	12	37	30	7	22	38	28	11	35	27	39	17	5	13	33	6	16	44	29	8	15	34	3	14	32	23	9	4	42	2	43	41	40
19	20	22	25	21	27	32	2	11	37	26	13	38	31	8	23	39	29	12	36	28	40	18	6	14	34	7	17	1	30	9	16	35	4	15	33	24	10	5	43	3	44	42	41
20	21	23	26	22	28	33	3	12	38	27	14	39	32	9	24	40	30	13	37	29	41	19	7	15	35	8	18	2	31	10	17	36	5	16	34	25	11	6	44	4	1	43	42
21	22	24	27	23	29	34	4	13	39	28	15	40	33	10	25	41	31	14	38	30	42	20	8	16	36	9	19	3	32	11	18	37	6	17	35	26	12	7	1	5	2	44	43
22	23	25	28	24	30	35	5	14	40	29	16	41	34	11	26	42	32	15	39	31	43	21	9	17	37	10	20	4	33	12	19	38	7	18	36	27	13	8	2	6	3	1	44
23	24	26	29	25	31	36	6	15	41	30	17	42	35	12	27	43	33	16	40	32	44	22	10	18	38	11	21	5	34	13	20	39	8	19	37	28	14	9	3	7	4	2	1
24	25	27	30	26	32	37	7	16	42	31	18	43	36	13	28	44	34	17	41	33	1	23	11	19	39	12	22	6	35	14	21	40	9	20	38	29	15	10	4	8	5	3	2
25	26	28	31	27	33	38	8	17	43	32	19	44	37	14	29	1	35	18	42	34	2	24	12	20	40	13	23	7	36	15	22	41	10	21	39	30	16	11	5	9	6	4	3
26	27	29	32	28	34	39	9	18	44	33	20	1	38	15	30	2	36	19	43	35	3	25	13	21	41	14	24	8	37	16	23	42	11	22	40	31	17	12	6	10	7	5	4
27	28	30	33	29	35	40	10	19	1	34	21	2	39	16	31	3	37	20	44	36	4	26	14	22	42	15	25	9	38	17	24	43	12	23	41	32	18	13	7	11	8	6	5
28	29	31	34	30	36	41	11	20	2	35	22	3	40	17	32	4	38	21	1	37	5	27	15	23	43	16	26	10	39	18	25	44	13	24	42	33	19	14	8	12	9	7	6
29	30	32	35	31	37	42	12	21	3	36	23	4	41	18	33	5	39	22	2	38	6	28	16	24	44	17	27	11	40	19	26	1	14	25	43	34	20	15	9	13	10	8	7
30	31	33	36	32	38	43	13	22	4	37	24	5	42	19	34	6	40	23	3	39	7	29	17	25	1	18	28	12	41	20	27	2	15	26	44	35	21	16	10	14	11	9	8
31	32	34	37	33	39	44	14	23	5	38	25	6	43	20	35	7	41	24	4	40	8	30	18	26	2	19	29	13	42	21	28	3	16	27	1	36	22	17	11	15	12	10	9
32	33	35	38	34	40	1	15	24	6	39	26	7	44	21	36	8	42	25	5	41	9	31	19	27	3	20	30	14	43	22	29	4	17	28	2	37	23	18	12	16	13	11	10
33	34	36	39	35	41	2	16	25	7	40	27	8	1	22	37	9	43	26	6	42	10	32	20	28	4	21	31	15	44	23	30	5	18	29	3	38	24	19	13	17	14	12	11
34	35	37	40	36	42	3	17	26	8	41	28	9	2	23	38	10	44	27	7	43	11	33	21	29	5	22	32	16	1	24	31	6	19	30	4	39	25	20	14	18	15	13	12
35	36	38	41	37	43	4	18	27	9	42	29	10	3	24	39	11	1	28	8	44	12	34	22	30	6	23	33	17	2	25	32	7	20	31	5	40	26	21	15	19	16	14	13
36	37	39	42	38	44	5	19	28	10	43	30	11	4	25	40	12	2	29	9	1	13	35	23	31	7	24	34	18	3	26	33	8	21	32	6	41	27	22	16	20	17	15	14
37	38	40	43	39	1	6	20	29	11	44	31	12	5	26	41	13	3	30	10	2	14	36	24	32	8	25	35	19	4	27	34	9	22	33	7	42	28	23	17	21	18	16	15
38	39	41	44	40	2	7	21	30	12	1	32	13	6	27	42	14	4	31	11	3	15	37	25	33	9	26	36	20	5	28	35	10	23	34	8	43	29	24	18	22	19	17	16
39	40	42	1	41	3	8	22	31	13	2	33	14	7	28	43	15	5	32	12	4	16	38	26	34	10	27	37	21	6	29	36	11	24	35	9	44	30	25	19	23	20	18	17
40	41	43	2	42	4	9	23	32	14	3	34	15	8	29	44	16	6	33	13	5	17	39	27	35	11	28	38	22	7	30	37	12	25	36	10	1	31	26	20	24	21	19	18
41	42	44	3	43	5	10	24	33	15	4	35	16	9	30	1	17	7	34	14	6	18	40	28	36	12	29	39	23	8	31	38	13	26	37	11	2	32	27	21	25	22	20	19
42	43	1	4	44	6	11	25	34	16	5	36	17	10	31	2	18	8	35	15	7	19	41	29	37	13	30	40	24	9	32	39	14	27	38	12	3	33	28	22	26	23	21	20
43	44	2	5	1	7	12	26	35	17	6	37	18	11	32	3	19	9	36	16	8	20	42	30	38	14	31	41	25	10	33	40	15	28	39	13	4	34	29	23	27	24	22	21
44	1	3	6	2	8	13	27	36	18	7	38	19	12	33	4	20	10	37	17	9	21	43	31	39	15	32	42	26	11	34	41	16	29	40	14	5	35	30	24	28	25	23	22

