Perfect Colourings

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Abstract. Given a graph G and nonnegative integer k, a map $\pi : V(G) \to \{1,\dots,k\}$ is a *perfect k-colouring* if the subgraph induced by each colour class is perfect. The *perfect chromatic number* of G is the least k for which G has a perfect k-colouring; such an invariant is a measure of a graph's imperfection. We study here the theory of perfect colourings. In particular, the existence of perfect k-chromatic graphs are shown for all k, and we draw attention to the associated extremal problem. We provide extentions to C. Berge's Strong Perfect Graph Conjecture, and prove the existence of graphs with only one perfect k-colouring (up to a permutation of colours). The type of approach taken here can be applied to studying any graph property closed under induced subgraphs.

1. Introduction

A graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, $\chi(H)$, is equal to the clique number of H, $\omega(H)$. The property of perfection has received considerable attention since its introduction by Berge in the 1960's [1]. Indeed, one of the foremost open problems is Berge's Strong Perfect Graph Conjecture (SPGC): "A graph G is perfect if and only if it does not contain as an induced subgraph an odd cycle of length at least 5 or its complement."

Much of the work has been devoted to determining whether a graph is perfect or not. What we propose here is a measure of imperfecton. Given a graph G and nonnegative integer k, a map π that assigns to each vertex of G a 'colour' from the set $\{1,\dots,k\}$ is a *perfect k-colouring of* G if the subgraphs induced by each colour class $\pi^{-1}(i)$ is perfect. The *perfect chromatic number of* G, $\chi(G :$ perfect$)$, is the least nonnegative integer k for which G has a perfect k-colouring. Note that a graph is perfect if and only if its perfect chromatic number is 1. Also, perfection is closed under graph complementation [22]; it follows that π is a perfect k-colouring of G if and only if it is a perfect k-colouring of $\overline{G}$, so G and $\overline{G}$ have the same perfect chromatic number. The perfect chromatic number will be our indicator of how far a graph is from being perfect.

The notions of a perfect k-colouring and perfect chromatic number are particular examples of P k-colourings and P chromatic numbers (see [8]). A graph property P is a class of graphs (closed under isomorphism) that contains the empty

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graph K_0 and the trivial graph K_1. A property P is called *hereditary* if it is closed under induced subgraphs, i.e. whenever $G \in P$ and H is an induced subgraph of G (written $H \trianglelefteq G$), then $H \in P$. Note that perfection is hereditary. We shall always assume that the properties under question are hereditary. Given a graph G, a hereditary property P and a nonnegative integer k, a *P k-colouring of G* is a map $\pi : V(G) \to \{1, \ldots, k\}$ such that the subgraph induced by each colour class $\pi^{-1}(i)$ belongs to P; the *P chromatic number of G*, $\chi(G : P)$, is the least k for which G has a P k-colouring. P colourings are discussed by S. Hedetniemi [20], E. Cockayne [10] and F. Harary [17]. A detailed study of the theory of P colourings was undertaken in [8] and to a greater extent in [7]. We propose here to study the particular property of perfection from such a vantage point.

In general, our notation follows [2] and [3]. The graphs P_n, C_n and K_n denote respectively the path, cycle and complete graph on n vertices (C_n is often called a *hole* and its complement, $\overline{C_n}$, an *antihole*). The *order* of a graph is its number of vertices. $H \trianglelefteq G$ denotes that H is an induced subgraph of G. For a fixed graph G of order at least 2, the property of being G-free is $-G = \{H : G \ntrianglelefteq H\}$; for a family $\mathcal{F}$ of graphs of order at least 2, $\mathcal{F}$ denotes the property $\bigcap\{-G : G \in \mathcal{F}\}$. Throughout we often omit the property P in the notation when $P = -K_2$, since the generalized chromatic notions coincide then with the standard ones.

We remark that Hell and Roberts [19] have defined a different measure of imperfection.

2 Examples of Perfect k-Chromatic Graphs

Any perfect graph is perfect 1-chromatic. The odd cycles of length at least 5 and their complements are examples of perfect 2-chromatic graphs, since they are imperfect but the removal of any vertex leaves a perfect graph. The existence of perfect k-chromatic graphs for all $k \geq 1$ is not entirely obvious; we present a few methods for constructing such graphs.

