On bicovers of pairs by quintuples: v odd, $v \not\equiv 3 \pmod{10}$

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Abstract

A bicover of pairs by quintuples of a v-set V is a family of 5-subsets of V (called blocks) with the property that every pair of distinct elements from V occurs in at least two blocks. If no other such bicover has fewer blocks, the bicover is said to be minimum, and the number of blocks in a minimum bicover is the covering number $C_2(v,5,2)$, or simply $C_2(v)$. It is well known that $C_2(v) \geq \lceil v\lceil (v-1)/2\rceil/5\rceil = B_2(v)$, where $\lceil x\rceil$ is the least integer not less than x. It is shown here that if v is odd and $v \not\equiv 3 \bmod 10$, $v \neq 9$ or 15, then $C_2(v) = B_2(v)$.

1 Introduction

Let V be a finite set of cardinality v. A (k,t)-*cover of index* λ is a family of k-subsets of V (called blocks), with the property that every t-subset of V occurs in at least λ of the blocks. The covering number $C_\lambda(v,k,t)$ is defined to be the number of blocks in a minimum (as opposed to minimal) (k,t)-cover of index λ of V.

For $v > k > t > 0$, let

$$B_\lambda(v,k,t) = \lceil v\lceil (v-1)\ldots\lceil (v-t+1)\lambda/(k-t+1)\rceil\ldots/(k-1)\rceil/k\rceil.$$

Then the quantity $B_\lambda(v,k,t)$ is a lower bound for $C_\lambda(v,k,t)$ (see [24]). Many researchers have been involved in determining the covering numbers known to date (see bibliography). Our interest here is in the case $k=5$, $t=2$, $\lambda=2$, v odd, $v \not\equiv 3 \pmod{10}$. For simplicity, let $C_2(v,5,2)$ be denoted by $C_2(v)$ and $B_2(v,5,2)$ be denoted by $B_2(v)$. Covers with $t=2$ and $\lambda=2$ are called *bicovers* of *pairs*, or *pair bicovers*. For $k=5$, these are then *bicovers of pairs by quintuples*. It was shown in [23] that if v is an even integer greater than or equal to six, then $C_2(v) = B_2(v)$. Unfortunately this is not true for odd values of v. For example, $C_2(9) \neq B_2(9)$, and $C_2(15) \neq B_2(15)$. However, $C_2(v) = B_2(v)$ for all other odd v greater than or equal to 5, $v \not\equiv 3 \pmod{10}$.

2 Some infinite families of bicovers

A balanced incomplete block design $BIBD(v, k, \lambda)$ is a pair (V, B) where V is a v-set and B is a family of subsets, each of size $k < v$, where every pair of distinct elements of V occurs in precisely λ blocks. A flat of a $BIBD$ is a subset F of V such that every block intersects F in 0, 1, or k points. It is well known (see [9]) that there exists a $BIBD(v, 5, \lambda)$ for all integers $v > 5$ which satisfy the relations

$$\lambda(v-1) \equiv 0 \bmod 4$$

and

$$\lambda v(v-1) \equiv 0 \bmod 20$$

with the exception of $v = 15$ and $\lambda = 2$. In particular, if $v \equiv 1$ or $5 \pmod{10}$ and $v \neq 15$, then there exists a $BIBD(v, 5, 2)$. It is immediate that if $v \equiv 1$ or $5 \pmod{10}$, $v \neq 15$, then $C_2(v) = B_2(v)$. The quantity referred to as "excess pair count" is useful in the following. Let C be a bicover of pairs by quintuples of a v-set V. If C contains c blocks, then the *excess pair count* $E(C)$ is defined by

$$E(C) = 10c - 2\binom{v}{2}.$$

Lemma 2.1 *Suppose that $v = 10m + 7$ or $10m + 9$, where m is a positive integer. Then $C_2(v) = B_2(v)$ if and only if there exists a bicover C of pairs by quintuples of a v-set such that $E(C) = 8$.*