Figure 48A

Tuscan-2, $N = 48$, symmetric polygonal path necessarily circular Tuscan-2

1	3	2	5	9	14	4	10	22	37	6	15	44	18	36	43	16	32	45	17	31	23	48	11	35	24	47	7	41	21	8	40	19	12	42	20	39	30	13	46	34	28	38	33	29	26	27	25
2	4	3	6	10	15	5	11	23	38	7	16	45	19	37	44	17	33	46	18	32	24	1	12	36	25	48	8	42	22	9	41	20	13	43	21	40	31	14	47	35	29	39	34	30	27	28	26
3	5	4	7	11	16	6	12	24	39	8	17	46	20	38	45	18	34	47	19	33	25	2	13	37	26	1	9	43	23	10	42	21	14	44	22	41	32	15	48	36	30	40	35	31	28	29	27
4	6	5	8	12	17	7	13	25	40	9	18	47	21	39	46	19	35	48	20	34	26	3	14	38	27	2	10	44	24	11	43	22	15	45	23	42	33	16	1	37	31	41	36	32	29	30	28
5	7	6	9	13	18	8	14	26	41	10	19	48	22	40	47	20	36	1	21	35	27	4	15	39	28	3	11	45	25	12	44	23	16	46	24	43	34	17	2	38	32	42	37	33	30	31	29
6	8	7	10	14	19	9	15	27	42	11	20	1	23	41	48	21	37	2	22	36	28	5	16	40	29	4	12	46	26	13	45	24	17	47	25	44	35	18	3	39	33	43	38	34	31	32	30
7	9	8	11	15	20	10	16	28	43	12	21	2	24	42	1	22	38	3	23	37	29	6	17	41	30	5	13	47	27	14	46	25	18	48	26	45	36	19	4	40	34	44	39	35	32	33	31
8	10	9	12	16	21	11	17	29	44	13	22	3	25	43	2	23	39	4	24	38	30	7	18	42	31	6	14	48	28	15	47	26	19	1	27	46	37	20	5	41	35	45	40	36	33	34	32
9	11	10	13	17	22	12	18	30	45	14	23	4	26	44	3	24	40	5	25	39	31	8	19	43	32	7	15	1	29	16	48	27	20	2	28	47	38	21	6	42	36	46	41	37	34	35	33
10	12	11	14	18	23	13	19	31	46	15	24	5	27	45	4	25	41	6	26	40	32	9	20	44	33	8	16	2	30	17	1	28	21	3	29	48	39	22	7	43	37	47	42	38	35	36	34
11	13	12	15	19	24	14	20	32	47	16	25	6	28	46	5	26	42	7	27	41	33	10	21	45	34	9	17	3	31	18	2	29	22	4	30	1	40	23	8	44	38	48	43	39	36	37	35
12	14	13	16	20	25	15	21	33	48	17	26	7	29	47	6	27	43	8	28	42	34	11	22	46	35	10	18	4	32	19	3	30	23	5	31	2	41	24	9	45	39	1	44	40	37	38	36
13	15	14	17	21	26	16	22	34	1	18	27	8	30	48	7	28	44	9	29	43	35	12	23	47	36	11	19	5	33	20	4	31	24	6	32	3	42	25	10	46	40	2	45	41	38	39	37
14	16	15	18	22	27	17	23	35	2	19	28	9	31	1	8	29	45	10	30	44	36	13	24	48	37	12	20	6	34	21	5	32	25	7	33	4	43	26	11	47	41	3	46	42	39	40	38
15	17	16	19	23	28	18	24	36	3	20	29	10	32	2	9	30	46	11	31	45	37	14	25	1	38	13	21	7	35	22	6	33	26	8	34	5	44	27	12	48	42	4	47	43	40	41	39
16	18	17	20	24	29	19	25	37	4	21	30	11	33	3	10	31	47	12	32	46	38	15	26	2	39	14	22	8	36	23	7	34	27	9	35	6	45	28	13	1	43	5	48	44	41	42	40