2.1 Via Triangle-free Graphs

Theorem 2.1. *Let G be a triangle-free graph. Then $\chi(G : \text{perfect}) = \lceil \frac{\chi(G)}{2} \rceil$.*

Proof: Let $\pi : V(G) \to \{1, \ldots, \chi(G)\}$ be a $\chi(G)$-colouring of G (in the usual sense). Then as every bipartite graph is perfect, we see that $\pi' : V(G) \to \{1, \ldots, \lceil \frac{\chi(G)}{2} \rceil\}$, where $\pi'(u) = \lceil \frac{\pi(u)}{2} \rceil$, is a perfect $\lceil \frac{\chi(G)}{2} \rceil$-colouring of G. Thus $\chi(G : \text{perfect}) \leq \lceil \frac{\chi(G)}{2} \rceil$. On the other hand, if $\rho : V(G) \to \{1, \ldots, \chi(G : \text{perfect})\}$ is a perfect $\chi(G : \text{perfect})$-colouring, then each colour class must be bipartite (we use the fact that G is triangle-free and that any non-bipartite graph contains an induced odd cycle). Thus by colouring each colour class with two new colours, we get a $2\chi(G:\ \text{perfect})$-colouring of G, so that $2\chi(G:\ \text{perfect}) \geq \chi(G)$. It follows that $\chi(G:\ \text{perfect}) \geq \lceil\frac{\chi(G)}{2}\rceil$, so we are done. ∎

Now it is well known that there exist k-chromatic triangle-free graphs for all k, so this, in conjunction with the theorem above, yields perfect k-chromatic graphs. In fact, from Erdös' theorem [11] on the existence of graphs with large girth and chromatic number, there are perfect k-chromatic graphs of girth at least g for all positive integers g and k.

2.2 Via the Substitution Operation

Given graphs G and H and a vertex v of G, the graph arising by substituting H for v in G is formed from $H \cup (G - v)$ by joining each vertex of H to every neighbour of v in G. It is known [22] that the property of perfection is closed under substitution. Now suppose we have constructed a perfect k-chromatic graph H_k. We substitute H_k for every vertex but one in a graph G that is either an odd cycle of length at least 5 or its complement. Then it follows from Theorem 2.8 of [8] that the new graph is perfect $(k + 1)$-chromatic. Starting with $H_1 = K_1$, we recursively build perfect k-chromatic graphs for all $k \geq 2$. An example of such a perfect 3-chromatic graph is shown in figure 1.

2.3 Via Ramsey Theory

Let H be a graph such that for assignment of one of $k - 1$ colours to the edges of H, there is a partial monochromatic subgraph isomorphic to an odd cycle of length at least 5 (the existence of such a graph H follows from Ramsey's theorem - see [15, pg. 120]). If G is the line graph of H, it is not hard to see that G is not perfect $(k - 1)$-colourable, and hence contains an induced perfect k-chromatic subgraph.

Another approach uses Erdös' result [12] that there is a graph G_n of order at least $2^{\frac{n}{2}} - 1$ that contains no K_n or $\overline{K_n}$. Consider a perfect k-colouring π of G_n. Lovász [23] proved that a graph H is perfect if and only if for every induced subgraph F of H, $\omega(F)\omega(\overline{F}) \geq |V(F)|$. Thus if π is a perfect k-colouring of G_n then each colour class $\pi^{-1}(i)$ has order at most $(n-1)^2$, so that

$$k(n-1)^2 \geq \sum_{i=1}^{k} |\pi^{-1}(i)| = |V(G_n)| \geq 2^{\frac{n}{2}} - 1$$

It follows that $\chi(G:\ \text{perfect}) \geq \frac{2^{\frac{n}{2}}-1}{(n-1)^2}$, so we have graphs of arbitrary large perfect chromatic number.

The methods mentioned above do not provide examples of perfect k-chromatic graphs with order smaller than a constant times k^2. We can, however, determine

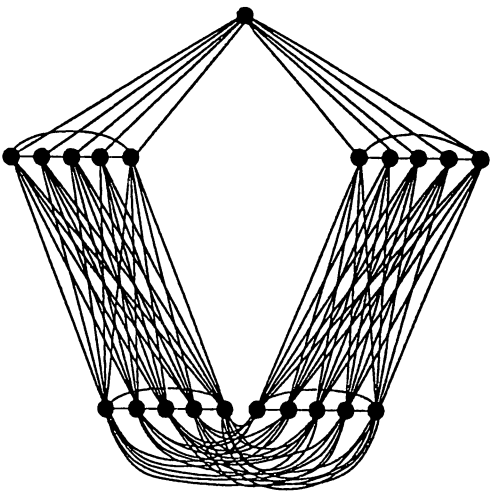


Figure 1

the asymptotic behaviour of $f(k:$ perfect$)$, the minimum order of a perfect k-chromatic graph.

Theorem 2.2. *There are positive constants C and C' such that*

$$Ck \log k \leq f(k: \textit{perfect}) \leq C'k \log k \quad (*)$$

for any $k \geq 3$.

Proof: In **[18]** it was started that for any graph G that is neither complete nor totally disconnected, there are positive constants K and K' such that

$$Kk \log k \leq f(k: -G) \leq K'k \log k \quad (**)$$

where $f(k: -G)$ is the minimum order of a $-G$ k-chromatic graph. Taking G to be a fixed imperfect graph and noting that any $-G$ k-chromatic graph is not perfect $(k-1)$-colourable, we have $f(k:$ perfect$) \leq f(k: -G) \leq K'k \log k$, and thus proves the existence of the upper bound constant C'.