Proof. The result follows from the identity

$$10B_2(v) - 2\binom{v}{2} = 8,$$

for such v. □

3 Some recursive constructions for bicovers

In this section we require several other types of combinatorial configuration. The definition of balanced incomplete block design (section 2) can be extended as follows. An *α-resolvable* balanced incomplete block design ($\alpha - RBIBD(v, k, \lambda)$) is a $BIBD(v, k, \lambda)$ together with a partition of the blocks into classes, called α-resolvable classes, which has the property that each point of the design occurs in precisely α blocks of each class. A *1-resolvable* balanced incomplete block design is simply said to be resolvable, and is denoted by $RBIBD$. Definitions of pairwise balanced design (PBD), group divisible design (GDD) and transversal design (TD) can be found in [27], and incomplete transversal design (ITD) can be found in [3]. Strictly speaking, the definitions given there are for the index $\lambda = 1$. To extend these to general index λ, the requirement that the pairs which occur in precisely one block of each of these configurations is to be replaced by the requirement that each such pair occur in precisely λ blocks. For the existence of transversal designs, our authority is [2] unless another reference is given. Similarly, for the existence of resolvable balanced incomplete block designs and balanced incomplete block designs, see [13]. For group divisible designs, we use the notation $GDD(g_1^{n_1} g_2^{n_2} \cdots g_s^{n_s}, K, \lambda)$ to represent such a design with n_i groups of size $g_i, i = 1, 2, \ldots, s$, whose block sizes lie in the set K, and whose index is λ. Two very important group divisible designs, namely a $GDD(2^5, \{5\}, 2)$ and a $GDD(2^6, \{5\}, 2)$ were found by Hanani [9], Lemma 4.15.

Lemma 3.1 *Let $v = 50m + 17$ or $v = 50m + 47$, where $m \geq 0$. If $v \neq 67$, then $C_2(v) = B_2(v)$.*

Proof. Let v be such an integer. Then $v = 5(10m + 3) + 2$ or $v = 5(10m + 9) + 2$. If $v \neq 17$, then there exists a transversal design $TD(5, 10m + 3)$ and a transversal design $TD(5, 10m + 9)$. By replacing each block of the transversal design with two copies of itself, group divisible designs $GDD((10m + 3)^5, \{5\}, 2)$ and $GDD((10m+9)^5, \{5\}, 2)$, respectively, are created. Let $\{x, y\}$ be a pair of points which do not occur in the group divisible designs. By replacing each group G of the group divisible design by a $BIBD(10m + 5, 5, 2)$ or a $BIBD(10m+11, 5, 2)$ respectively, defined on the set $G \cup \{x, y\}$, bicovers of $50m + 17$ and $50m + 47$ points are created. (This latter operation is possible provided $10m + 3 \neq 13$, hence the exception $v = 67$). Since the excess frequency of such a cover is clearly 8, these covers are minimum covers.

This leaves only the case of $v = 17$. A group divisible design, $GDD(3^5, \{5\}, 2)$ follows from Hanani [9], Theorem 3.11. By adjoining two new points $\{x, y\}$ to each group, then replacing the resulting blocks by two copies of themselves, and adjoining these to the blocks of the group divisible design, a minimum bicover of 17 points is obtained. Thus $C_2(17) = B_2(17)$. □

Lemma 3.2 *Let m and t be integers such that $0 \leq t \leq m$, $m \equiv 0$ or $2 \pmod 5$, $m \neq 7$. Suppose there exists a $TD(6, m)$, or let $m = 10$. If $C_2(2t + 1) = B_2(2t + 1)$, then $C_2(10m + 2t + 1) = B_2(10m + 2t + 1)$.*

Proof. If $m \neq 10$, by deleting $m - t$ points from one of the groups of the transversal designs and inflating each point by a factor of two (using the group divisible designs $GDD(2^5, \{5\}, 2)$ and $GDD(2^6, \{5\}, 2)$ and applying Wilson's fundamental theorem [27]), a group divisible design $GDD((2m)^5(2t)^1, \{5\}, 2)$ is obtained. If a new point is adjoined to each group, and if each resulting block B of size $2m + 1$ is replaced by a $BIBD(2m + 1, 5, 2)$ on the points of

B, and if the block B of size $2t+1$ is replaced by a minimum bicover of the points of B, the resulting configuration is a minimum cover of $10m+2t+1$ points which has $B_2(10m+2t+1)$ blocks. If $m=10$, the proof is as that of the same case in Lemma 3.2 of [23] mutatis mutandis. □

It is worth noting that for positive $m \equiv 0 \pmod 5$, the $TD(6,m)$ required in the above theorem is known to exist with the exception of $m=30$, and for positive $m \equiv 2 \pmod 5$, it exists except possibly for $m \epsilon \{2,22,42,52\}$, as is shown in [2] and [26].