17	19	18	21	25	30	20	26	38	5	22	31	12	34	4	11	32	48	13	33	47	39	16	27	3	40	15	23	9	37	24	8	35	28	10	36	7	46	29	14	2	44	6	1	45	42	43	41
18	20	19	22	26	31	21	27	39	6	23	32	13	35	5	12	33	1	14	34	48	40	17	28	4	41	16	24	10	38	25	9	36	29	11	37	8	47	30	15	3	45	7	2	46	43	44	42
19	21	20	23	27	32	22	28	40	7	24	33	14	36	6	13	34	2	15	35	1	41	18	29	5	42	17	25	11	39	26	10	37	30	12	38	9	48	31	16	4	46	8	3	47	44	45	43
20	22	21	24	28	33	23	29	41	8	25	34	15	37	7	14	35	3	16	36	2	42	19	30	6	43	18	26	12	40	27	11	38	31	13	39	10	1	32	17	5	47	9	4	48	45	46	44
21	23	22	25	29	34	24	30	42	9	26	35	16	38	8	15	36	4	17	37	3	43	20	31	7	44	19	27	13	41	28	12	39	32	14	40	11	2	33	18	6	48	10	5	1	46	47	45
22	24	23	26	30	35	25	31	43	10	27	36	17	39	9	16	37	5	18	38	4	44	21	32	8	45	20	28	14	42	29	13	40	33	15	41	12	3	34	19	7	1	11	6	2	47	48	46
23	25	24	27	31	36	26	32	44	11	28	37	18	40	10	17	38	6	19	39	5	45	22	33	9	46	21	29	15	43	30	14	41	34	16	42	13	4	35	20	8	2	12	7	3	48	1	47
24	26	25	28	32	37	27	33	45	12	29	38	19	41	11	18	39	7	20	40	6	46	23	34	10	47	22	30	16	44	31	15	42	35	17	43	14	5	36	21	9	3	13	8	4	1	2	48
25	27	26	29	33	38	28	34	46	13	30	39	20	42	12	19	40	8	21	41	7	47	24	35	11	48	23	31	17	45	32	16	43	36	18	44	15	6	37	22	10	4	14	9	5	2	3	1
26	28	27	30	34	39	29	35	47	14	31	40	21	43	13	20	41	9	22	42	8	48	25	36	12	1	24	32	18	46	33	17	44	37	19	45	16	7	38	23	11	5	15	10	6	3	4	2
27	29	28	31	35	40	30	36	48	15	32	41	22	44	14	21	42	10	23	43	9	1	26	37	13	2	25	33	19	47	34	18	45	38	20	46	17	8	39	24	12	6	16	11	7	4	5	3
28	30	29	32	36	41	31	37	1	16	33	42	23	45	15	22	43	11	24	44	10	2	27	38	14	3	26	34	20	48	35	19	46	39	21	47	18	9	40	25	13	7	17	12	8	5	6	4
29	31	30	33	37	42	32	38	2	17	34	43	24	46	16	23	44	12	25	45	11	3	28	39	15	4	27	35	21	1	36	20	47	40	22	48	19	10	41	26	14	8	18	13	9	6	7	5
30	32	31	34	38	43	33	39	3	18	35	44	25	47	17	24	45	13	26	46	12	4	29	40	16	5	28	36	22	2	37	21	48	41	23	1	20	11	42	27	15	9	19	14	10	7	8	6
31	33	32	35	39	44	34	40	4	19	36	45	26	48	18	25	46	14	27	47	13	5	30	41	17	6	29	37	23	3	38	22	1	42	24	2	21	12	43	28	16	10	20	15	11	8	9	7
32	34	33	36	40	45	35	41	5	20	37	46	27	1	19	26	47	15	28	48	14	6	31	42	18	7	30	38	24	4	39	23	2	43	25	3	22	13	44	29	17	11	21	16	12	9	10	8