To prove the lower bound, we use the same type of argument as was used to prove the lower bound of (**) in [18]. Let G be a graph of order n (n large). It is known (see [15], page 77) that (for large n) every graph of order n contains either a $K_{\lfloor \log_4 n \rfloor}$ or a $\overline{K_{\lfloor \log_4 n \rfloor}}$. Now

$$\lfloor \log_4(n - i\lceil \log_{16} n \rceil) \rfloor \geq \lceil \log_{16} n \rceil \qquad (***)$$

provided $i \leq \frac{n-16\sqrt{n}}{\lceil \log_{16} n \rceil}$, and for such an i, $n - i\lceil \log_{16} n \rceil \geq 16\sqrt{n}$. Therefore, for large n we can recursively remove a $K_{\lceil \log_{16} n \rceil}$ or a $\overline{K_{\lceil \log_{16} n \rceil}}$ from G $\lceil \frac{n-16\sqrt{n}}{\lceil \log_{16} n \rceil} \rceil$ times since (***) holds. We colour each such clique or independent set with a different colour, and the remaining

$$n - \lceil \frac{n - 16\sqrt{n}}{\lceil \log_{16} n \rceil} \rceil \cdot \lceil \log_{16} n \rceil \leq 16\sqrt{n}$$

vertices each with a different colour. The number of colours used in this perfect colouring is at most $\lceil \frac{n-16\sqrt{n}}{\lceil \log_{16} n \rceil} \rceil + 16\sqrt{n} < C'' \frac{n}{\log_{16} n}$ (C'' a positive constant). For large k, we choose $n = \lfloor \frac{k}{C''} \log_{16} k \rfloor$. Then $\frac{C'' n}{\log_{16} n} \leq k$, so any graph of order at most n is perfect k-colourable. Thus $f(k : \text{ perfect}) \geq \lfloor \frac{(k-1)}{C''} \log_{16}(k-1) \rfloor \geq Ck \log k$ for some positive constant C. ∎

We remark that a precise calculation of $f(k : \text{ perfect})$ appears difficult for any $k \geq 3$. The smallest perfect 3-chromatic graph we know is shown in figure 1.

3 Vertex Perfect k-Critical Graphs and the SPGC

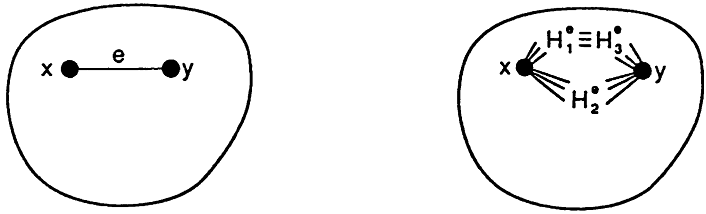
Many of the attacks on the SPGC have been directed at *minimally imperfect graphs*, that is, those graphs that are imperfect but the removal of any vertex leaves a perfect graph. The structure of minimally imperfect graphs has been thoroughly investigated (see [14], for example). We now define a subclass of perfect k-chromatic graphs that includes as a special case ($k = 2$) minimally imperfect graphs. In [8] a graph G was defined to be *vertex P k-critical* if G is P k-chromatic but $G - v$ is P $(k-1)$-colourable for all vertices v of G. Hence:

Definition 3.1. *If G is perfect k-chromatic but for every vertex v of $G, G - v$ is perfect $(k-1)$-colourable, then G is called vertex perfect k-critical.*

The reader can verify that the graph of figure 1 is indeed vertex perfect 3-critical.

It should be clear that any perfect k-chromatic graph contains an induced vertex perfect k-critical subgraph. Thus the constructions of section 2 yield examples of vertex perfect k-critical graphs. It is of interest (as it is in the standard chromatic case [27],[29],[28]), however, to construct explicitly such graphs.

As mentioned earlier, the property of perfection is closed under substitution, and in [8] it was shown that if P is any property closed under substitution, $H_1, \ldots, H_n$ are vertex P k-critical graphs and G is a vertex P 2-critical graph on vertices



F   F$_k$

Figure 2:

$v_0, v_1, \ldots, v_n$, then the graph G_{k+1} arising by substituting successively H_i for v_i in G $(i = 1, \ldots, n)$ is vertex P $(k+1)$-critical.