Lemma 3.3 *Suppose that there exists a $BIBD(v,5,1)$ with s disjoint resolution classes. Let t be an integer satisfying $0 \le t \le s-1$. Suppose further that $C_2(2t+1) = B_2(2t+1)$, then $C_2(2v+2t+1) = B_2(2v+2t+1)$.*

Proof. Let $u = v/5$. Let $C_1, C_2, \ldots, C_s$ be the s resolution classes, and let $x_1, x_2, \ldots, x_t$ be t points not occurring in the block design. Adjoin x_i to each block of $C_i, i = 1,2,\ldots,t$, and then adjoin a block $B* = \{x_1, x_2, \ldots, x_t\}$, and take the blocks of the class C_{t+1} together with $B*$ as the groups of a $GDD(5^u t^1, \{5,6\}, 1)$. Inflate each point by factor of two to obtain a $GDD(10^u (2t)^1, \{5\}, 2)$, say D. Let the groups of this design be $G_1, G_2, \ldots, G_u, G_{u+1}$ where $|G_i| = 10$, for $i = 1,2,\ldots,u$, and $|G_{u+1}| = 2t$. Let ∞ be a point not in D. For each $i, i = 1,2,\ldots,u$, adjoin the blocks of a $BIBD(11,5,2)$ to the blocks of D defined on the set $G_i \cup \{\infty\}$. Then adjoin the blocks of a bicover of $G_{u+1} \cup \{\infty\}$ with $B_2(2t+1)$ blocks. The resulting set of blocks are the blocks of the required bicover. □

Lemma 3.4 *Suppose there exists a $BIBD(v,6,1)$ with a flat of order w. Let t be any integer satisfying $0 \le t \le w-1$. Suppose that $C_2(2t+1) = B_2(2t+1)$. Then $C_2(2v-2w+2t+1) = B_2(2v-2w+2t+1)$.*

Proof. Let x be a distinguished point lying in the flat of the $BIBD$. Let $u = (v-w)/5$. Replacing the blocks of the flat by a single block and deleting x from the resulting design yields a set of groups of a $GDD(5^u (w-1)^1, \{6\}, 1)$. Deleting $w-1-t$ more points of the flat then yields a $GDD\{5^u t^1, \{5,6\}, 1\}$ which can be inflated by a factor of two to produce a $GDD(10^u (2t)^1, \{5\}, 2)$. By proceeding as in Lemma 3.3, the required bicover is obtained. □

Lemma 3.5 *For $v \in \{7,27,37,67\}, C_2(v) = B_2(v)$.*

Proof. For $v=7$, let V be the points $\{1,2,\ldots,7\}$. Then the blocks of the bicover are

12345 12567
12346 34567
12347

For $v = 27$, let V be the set of points $\{(i,j) : i,j \in \mathbb{Z}_5\} \cup \{X,Y\}$. Then the blocks of the bicover are

(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	
(0,0)	(1,0)	(0,3)	(0,4)	(2,4)	mod (5,−)
(0,0)	(2,0)	(0,2)	(0,3)	(3,4)	mod (5,−)
(0,0)	(0,2)	(1,2)	(2,2)	(4,3)	mod (5,−)
(0,0)	(0,1)	(1,2)	(3,2)	(0,4)	mod (5,−)
(0,0)	(0,1)	(1,1)	(3,3)	(2,4)	mod (5,−)
(0,0)	(1,1)	(3,1)	(1,3)	(2,3)	mod (5,−)
(0,0)	(2,1)	(3,1)	(4,2)	(4,4)	mod (5,−)
(0,0)	(2,1)	(4,1)	(2,2)	(1,3)	mod (5,−)
(0,2)	(0,3)	(1,3)	(3,3)	(1,4)	mod (5,−)
(0,1)	(2,2)	(0,4)	(3,4)	(4,4)	mod (5,−)
X	(0,1)	(4,2)	(0,3)	(4,4)	mod (5,−)
Y	(0,1)	(4,2)	(1,3)	(3,4)	mod (5,−)
X	Y	(0,0)	(4,1)	(4,2)	mod (5,−)
X	Y	(0,0)	(2,3)	(3,4)	mod (5,−)