33	35	34	37	41	46	36	42	6	21	38	47	28	2	20	27	48	16	29	1	15	7	32	43	19	8	31	39	25	5	40	24	3	44	26	4	23	14	45	30	18	12	22	17	13	10	11	9
34	36	35	38	42	47	37	43	7	22	39	48	29	3	21	28	1	17	30	2	16	8	33	44	20	9	32	40	26	6	41	25	4	45	27	5	24	15	46	31	19	13	23	18	14	11	12	10
35	37	36	39	43	48	38	44	8	23	40	1	30	4	22	29	2	18	31	3	17	9	34	45	21	10	33	41	27	7	42	26	5	46	28	6	25	16	47	32	20	14	24	19	15	12	13	11
36	38	37	40	44	1	39	45	9	24	41	2	31	5	23	30	3	19	32	4	18	10	35	46	22	11	34	42	28	8	43	27	6	47	29	7	26	17	48	33	21	15	25	20	16	13	14	12
37	39	38	41	45	2	40	46	10	25	42	3	32	6	24	31	4	20	33	5	19	11	36	47	23	12	35	43	29	9	44	28	7	48	30	8	27	18	1	34	22	16	26	21	17	14	15	13
38	40	39	42	46	3	41	47	11	26	43	4	33	7	25	32	5	21	34	6	20	12	37	48	24	13	36	44	30	10	45	29	8	1	31	9	28	19	2	35	23	17	27	22	18	15	16	14
39	41	40	43	47	4	42	48	12	27	44	5	34	8	26	33	6	22	35	7	21	13	38	1	25	14	37	45	31	11	46	30	9	2	32	10	29	20	3	36	24	18	28	23	19	16	17	15
40	42	41	44	48	5	43	1	13	28	45	6	35	9	27	34	7	23	36	8	22	14	39	2	26	15	38	46	32	12	47	31	10	3	33	11	30	21	4	37	25	19	29	24	20	17	18	16
41	43	42	45	1	6	44	2	14	29	46	7	36	10	28	35	8	24	37	9	23	15	40	3	27	16	39	47	33	13	48	32	11	4	34	12	31	22	5	38	26	20	30	25	21	18	19	17
42	44	43	46	2	7	45	3	15	30	47	8	37	11	29	36	9	25	38	10	24	16	41	4	28	17	40	48	34	14	1	33	12	5	35	13	32	23	6	39	27	21	31	26	22	19	20	18
43	45	44	47	3	8	46	4	16	31	48	9	38	12	30	37	10	26	39	11	25	17	42	5	29	18	41	1	35	15	2	34	13	6	36	14	33	24	7	40	28	22	32	27	23	20	21	19
44	46	45	48	4	9	47	5	17	32	1	10	39	13	31	38	11	27	40	12	26	18	43	6	30	19	42	2	36	16	3	35	14	7	37	15	34	25	8	41	29	23	33	28	24	21	22	20
45	47	46	1	5	10	48	6	18	33	2	11	40	14	32	39	12	28	41	13	27	19	44	7	31	20	43	3	37	17	4	36	15	8	38	16	35	26	9	42	30	24	34	29	25	22	23	21
46	48	47	2	6	11	1	7	19	34	3	12	41	15	33	40	13	29	42	14	28	20	45	8	32	21	44	4	38	18	5	37	16	9	39	17	36	27	10	43	31	25	35	30	26	23	24	22
47	1	48	3	7	12	2	8	20	35	4	13	42	16	34	41	14	30	43	15	29	21	46	9	33	22	45	5	39	19	6	38	17	10	40	18	37	28	11	44	32	26	36	31	27	24	25	23
48	2	1	4	8	13	3	9	21	36	5	14	43	17	35	42	15	31	44	16	30	22	47	10	34	23	46	6	40	20	7	39	18	11	41	19	38	29	12	45	33	27	37	32	28	25	26	24