A second construction can be described as follows. Let F be a k-critical graph, i.e. F is k-chromatic, but the removal of any edge or vertex leaves a $(k-1)$-colourable subgraph (constructions of such graphs are plentiful; see [28],[29]). For each edge $e = xy$ of G, we take three disjoint copies H_1^e,H_2^e, and H_3^e of any vertex perfect $(k-1)$-critical graphs, and add in all edges between x and H_1^e, x and H_2^e, H_1^e and H_3^e, y and H_2^e, and y and H_3^e (see figure 2); let this new graph be F_k. In any attempt to perfect $(k-1)$-colour F_k, some edge $e = xy$ of F has both endpoints the same colour c; as each colour must be used on each of the vertex perfect $(k-1)$-critical graphs H_i^e, colour class c has a monochromatic 5-cycle C_5, a contradiction. Thus F_k is not perfect $(k-1)$-colourable. Let v be any vertex of F_k. If $v \in G$, then let $\pi : V(F_k) \to \{1, \ldots, k-1\}$ be any map such that the restriction of π to $G - v$ is a $(k-1)$-colouring of $G - v$, and the restriction of π to each H_j^e is a perfect $(k-1)$-colouring of H_j^e; if $v \in H_j^e$, then let $\pi : V(F_k) \to \{1, \ldots, k-1\}$ be any map such that the restriction of π to G is a $(k-1)$-colouring of $G - e$ (with the endpoints of e having the same colour c), the restriction of π to each $H_l^f((f, l) \neq (e, j))$ is a perfect $(k-1)$-colouring of H_l^f, and the restriction of π to $H_j^e - v$ is a perfect $(k-2)$-colouring of $H_j^e - v$ not using the colour c. In either case, each colour class of π arises by substituting perfect graphs for the vertices of a forest. As any forest is perfect we see that $F_k - v$ is perfect $(k-1)$-colourable for any vertex v of F_k. It follows that F_k is vertex perfect k-critical.

Let $k = 3$. By taking $G = C_5$ and each H_i to be an odd hole of length at least 5 in the first construction, we see that there are vertex perfect 3-critical graphs of all odd orders greater than or equal to 31. In the second construction, by choosing F to be a 5-cycle, and each H_i^e's to be an odd hole of length at least 5, we derive vertex perfect 3-critical graphs of all even orders greater than or equal to 80. Thus there are vertex perfect 3-critical graphs of all orders greater than or equal to 79. It is not hard to see from the first construction (with $G = C_5$) that if there are vertex perfect k-critical graphs of all orders $\geq m$, then there are vertex perfect $(k+1)$-critical graphs of all orders $\geq 4m + 1$. Thus we have shown the following.

Theorem 3.1. *For every $k \geq 3$, there are vertex perfect k-critical graphs of every order $\geq 78 \cdot 4^{k-3} + \frac{1}{3}(4^{k-2} - 1)$.* ∎

Therefore we have a method for explicitly constructing vertex perfect k-critical graphs of all large orders for all $k \geq 3$ (while it is easy to deduce from the result of Lovász mentioned in section 2.3 that there are no vertex perfect 2-critical graphs of order $p + 1$, where p is a prime). All of these contain an induced $K_{1,3}$ and are therefore not line graphs, so that the first construction of perfect k-chromatic graphs via Ramsey theory in section 2.3 yields a different family of vertex perfect k-critical graphs (other vertex perfect k-critical graphs can be found by taking vertex $(2k-1)$-critical graphs of girth at least 4).

We now turn our attention to vertex perfect k-critical graphs and the SPGC. Let Hole = $\{C_{2n+1} : n \geq 2\} \cup \{\overline{C_{2n+1}} : n \geq 2\}$. Then the SPGC is equivalent to the statement that the vertex perfect 2-critical graphs are precisely the vertex $-$Hole 2-critical graphs. Our next result states that equality in any level of the criticality hierarchy is equivalent to the SPGC.

Theorem 3.2. *The SPGC is equivalent to the statement:*

There is a $k \geq 2$ such that the set of vertex perfect k-critical graphs is precisely the set of vertex $-$Hole *k-critical graphs.*

Proof: If the SPGC fails, then there is a minimal imperfect graph G that does not contain an induced C_{2n+1} or $\overline{C_{2n+1}}$ $(n \geq 2)$, so $G \in -$Hole. Using the substitution construction recursively, we construct vertex perfect k-critical graphs G_k $(k \geq 2)$ by setting $G_2 = G$ and forming G_{k+1} by substituting G_k for all but one vertex of G. It is easy to verify that $-$Hole is closed under substitution, so as $G_2 = G \in -$Hole, we see inductively that $G_k \in -$Hole for all $k \geq 2$. In particular, there is a graph that is vertex perfect k-critical but not vertex $-$Hole k-critical for any $k \geq 2$. ∎

We remark that while clearly a property P is determined by its vertex P 2-critical graphs, in general it is not true that P is determined by its vertex P k-critical graphs for any fixed $k \geq 3$. For example, if $Q = \{G : |V(G)| \leq 4\}$ and $P = Q - \{P_4\}$, then the vertex P k-critical graphs are the same as the vertex Q k-critical graphs for any $k \geq 3$, namely the set $\{G : |V(G)| = 4k - 3\}$; the lengthy details are omitted.