For $v = 37$, let V be the set of points $\{(i,j) : i \in \mathbb{Z}_7, j \in \mathbb{Z}_5\} \cup \{X,Y\}$. Then the blocks of the bicover are

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(*taken twice*)	
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(*taken twice*)	
X	Y	(4,0)	$(5,4j)$	$(6,4j)$	mod (−,5),	$j = 1,2$
X	(0,0)	$(1,4j)$	$(2,2j)$	$(3,3j)$	mod (−,5),	$j = 1,2$
Y	(0,0)	$(1,4j)$	$(2,3j)$	$(3,2j)$	mod (−,5),	$j = 1,2$
(2,0)	(3,0)	$(4,j)$	$(5,2j)$	$(6,j)$	mod (−,5),	$j = 1,2$
(1,0)	(3,0)	$(3,2j)$	(6,0)	$(6,4j)$	mod (−,5),	$j = 1,2$
(1,0)	(2,0)	$(3,j)$	$(5,j)$	$(5,4j)$	mod (−,5),	$j = 1,2$
(1,0)	$(2,j)$	$(2,2j)$	$(4,j)$	$(4,4j)$	mod (−,5),	$j = 1,2$
(0,0)	(3,0)	$(3,j)$	(4,0)	$(4,3j)$	mod (−,5),	$j = 1,2$
(0,0)	(2,0)	$(3,4j)$	(5,0)	$(5,3j)$	mod (−,5),	$j = 1,2$
(0,0)	$(2,j)$	$(2,4j)$	$(6,3j)$	$(6,4j)$	mod (−,5),	$j = 1,2$
(0,0)	(1,0)	$(4,2j)$	$(5,4j)$	$(6,j)$	mod (−,5),	$j = 1,2$
(0,0)	$(1,j)$	$(4,j)$	$(5,j)$	$(6,2j)$	mod (−,5),	$j = 1,2$
(0,0)	$(1,j)$	$(4,4j)$	$(5,2j)$	(6,0)	mod (−,5),	$j = 1,2$

The case $v = 67$ is more complicated to describe. Let V be the set of points $\{(i,j) : i \in \mathbb{Z}_{15}, j \in \mathbb{Z}_4\} \cup \{x_1, x_2, \ldots, x_7\}$. Since there are four mutually orthogonal Latin squares of order 15, there is a resolvable transversal design with groups

$$(0,0) \quad (1,0) \quad \cdots \quad (14,0) \quad \text{mod}(-,4).$$

Let $C_0, C_1, \ldots, C_{14}$ be the resolution classes of this transversal design. We can suppose that C_{14} consists of the fifteen blocks

$$(0,0) \quad (0,1) \quad (0,2) \quad (0,3) \quad \text{mod}(15,-).$$

We now discard C_{14} and adjoin x_{i+1} to the quadruples in the resolution classes C_{2i} and $C_{2i+1}, 0 \le i \le 6$. To this set of 210 quintuples we add the three blocks