Figure 50A

Tuscan-2, $N = 50$, symmetric polygonal path

necessarily circular Tuscan-2

1	2	4	7	3	9	14	5	20	46	36	24	13	43	6	23	44	37	15	47	33	10	41	25	17	42	50	16	35	8	22	40	12	19	48	31	18	38	49	11	21	45	30	39	34	28	32	29	27	26
2	3	5	8	4	10	15	6	21	47	37	25	14	44	7	24	45	38	16	48	34	11	42	26	18	43	1	17	36	9	23	41	13	20	49	32	19	39	50	12	22	46	31	40	35	29	33	30	28	27
3	4	6	9	5	11	16	7	22	48	38	26	15	45	8	25	46	39	17	49	35	12	43	27	19	44	2	18	37	10	24	42	14	21	50	33	20	40	1	13	23	47	32	41	36	30	34	31	29	28
4	5	7	10	6	12	17	8	23	49	39	27	16	46	9	26	47	40	18	50	36	13	44	28	20	45	3	19	38	11	25	43	15	22	1	34	21	41	2	14	24	48	33	42	37	31	35	32	30	29
5	6	8	11	7	13	18	9	24	50	40	28	17	47	10	27	48	41	19	1	37	14	45	29	21	46	4	20	39	12	26	44	16	23	2	35	22	42	3	15	25	49	34	43	38	32	36	33	31	30
6	7	9	12	8	14	19	10	25	1	41	29	18	48	11	28	49	42	20	2	38	15	46	30	22	47	5	21	40	13	27	45	17	24	3	36	23	43	4	16	26	50	35	44	39	33	37	34	32	31
7	8	10	13	9	15	20	11	26	2	42	30	19	49	12	29	50	43	21	3	39	16	47	31	23	48	6	22	41	14	28	46	18	25	4	37	24	44	5	17	27	1	36	45	40	34	38	35	33	32
8	9	11	14	10	16	21	12	27	3	43	31	20	50	13	30	1	44	22	4	40	17	48	32	24	49	7	23	42	15	29	47	19	26	5	38	25	45	6	18	28	2	37	46	41	35	39	36	34	33
9	10	12	15	11	17	22	13	28	4	44	32	21	1	14	31	2	45	23	5	41	18	49	33	25	50	8	24	43	16	30	48	20	27	6	39	26	46	7	19	29	3	38	47	42	36	40	37	35	34
10	11	13	16	12	18	23	14	29	5	45	33	22	2	15	32	3	46	24	6	42	19	50	34	26	1	9	25	44	17	31	49	21	28	7	40	27	47	8	20	30	4	39	48	43	37	41	38	36	35
11	12	14	17	13	19	24	15	30	6	46	34	23	3	16	33	4	47	25	7	43	20	1	35	27	2	10	26	45	18	32	50	22	29	8	41	28	48	9	21	31	5	40	49	44	38	42	39	37	36
12	13	15	18	14	20	25	16	31	7	47	35	24	4	17	34	5	48	26	8	44	21	2	36	28	3	11	27	46	19	33	1	23	30	9	42	29	49	10	22	32	6	41	50	45	39	43	40	38	37
13	14	16	19	15	21	26	17	32	8	48	36	25	5	18	35	6	49	27	9	45	22	3	37	29	4	12	28	47	20	34	2	24	31	10	43	30	50	11	23	33	7	42	1	46	40	44	41	39	38
14	15	17	20	16	22	27	18	33	9	49	37	26	6	19	36	7	50	28	10	46	23	4	38	30	5	13	29	48	21	35	3	25	32	11	44	31	1	12	24	34	8	43	2	47	41	45	42	40	39
15	16	18	21	17	23	28	19	34	10	50	38	27	7	20	37	8	1	29	11	47	24	5	39	31	6	14	30	49	22	36	4	26	33	12	45	32	2	13	25	35	9	44	3	48	42	46	43	41	40
16	17	19	22	18	24	29	20	35	11	1	39	28	8	21	38	9	2	30	12	48	25	6	40	32	7	15	31	50	23	37	5	27	34	13	46	33	3	14	26	36	10	45	4	49	43	47	44	42	41