We end this section by stating the SPGC in terms of another property. The reader can verify that $-\{P_3, \overline{P_3}\} = \{K_n : n \geq 0\} \cup \{\overline{K_n} : n \geq 0\}$. $-\{P_3, \overline{P_3}\}$-colourings have been investigated in the literature under the name *cocolourings* (see [21],[26],[5],[6]). In Broere and Burger's notation [6] vertex $-\{P_3, \overline{P_3}\}$-k-critical graphs are called *critically k-cochromatic* graphs; we relate such graphs to the SPGC.

Proposition 3.3. *Let G be a minimal imperfect graph with clique number ω and independence number α. Let $\beta = \min\{\alpha, \omega\}$. Then G is critically $(\beta + 1)$-cochromatic. Thus the SPGC is equivalent to the statement that every minimally imperfect graph is critically 3-cochromatic.*

Proof: Suppose $\pi : V(G) \to \{1, \ldots, \beta\}$ were a β-cocolouring of G. Let the order of G be n; by a theorem of Lovász [23], (see also [14, pg. 58]) $n = \alpha\omega + 1$. As each colour class of π is independent or complete, $|\pi^{-1}(i)| \leq \max\{\alpha, \omega\}$ for all i, so $n = \sum_{i=1}^{\beta} |\pi^{-1}(i)| \leq \beta \cdot \max\{\alpha, \omega\} = \alpha\omega$, a contradiction. Thus G is not β-cocolourable.

Let v be a vertex of G; it suffices to show that $G - v$ is β-cocolourable. Note that $\overline{G}$ is also minimally imperfect and $\beta = \min\{\alpha(\overline{G}) = \omega, \omega(\overline{G}) = \alpha\}$. Furthermore, G is critically $(\beta + 1)$-cochromatic if and only if $\overline{G}$ is, so we may assume $\beta = \omega$. $G - v$ is perfect with clique number at most $\beta = \omega$, so let π be a β-colouring of $G - v$. Then clearly π is also a cocolouring of $G - v$ as well. ∎

4 Uniquely Perfect k-colourable Graphs

We now turn our attention to the existence of graphs with unique perfect k-colourings.

Definition 4.1. *A graph G is uniquely perfect k-colourable if and only if it is perfect k-chromatic and, up to a permutation of colours, G has only one perfect k-colouring.*

This definition is a particular instance of the notion of a *uniquely P k-colourable graph* [8], i.e. those graphs that are P k-chromatic and have only one P k-colouring up to a permutation of colours. For some properties P, uniquely P k-colourable graphs need not exist for any $k \geq 2$, for example, for $P = \{G : |V(G)| < n\}$ where $n \geq 3$ is fixed [8]. Another interesting example is afforded by P = perfect $\cap -K_3$. Note that perfect $\cap -K_3$ = bipartite. By the proof of Theorem 2.1, if G is (perfect $\cap -K_3$) k-chromatic ($k \geq 2$) then G is $(2k - 1)$-chromatic or $2k$-chromatic. Since any pairing of colour classes of a $2k$-colouring of G yields a bipartite k-colouring, we see that G is not uniquely (perfect $\cap -K_3$) k-colourable as $2k \geq 4$.

We devote this section to proving the existence of uniquely perfect k-colourable graphs for $k \geq 2$ (for $k = 0, 1$ we may take K_k). We begin with a lemma about the existence of certain uniquely k-colourable graphs. For a graph G and vertex v, we denote the *neighbourhood of* v *in* G by $N(v : G) = \{w \in V(G) : wv \in E(G)\}$, and the degree of v in G by $\deg(v : G)$. For a subset S of $V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S.

Lemma 4.1. *For all* $k \geq 2$ *and* $g \geq 3$ *there is a uniquely* k*-colourable graph* G *with colour classes* $V_1, \ldots, V_k$ *and of girth at least* g *such that for all* $1 \leq i < j \leq k$ *and all* $w \in V_i \cup V_j$,

$$\deg(w : \langle V_i \cup V_j \rangle) < 3n^{\varepsilon},$$

where $\varepsilon \in (0, \frac{1}{4g})$ *is fixed and* $n = |V_i|$ *for all* i.