$$(i,0)\quad (i+3,0)\quad (i+6,0)\quad (i+9,0)\quad (i+12,0)\quad i=0,1,2,$$

and the 225 blocks

$$\begin{array}{llllll}
(0,0) & (5,0) & (6,0) & (5,1) & (5,2) & \bmod (15,-)\\
(0,0) & (1,0) & (5,0) & (11,1) & (1,2) & \bmod (15,-)\\
(0,0) & (11,0) & (3,1) & (3,2) & (11,3) & \bmod (15,-)\\
(0,0) & (2,0) & (0,1) & (10,2) & (2,3) & \bmod (15,-)\\
(0,0) & (7,0) & (8,1) & (5,3) & (6,3) & \bmod (15,-)\\
(0,0) & (2,0) & (4,1) & (3,3) & (10,3) & \bmod (15,-)\\
(0,0) & (12,0) & (9,1) & (4,3) & (9,3) & \bmod (15,-)\\
(0,0) & (7,0) & (4,2) & (9,2) & (13,2) & \bmod (15,-)\\
(0,1) & (5,1) & (11,1) & (12,1) & (8,2) & \bmod (15,-)\\
(0,1) & (1,1) & (5,1) & (7,1) & (1,3) & \bmod (15,-)\\
(0,1) & (2,1) & (6,2) & (9,2) & (5,3) & \bmod (15,-)\\
(0,1) & (12,1) & (13,2) & (14,2) & (4,3) & \bmod (15,-)\\
(0,2) & (2,2) & (5,2) & (13,2) & (0,3) & \bmod (15,-)\\
(0,2) & (1,2) & (7,2) & (1,3) & (4,3) & \bmod (15,-)\\
(0,3) & (2,3) & (4,3) & (5,3) & (11,3) & \bmod (15,-).
\end{array}$$

By adjoining the blocks of a minimum bicover of seven points defined on the set $\{x_1, x_2, \ldots; x_7\}$, the required bicover of 67 points is obtained. □

In [5], S. Furino shows that there exist a 2-resolvable $BIBD(12s+10,4,2)$ for all but a finite number of positive integers s. For completeness, a special case of that result is given here.

Lemma 3.6 *If q is a prime power, $q \equiv 3 \ (mod\, 4)$, $q > 3$, then a 2-resolvable $BIBD(3q+1,4,2)$ exists.*

Proof. The points of V are a point at infinity X and the $3q$ ordered pairs (i,α), where i is an integer modulo 3, and $\alpha \in GF(q)$. Let β be a fixed element of $GF(q), \beta \neq 0, \pm 1$. For each α in $GF(q)$ the block

$$X\quad (0,\alpha)\quad (1,\alpha)\quad (2,\alpha),$$

taken twice, and the $3(q-1)/2$ blocks

$$(i,\alpha+\omega)\quad (i,\alpha-\omega)\quad (i+1,\alpha+\omega\beta)\quad (i+1,\alpha-\omega\beta),$$

where i is an integer modulo 3, and ω is a non-zero square in $GF(q)$, form a 2-resolution class. The collection of all these classes is a 2-resolvable $BIBD(3q+1,4,2)$. □

Lemma 3.7 *If* $v \equiv 7, 27,$ *or* $37 (\text{mod } 50)$, *then* $C_2(v) = B_2(v)$.

Proof. For $v \equiv 7 (\text{mod } 50)$, applying Lemmas 3.2 and 3.5 with $m \equiv 0 \ (\text{mod } 5)$ and $t = 3$ yields the result for all cases except for $v = 307$. In this case apply Lemma 3.4 to a $BIBD(156,6,1)$, taking a block of size 6 as a flat and $t = 3$.

For $v \equiv 27 (\text{mod } 50)$, applying Lemmas 3.2 and 3.5 with $m \equiv 0 \ (\text{mod } 5)$ and $t = 13$ yields the result for all cases except $v = 77, 127,$ or 327.

The case $v = 77$ is handled recursively, based on the fact that $C_2(19) = B_2(19)$, as is shown below.

For $v = 19$, let V be the points $\{1, 2, \ldots, 19\}$. Then the blocks of the required bicover are

0	1	2	3	4	1	7	13	14	16	3	6	9	15	18
0	1	2	5	6	1	8	10	12	15	3	6	12	16	17
0	3	4	7	8	1	8	11	16	18	3	7	15	16	18
0	5	7	8	9	1	9	15	17	18	3	8	10	11	14
0	6	10	11	15	2	3	9	11	13	4	5	11	14	15
0	9	10	12	16	2	4	11	12	18	4	5	15	16	17
0	11	13	16	17	2	5	10	16	18	4	6	8	12	13
0	12	14	17	18	2	6	7	10	14	4	6	9	14	16
0	13	14	15	18	2	7	12	13	15	4	7	10	17	18
1	3	5	12	14	2	8	9	14	17	5	6	8	13	18
1	4	9	10	13	2	8	15	16	17	5	7	9	11	12
1	6	7	11	17	3	5	10	13	17					

For $v = 77$, by Lemma 3.6, there exists a 2-resolvable $BIBD(58,4,2)$, say D. Let $x_1, x_2, \ldots, x_{19}$ be points which do not occur in D. To each block of the *i'th* 2-resolution class of D, adjoin the point x_i, and to the resulting set of blocks, adjoin the blocks of a minimal bicover of 19 points defined on the set $\{x_1, x_2, \ldots, x_{19}\}$. The result is a bicover of 77 points with $B_2(77)$ blocks. For the cases $v = 127$ and $v = 327$, apply Lemmas 3.2 and 3.5 with $t = 3$, and $m = 12$ and 32, respectively.