17	18	20	23	19	25	30	21	36	12	2	40	29	9	22	39	10	3	31	13	49	26	7	41	33	8	16	32	1	24	38	6	28	35	14	47	34	4	15	27	37	11	46	5	50	44	48	45	43	42
18	19	21	24	20	26	31	22	37	13	3	41	30	10	23	40	11	4	32	14	50	27	8	42	34	9	17	33	2	25	39	7	29	36	15	48	35	5	16	28	38	12	47	6	1	45	49	46	44	43
19	20	22	25	21	27	32	23	38	14	4	42	31	11	24	41	12	5	33	15	1	28	9	43	35	10	18	34	3	26	40	8	30	37	16	49	36	6	17	29	39	13	48	7	2	46	50	47	45	44
20	21	23	26	22	28	33	24	39	15	5	43	32	12	25	42	13	6	34	16	2	29	10	44	36	11	19	35	4	27	41	9	31	38	17	50	37	7	18	30	40	14	49	8	3	47	1	48	46	45
21	22	24	27	23	29	34	25	40	16	6	44	33	13	26	43	14	7	35	17	3	30	11	45	37	12	20	36	5	28	42	10	32	39	18	1	38	8	19	31	41	15	50	9	4	48	2	49	47	46
22	23	25	28	24	30	35	26	41	17	7	45	34	14	27	44	15	8	36	18	4	31	12	46	38	13	21	37	6	29	43	11	33	40	19	2	39	9	20	32	42	16	1	10	5	49	3	50	48	47
23	24	26	29	25	31	36	27	42	18	8	46	35	15	28	45	16	9	37	19	5	32	13	47	39	14	22	38	7	30	44	12	34	41	20	3	40	10	21	33	43	17	2	11	6	50	4	1	49	48
24	25	27	30	26	32	37	28	43	19	9	47	36	16	29	46	17	10	38	20	6	33	14	48	40	15	23	39	8	31	45	13	35	42	21	4	41	11	22	34	44	18	3	12	7	1	5	2	50	49
25	26	28	31	27	33	38	29	44	20	10	48	37	17	30	47	18	11	39	21	7	34	15	49	41	16	24	40	9	32	46	14	36	43	22	5	42	12	23	35	45	19	4	13	8	2	6	3	1	50
26	27	29	32	28	34	39	30	45	21	11	49	38	18	31	48	19	12	40	22	8	35	16	50	42	17	25	41	10	33	47	15	37	44	23	6	43	13	24	36	46	20	5	14	9	3	7	4	2	1
27	28	30	33	29	35	40	31	46	22	12	50	39	19	32	49	20	13	41	23	9	36	17	1	43	18	26	42	11	34	48	16	38	45	24	7	44	14	25	37	47	21	6	15	10	4	8	5	3	2
28	29	31	34	30	36	41	32	47	23	13	1	40	20	33	50	21	14	42	24	10	37	18	2	44	19	27	43	12	35	49	17	39	46	25	8	45	15	26	38	48	22	7	16	11	5	9	6	4	3
29	30	32	35	31	37	42	33	48	24	14	2	41	21	34	1	22	15	43	25	11	38	19	3	45	20	28	44	13	36	50	18	40	47	26	9	46	16	27	39	49	23	8	17	12	6	10	7	5	4
30	31	33	36	32	38	43	34	49	25	15	3	42	22	35	2	23	16	44	26	12	39	20	4	46	21	29	45	14	37	1	19	41	48	27	10	47	17	28	40	50	24	9	18	13	7	11	8	6	5
31	32	34	37	33	39	44	35	50	26	16	4	43	23	36	3	24	17	45	27	13	40	21	5	47	22	30	46	15	38	2	20	42	49	28	11	48	18	29	41	1	25	10	19	14	8	12	9	7	6
32	33	35	38	34	40	45	36	1	27	17	5	44	24	37	4	25	18	46	28	14	41	22	6	48	23	31	47	16	39	3	21	43	50	29	12	49	19	30	42	2	26	11	20	15	9	13	10	8	7
33	34	36	39	35	41	46	37	2	28	18	6	45	25	38	5	26	19	47	29	15	42	23	7	49	24	32	48	17	40	4	22	44	1	30	13	50	20	31	43	3	27	12	21	16	10	14	11	9	8