Proof: Let $\mathcal{G}$ be the set of all k-partite graphs on vertex set $V_1 \cup \cdots \cup V_k$ ($\langle V_i \rangle$ is independent for all i) with $m = \lfloor \binom{k}{2} n^{1+\varepsilon} \rfloor$ edges. Bollobás and Sauer [4] have shown that (for all large n) there is a $\mathcal{G}' \subseteq \mathcal{G}$ such that

$$\frac{|\mathcal{G}'|}{|\mathcal{G}|} \geq 1 - n^{-\frac{\varepsilon}{4}}$$

and for any $G \in \mathcal{G}'$ we can omit a set of $\lfloor n^{g\varepsilon} \rfloor$ independent edges to get a graph G' of girth at least g that is uniquely k-colourable. Let $\mathcal{G}''$ be the collection of $G \in \mathcal{G}$ that have no vertex w in some V_i such that for some $j \neq i$, $\deg(w : \langle V_i \cup V_j \rangle) \geq 3 \lfloor n^{\varepsilon} \rfloor$. Then for large n

$$\begin{aligned}
\frac{|\mathcal{G}| - |\mathcal{G}''|}{|\mathcal{G}|} &\leq kn(k-1) \binom{n}{3\lfloor n^{\varepsilon} \rfloor} \binom{\binom{k}{2}n^2 - 3\lfloor n^{\varepsilon} \rfloor}{m - 3\lfloor n^{\varepsilon} \rfloor} \binom{\binom{k}{2}n^2}{m}^{-1} \\
&\leq k(k-1)n \left(\frac{en}{3\lfloor n^{\varepsilon} \rfloor} \right)^{3\lfloor n^{\varepsilon} \rfloor} \left(\frac{\lfloor \binom{k}{2} n^{1+\varepsilon} \rfloor}{\binom{k}{2} n^2} \right)^{3\lfloor n^{\varepsilon} \rfloor} \\
&\leq k(k-1)n \left(\frac{e \lfloor \binom{k}{2} n^{1+\varepsilon} \rfloor}{3\binom{k}{2} n \lfloor n^{\varepsilon} \rfloor} \right)^{3\lfloor n^{\varepsilon} \rfloor} \\
&\leq k(k-1)n \left(\frac{e}{3} + o(1) \right)^{3\lfloor n^{\varepsilon} \rfloor} \\
&= o(1)
\end{aligned}$$

Therefore $\frac{|\mathcal{G}''|}{|\mathcal{G}|} \to 1$ as $n \to \infty$. In particular, it follows for large n that $\mathcal{G}'' \cap \mathcal{G}' \neq \emptyset$, so we can find a uniquely k-colourable graph of girth at least g such that for all $1 \leq i < j \leq k$ and all $w \in V_i \cup V_j$, $\deg(w : \langle V_i \cup V_j \rangle) < 3\lfloor n^{\varepsilon} \rfloor \leq 3n^{\varepsilon}$. ∎

We now prove our main result.

Theorem 4.2. *For all $k \geq 1$ there are infinitely many uniquely perfect k-colourable graphs.*

Proof: For $k = 1$ the result is clear as there are infinitely many perfect graphs. We assume now $k \geq 2$. Let $G_{k,n}$ be a uniquely $2k$-colourable graph of order $2kn$ and girth at least 4, as afforded by the previous lemma, so that for all $1 \leq i < j \leq k$ and all $w \in V_i \cup V_j$, $\deg(w : \langle V_i \cup V_j \rangle) < 3n^{\varepsilon}$. As $\langle V_1 \cup V_2 \rangle$ is uniquely 2-colourable, we can pick an edge $w_1^1 w_2^1$ with $w_1^1 \in V_1$ and $w_2^1 \in V_2$. We recursively pick $w_i^1 \in V_i \quad (i = 3, \ldots, 2k)$ such that the subgraph of $G_{k,n}$ induced by $\{w_1^1, w_2^1, \ldots, w_{2k}^1\}$ is isomorphic to $K_2 \cup (2k-2)K_1$. This can be done as for any $i \geq 3$

$$\begin{aligned}\left|\bigcup_{j=1}^{i-1} (N(w_j^1 : G_{k,n}) \cap V_i)\right| &= \sum_{j=1}^{i-1} \deg(w_j^1 : \langle V_j \cup V_i \rangle) \\ &< (i-1)3n^{\varepsilon} \\ &\leq 3(2k-1)n^{\varepsilon} \\ &< n = |V_i|\end{aligned}$$

(for n large), so there is a w_i^1 not joined to $w_1^1, \ldots, w_{i-1}^1$. Similarly, for $l = 2, \ldots, k$ there are points $w_1^l, \ldots, w_{2k}^l$ such that $w_i^l w_j^l$ is an edge of $G_{k,n}$ if and only if $\{i, j\} = \{2l-1, 2l\}$.

Now we form $F_{k,n}$ from $G_{k,n}$ by taking new points $x_1, \ldots, x_k$ and joining x_l to $w_1^l, \ldots, w_{2k}^l$ (see figure 3). We claim that $F_{k,n}$ is uniquely perfect k-colourable.