For $v \equiv 37 (\text{mod } 50)$, applying Lemmas 3.2, 3.5 and 3.1 with $m \equiv 0 \ (\text{mod } 5)$ and $t = 18$ or $m \equiv 2 (\text{mod } 5)$ and $t = 8$ yields all cases except for $v = 87$.

For $v = 87$, proceed as follows. Delete 7 points from one group of a $TD(6,8)$ to obtain a $GDD(8^5 1^1, \{5,6\}, 1)$ and inflate each point by a factor of two to obtain a $GDD(16^5 2^1, \{5\}, 2)$, say D. Let the groups of D be $G_1, G_2, \ldots, G_6$ where $|G_i| = 16$ for $i = 1, 2, \ldots, 5$ and $|G_6| = 2$. Let X be a set of five points not occurring in D. For each i, $i = 1,2,\ldots,5$, form a copy of a $BIBD(21,5,1)$ D_i on $G_i \cup X$ in such a way that X occurs as a block, and adjoin two copies of each block of D_i except X to the blocks of D. Finally, adjoin the blocks of a bicover of $G_6 \cup X$ with $B_2(7) = 5$ blocks to the above set of blocks to obtain the desired bicover. □

The foregoing is summarized in the following theorem.

Theorem 3.8 Let v be a positive integer congruent to 7 (mod 10). Then $C_2(v) = B_2(v)$.

4 Bicovers for orders congruent to 9 (mod 10).

In this section, bicovering numbers for orders $v = 10m + 9$ are determined.

Lemma 4.1 *Let* $v = 20m + 19$, *where* $m \geq 1$. *If* $C_1(v-2,5,2) = B_1(v-2,5,2)$, *then* $C_2(v) = B_2(v)$.

Proof. Let D_1 be a $(5,2)$ cover of index one of $20m + 17$ points which has precisely $B_1(20m+17,5,2)$ blocks. Then elementary counting shows that D_1 contains two special points, x and y, which have the property that the pair $\{x,y\}$ occurs in precisely five blocks of D_1, whereas every other pair occurs in precisely one block of D_1. Let us assume, without loss of generality, that the remaining points of D_1 are $1,2,\ldots,20m+15$, and that D_1 contains the blocks $B_1 = \{x,y,1,2,3\}$ and $B_2 = \{x,y,4,5,6\}$.

Now let D_2 be a $BIBD(20m+21,5,1)$ (recall that such a $BIBD$ exists in view of [9]). Without loss of generality, suppose that D_2 is defined on the set $\{1,2,\ldots 12m+15,x,y,a,b,c,d\}$ and contains the blocks $B_3 = \{a,c,1,2,3\}$ and $B_4 = \{b,d,4,5,6\}$. Create a new set of blocks D_3 on the set $\{1,2,\ldots,12m+15,x,y,a,b\}$ from the blocks of D_2 as follows: Delete the blocks B_3 and B_4 from D_2, then replace all occurences of the symbol c in the remaining blocks by the symbol a, and replace all occurences of the symbol d by the symbol b. To this set of blocks, adjoin all blocks of D_1 except for B_1 and B_2. Finally adjoin the blocks $\{x,a,1,2,3\}$, $\{y,a,1,2,3\}$, $\{x,b,4,5,6\}$, and $\{y,b,4,5,6\}$. Then this collection of blocks forms a bicover of the set $\{1,2,\ldots,20m+15,x,y,a,b\}$ in which the pair $\{x,y\}$ occurs in exactly four blocks, the pair $\{a,b\}$ occurs in exactly four blocks, and each of the pairs $\{x,a\},\{x,b\},\{y,a\},\{y,b\}$ occurs in exactly three blocks. All other pairs occur in precisely two blocks. Therefore the bicover contains exactly eight extra pairs, so by Lemma 2.1, $C_2(v) = B_2(v)$. □

Corollary 4.1.1 Suppose that $v = 100m + 99$ or $v = 100m + 19$ where $m \geq 0$. Then $C_2(v) = B_2(v)$.