34	35	37	40	36	42	47	38	3	29	19	7	46	26	39	6	27	20	48	30	16	43	24	8	50	25	33	49	18	41	5	23	45	2	31	14	1	21	32	44	4	28	13	22	17	11	15	12	10	9
35	36	38	41	37	43	48	39	4	30	20	8	47	27	40	7	28	21	49	31	17	44	25	9	1	26	34	50	19	42	6	24	46	3	32	15	2	22	33	45	5	29	14	23	18	12	16	13	11	10
36	37	39	42	38	44	49	40	5	31	21	9	48	28	41	8	29	22	50	32	18	45	26	10	2	27	35	1	20	43	7	25	47	4	33	16	3	23	34	46	6	30	15	24	19	13	17	14	12	11
37	38	40	43	39	45	50	41	6	32	22	10	49	29	42	9	30	23	1	33	19	46	27	11	3	28	36	2	21	44	8	26	48	5	34	17	4	24	35	47	7	31	16	25	20	14	18	15	13	12
38	39	41	44	40	46	1	42	7	33	23	11	50	30	43	10	31	24	2	34	20	47	28	12	4	29	37	3	22	45	9	27	49	6	35	18	5	25	36	48	8	32	17	26	21	15	19	16	14	13
39	40	42	45	41	47	2	43	8	34	24	12	1	31	44	11	32	25	3	35	21	48	29	13	5	30	38	4	23	46	10	28	50	7	36	19	6	26	37	49	9	33	18	27	22	16	20	17	15	14
40	41	43	46	42	48	3	44	9	35	25	13	2	32	45	12	33	26	4	36	22	49	30	14	6	31	39	5	24	47	11	29	1	8	37	20	7	27	38	50	10	34	19	28	23	17	21	18	16	15
41	42	44	47	43	49	4	45	10	36	26	14	3	33	46	13	34	27	5	37	23	50	31	15	7	32	40	6	25	48	12	30	2	9	38	21	8	28	39	1	11	35	20	29	24	18	22	19	17	16
42	43	45	48	44	50	5	46	11	37	27	15	4	34	47	14	35	28	6	38	24	1	32	16	8	33	41	7	26	49	13	31	3	10	39	22	9	29	40	2	12	36	21	30	25	19	23	20	18	17
43	44	46	49	45	1	6	47	12	38	28	16	5	35	48	15	36	29	7	39	25	2	33	17	9	34	42	8	27	50	14	32	4	11	40	23	10	30	41	3	13	37	22	31	26	20	24	21	19	18
44	45	47	50	46	2	7	48	13	39	29	17	6	36	49	16	37	30	8	40	26	3	34	18	10	35	43	9	28	1	15	33	5	12	41	24	11	31	42	4	14	38	23	32	27	21	25	22	20	19
45	46	48	1	47	3	8	49	14	40	30	18	7	37	50	17	38	31	9	41	27	4	35	19	11	36	44	10	29	2	16	34	6	13	42	25	12	32	43	5	15	39	24	33	28	22	26	23	21	20
46	47	49	2	48	4	9	50	15	41	31	19	8	38	1	18	39	32	10	42	28	5	36	20	12	37	45	11	30	3	17	35	7	14	43	26	13	33	44	6	16	40	25	34	29	23	27	24	22	21
47	48	50	3	49	5	10	1	16	42	32	20	9	39	2	19	40	33	11	43	29	6	37	21	13	38	46	12	31	4	18	36	8	15	44	27	14	34	45	7	17	41	26	35	30	24	28	25	23	22
48	49	1	4	50	6	11	2	17	43	33	21	10	40	3	20	41	34	12	44	30	7	38	22	14	39	47	13	32	5	19	37	9	16	45	28	15	35	46	8	18	42	27	36	31	25	29	26	24	23
49	50	2	5	1	7	12	3	18	44	34	22	11	41	4	21	42	35	13	45	31	8	39	23	15	40	48	14	33	6	20	38	10	17	46	29	16	36	47	9	19	43	28	37	32	26	30	27	25	24
50	1	3	6	2	8	13	4	19	45	35	23	12	42	5	22	43	36	14	46	32	9	40	24	16	41	49	15	34	7	21	39	11	18	47	30	17	37	48	10	20	44	29	38	33	27	31	28	26	25