First we show that $F_{k,n}$ is perfect k-colourable. For $v \in V(F_{k,n})$, set

$$\pi(v) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{if } v \in V_i \\ i & \text{if } v = x_i \end{cases}$$

that is, the colour classes of π are $\{x_i\} \cup V_{2i-1} \cup V_{2i}$, $i = 1, \ldots, k$. Let F_i be the subgraph of $F_{k,n}$ generated by the i^{th} colour class of π. Tucker [30] has shown the validity of the SPGC for graphs containing no K_4, so if F_i is not perfect, then it must contain an odd hole or antihole X; in fact, X is an odd antihole of length at least 7 as it is easy to verify that F_i contains no odd hole of length at least 5. X contains at most one K_3 as does F_i, a contradiction since every odd antihole of length at least 7 has more than one K_3. Thus F_i is perfect, and we see that π is a perfect k-colouring of $F_{k,n}$. In fact, $F_{k,n}$ is perfect k-chromatic, since by Theorem 2.1,

$$\chi(F_{k,n} : \text{perfect}) \geq \chi(G_{k,n} : \text{perfect}) \geq k.$$

It remains to show that if ρ is a perfect k-colouring of $F_{k,n}$ then ρ arises from π by a permutation of colours. Now the restriction ρ' of ρ to $G_{k,n}$ is a perfect

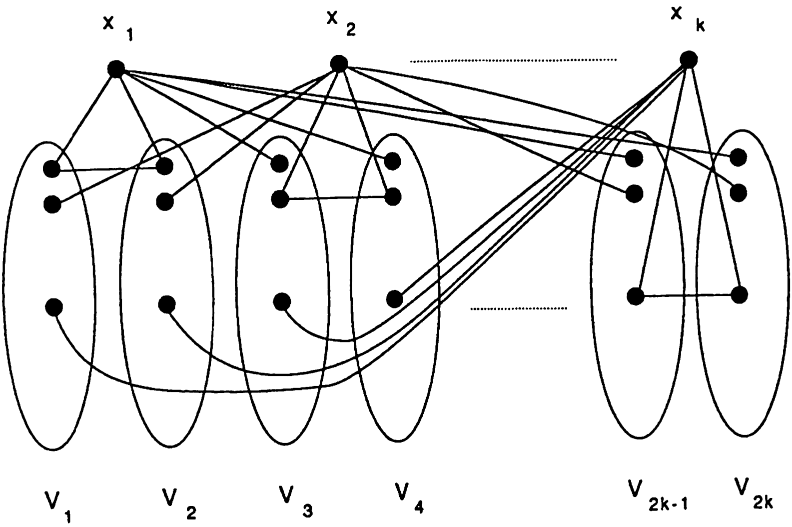


Figure 3: $F_{k,n}$

k-colouring of of $G_{k,n}$, so by the proof of Theorem 2.1, each colour class of ρ' is bipartite, and hence of the form $V_i \cup V_j \quad (i \neq j)$ by the unique 2 k-colourability of $G_{k,n}$. Consider the colour class C^l containing x_l. It also contains $V_{l_1} \cup V_{l_2}$ for some $l_1 \neq l_2$. If $\{l_1, l_2\} \neq \{2l-1, 2l\}$, then $w^l_{l_1}$ and $w^l_{l_2}$ are not adjacent in $F_{k,n}$. Since $V_{l_1} \cup V_{l_2}$, generates a uniquely 2-colourable subgraph of $G_{k,n}$, and hence of $F_{k,n}$, there is an induced path P' in $V_{l_1} \cup V_{l_2}$ with an even number of points that joins $w^l_{l_1}$ and $w^l_{l_2}$. As $w^l_{l_1}$ and $w^l_{l_2}$ are nonadjacent, P' contains at least 4 points, and so x_l together with P' generates an odd hole in C^l, contradicting the fact that C^l is perfect. Thus C^l contains $\{x_l\} \cup V_{2l-1} \cup V_{2l}$ and no more, since the restriction of C^l to $G_{k,n}$ is of the form $V_i \cup V_j$, and if $x_j \in C_l \quad (j \neq l)$, then $V_{2j_1} \cup V_{2j} \subseteq C^l$ follows as well from the argument above, a contradiction. Thus ρ has the same colour classes as π.

It follows that $F_{k,n}$ is uniquely k-colourable. As there are infinitely many graphs $G_{k,n}$ (for all large n), there are infinitely many uniquely perfect k-graphs $F_{k,n}$. ∎

Note that by the remark preceding Lemma 4.1, the triangles are essential in the construction; there do not exist uniquely perfect k-colourable graphs of girth at least g for any $k \geq 2$ and $g \geq 4$.

5 Further Remarks

We conclude our discussion of perfect k-colourings with two results. First, it is natural to inquire about the complexity of perfect k-colourability:

INPUT: A graph G.

QUESTION: Is G perfect k-colourable?

It is not known whether perfection (i.e. perfect 1-colourability) belongs to NP, and we do not know if perfect k-colourability belongs to NP for any k (we remark that Edmonds and Cameron [9] observed that perfection belongs to co-NP). However, we can show that perfect k-colourability is NP-hard for $k \geq 2$.

Theorem 5.1. *For all fixed $k \geq 2$, perfect k-colourability is NP-hard.*

Proof: We find a polynomial transformation of $2k$-colourability into perfect k-colourabilty (the former is known to be NP-complete for all fixed $k \geq 2$, see [13]). Let W be a fixed $(2k+1)$-critical graph without any triangles, and let $e = xy$ be any fixed edge of W. Given a graph G, we construct a graph G' by taking, for every edge $f = uv$ of G, a copy W_f of $W - e$, identifying one endpoint u of f with x_f, removing edge f from G, and joining the other endpoint v of f with y_f (this is the repeated use of Hajös' well known construction [16]). It is easy to see that G' is $2k$-colourable if and only if G is, and that G' is triangle-free. By Theorem 2.1, G' is $2k$-colourable if and only if it is perfect k-colourable, so we have that G is $2k$-colourable if and only if G' is perfect k-colourable. The construction is clearly polynomial since the order of W depends only on k (and not on G). ∎

Our other result of this final section asks under what conditions does there exist G-free graphs that are perfect k-colourable. The analogous problem for k-colourability and $G = C_n$ is Erdös' classic result on the existence of k-chromatic graphs of large girth. Mynhardt and Broere [24] posed the problem of whether for all graphs F and G of order at least 2 with $F \not\leq G$ and all positive integers k there are $-G$ k-chromatic graphs that are F-free. They provided partial results, and further results can be found in [7]. We determine precisely when there are F-free perfect k-chromatic graphs.

Theorem 5.2. *Let F be a graph of order at least 2. Then for all $k \geq 1$ there exists an F-free perfect k-chromatic graph if and only if F is not one of $K_2, \overline{K_2}, P_3, \overline{P_3}, P_4$.*

Proof: First note that as $-P_4 \subseteq$ perfect [25], any vertex perfect 2-critical graph (and hence any perfect k-chromatic graph for $k \geq 2$) contains a P_4. It follows that if G is any one of the subgraphs $K_2, \overline{K_2}, P_3, \overline{P_3}, P_4$ of P_4, then for any $k \geq 2$ there is no perfect k-chromatic graph that is F-free.

Conversely, suppose F is not one of $K_2, \overline{K_2}, P_3, \overline{P_3}$, or P_4. Note that $\overline{F}$ is also not equal to $K_2, \overline{K_2}, P_3, \overline{P_3}$, or P_4, so if we can find a perfect k-chromatic graph H that is F-free, then $\overline{H}$ is perfect k-chromatic and F-free. Since one of F and $\overline{F}$ is connected, we may assume F is connected. First assume that $\Delta(F) = \max\{\deg(v) : v \in V(F)\} \geq 3$, so that F contains either a $K_{1,3}$ or a K_3. If $K_3 \trianglelefteq F$ then by Theorem 2.1 there is a K_3-free graph, and hence an F-free graph, that is perfect k-chromatic for all $k \geq 1$. On the other hand, if $K_{1,3} \trianglelefteq F$, then as noted earlier, there are perfect k-chromatic graphs that are line graphs, and such graphs are $K_{1,3}$- (and therefore F-) free.

We may now assume $\Delta(F) \leq 2$, so as F is connected, F must be either a C_l $(l \geq 3)$ or a path P_m $(m \geq 5)$. If $F = C_l$, then we are done by the remark following Theorem 2.1. Finally, if $F = P_m (m \geq 5)$, then note that $-F$ is closed under substitution. As $C_5 \in -F$, we recursively see that the vertex perfect k-critical graphs formed from C_5 via the substitution construction are all F-free, and we are done. ∎

We have illustrated how the property of perfection can be studied in relation to the class of all graphs. Such an investigation can be carried out for any hereditary property. For nonhereditary properties (as shown in [8]) there need not exist Q k-chromatic graphs for all k, and the natural observation that the Q chromatic function is nondecreasing from the class of graphs under induced subgraphs to the natural numbers under $\leq$ may fail. However, for a nonhereditary property Q, one can form a hereditary property in much the same way that Berge formed the hereditary property of perfection from the nonhereditary property good $\equiv \{G : \chi(G) = \omega(G)\}$. Namely, we form the *filter* of Q,

$$\widehat{Q} \equiv \{G : \text{ for all induced subgraphs } H \text{ of } G, \ H \in Q\}.$$

Observe that $\widehat{Q}$ is the largest hereditary property contained in Q. The property of perfection is precisely the filter of the property of good. Even for a simple nonhereditary property such as Eulerian, the filter can be quite interesting, and the analogue of the SPGC, to characterize the vertex $\widehat{Q}$ k-critical graphs for small $k \geq 2$, is an avenue to explore.

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