Figure E

Circular Tuscan-2 squares by symmetric polygonal path construction none are Tuscan-3 except for $N = 12$, where the only existing example is a Vatican square

N=10

10	1	3	7	4
9	2	8	6	5

N=12

12	1	4	2	9	5
11	3	8	10	7	6

N=14

14	2	13	5	1	10	11
4	3	8	12	6	9	7

N=16

16	2	1	12	5	15	3	6
14	11	7	13	4	9	10	8

N=18

18	1	3	15	5	16	11	8	4
13	17	2	7	14	6	12	10	9

N=20

20	1	4	13	19	6	8	12	7	15
5	17	2	18	16	9	3	14	11	10

N=22

22	2	1	8	4	18	6	9	14	5	21
10	16	3	20	17	7	15	19	12	13	11

N=24

24	2	1	6	16	23	10	7	15	9	5	20
8	17	21	3	19	22	11	4	18	13	14	12

N=26

26	1	3	6	2	22	11	4	18	10	20	25	8
21	12	7	23	5	17	24	9	15	19	16	14	13

N=28

28	1	3	6	10	22	9	4	26	19	2	11	21	13
27	7	25	16	5	12	18	23	8	24	20	17	15	14

Figure E (continued)

N=30
30 2 1 4 8 24 14 5 11 22 27 10 18 6 13
28 21 3 25 12 7 26 20 29 9 23 19 16 17 15

N=32
32 2 1 4 9 26 30 7 27 3 21 15 28 6 13 24
8 29 22 12 31 5 19 11 23 14 10 25 20 17 18 16

N=34
34 1 3 6 2 12 21 32 25 5 13 31 26 7 28 16 10
27 33 11 24 9 14 30 22 8 15 4 29 19 23 20 18 17

N=36
36 1 3 6 2 8 28 17 34 29 5 15 30 7 14 22 31 9
27 13 4 32 25 12 33 23 11 16 35 10 26 20 24 21 19 18

N=38
38 2 1 4 8 13 22 34 18 26 36 11 25 14 31 24 9 29 35
16 10 28 5 12 33 6 30 17 7 37 15 3 32 27 23 20 21 19

N=40
40 2 1 4 8 13 26 34 11 39 23 5 15 9 38 12 27 36 17 10
30 37 16 7 32 18 29 35 25 3 19 31 14 6 33 28 24 21 22 20

N=42
42 1 3 6 2 8 20 12 35 28 4 36 16 11 26 39 13 30 19 10 38
17 31 40 9 34 18 5 32 37 15 25 7 14 33 41 29 23 27 24 22 21

N=44
44 1 3 6 2 8 13 27 36 18 7 38 19 12 33 4 20 10 37 17 9 21
43 31 39 15 32 42 26 11 34 41 16 29 40 14 5 35 30 24 28 25 23 22

N=46
46 2 1 4 8 13 3 17 29 45 16 9 28 34 42 7 20 38 14 35 44 18 33
10 41 21 12 37 15 43 30 19 11 5 32 39 22 6 40 26 36 31 27 24 25 23

N=48
48 2 1 4 8 13 3 9 21 36 5 14 43 17 35 42 15 31 44 16 30 22 47 10
34 23 46 6 40 20 7 39 18 11 41 19 38 29 12 45 33 27 37 32 28 25 26 24

N=50
50 1 3 6 2 8 13 4 19 45 35 23 12 42 5 22 43 36 14 46 32 9 40 24 16
41 49 15 34 7 21 39 11 18 47 30 17 37 48 10 20 44 29 38 33 27 31 28 26 25

References

1. S.W. Golomb and H. Taylor, *Tuscan squares – a new family of combinatorial designs*, Ars Combinatoria **20-B** (1985), 115-132.
2. J. Denes and A.D. Keedwell, "Latins Squares and Their Applications", Academic Press, New York, 1974.
3. H.J. Ryser, "Combinatorial Mathematics", CAM-14, Math. Assoc. of America, 1963.
4. C.F. Gauss, "Disquisitiones Arithmeticae", Springer-Verlag, New York, 1985.
5. K. Ireland and M. Rosen, "A Classical Introduction to Modern Number Theory", Springer-Verlag, New York, 1982.
6. S.W. Golomb and H. Taylor, *Constructions and properties of Costas arrays*, Proc. IEEE **72, No.9** (September 1984), 1143-1163.
7. T.W. Tillson, *A Hamiltonian decomposition of* K^{*}_{2m}, $2m$ *greater than or equal to 8*, J. Combin. Theory, Series B **29** (1980), 68-74.
8. G.S. Bloom and D.F. Hsu, *Graceful directed graphs*, Technical Report **CCNY-CS283** (November 1983).