Proof. It is shown in [7] that $C_1(v-2,5,2) = B_1(v-2,5,2)$ for these values of v with the exception of $v = 19$. The case $v = 19$ is handled in the proof of Lemma 3.7. □

The next theorem involves the notion of an incomplete bicover. Let v and w be positive integers, where both v and w are odd. By an incomplete bicover of type (v, w), denoted by $IB(v, w)$, we mean a triple (V, W, F) where V and W are disjoint sets of cardinality $v - w$ and w respectively, and F is a family of subsets (blocks), each of size five, from $S = V \cup W$, which has the following properties:

(i) each pair of distinct elements $\{x_1, x_2\}$, where at least one of x_1 or x_2 does not lie in W, occurs in exactly two blocks of F; and

(ii) no pair of distinct elements $\{x_1, x_2\}$, each of which lies in W, occurs in any block of F.

Clearly if $C_2(w) = B_2(w)$, then by adjoining the blocks of an appropriate bicover of W to the blocks of an incomplete bicover (V, W, F), a bicover of $S = V \cup W$ with $B_2(v)$ blocks is obtained.

Lemma 4.2 *Let v and w be positive integers, where both v and w are odd. Suppose that there exists an incomplete bicover $IB(v, w)$, and let $d = (v - w)/2$. Suppose that there exists a transversal design $TD(6, d)$. Let t be an integer satisfying $0 \leq t \leq d$, and suppose that $C_2(2t + w) = B_2(2t + w)$. Then $C_2(10d + 2t + w) = B_2(10d + 2t + w)$.*

Proof. Delete $d - t$ points from one group of the transversal design $TD(6, d)$ to obtain a group divisible design $GDD(d^5t^1, \{5, 6\}, 1)$. Inflate each point by a factor of two to obtain a group divisible design $G = GDD((v - w)^5(2t)^1, \{5\}, 2)$. Let the groups of this design be $G_1, G_2, \ldots, G_6$, where $|G_i| = v - w$, $i = 1, 2, \ldots, 5$ and $|G_6| = 2t$. Let W be a set of cardinality w which is disjoint from the point-set of G. For $i = 1, 2, \ldots, 5$, form an incomplete bicover (G_i, W, F_i) and adjoin the blocks of F_i to the blocks of G. Then adjoin the blocks of a bicover of $G_6 \cup W$ with $B_2(2t + w)$ blocks. The resulting set of blocks is a bicover of $10d + 2t + w$ points with $B_2(10d + 2t + w)$ blocks. □

The set W of an incomplete bicover (V, W, F) is referred to as the *hole* of the bicover. An incomplete transversal design of index λ, denoted by $TD_\lambda(k, v) - TD_\lambda(k, w)$, is a triple (G, H, F), where G is a set $\{G_1, G_2, \ldots, G_k\}$ of disjoint v-subsets, H is a collection $\{H_1, H_2, \ldots, H_k\}$ of w-subsets called holes, with the property that $H_i \subset G_i, i = 1, 2, \ldots, k$, and if $S_1 = \cup_{i=1}^k G_i$ and $S_2 = \cup_{i=1}^k H_i$, then F is a family of k-subsets (called blocks) from S_1 which has the property that any pair of elements x_1 and x_2 in distinct G_i, where at least one of x_1 and x_2 does not lie in S_2, lies in precisely λ blocks, whereas no pair of distinct elements x_1 and x_2, where x_1 and x_2 both are in S_2, lies in any block. Clearly a $TD_1(k,v) - TD_1(k,w)$ is the same object as a $TD(k,v) - TD(k,w)$.

Lemma 4.3 *There exist incomplete bicovers $IB(29,7), IB(39,7), IB(49,7)$, $IB(59,7), IB(69,17)$, $IB(79,7)$, $IB(89,17)$, and $IB(109,27)$. Therefore $C_2(29) = B_2(29), C_2(39) = B_2(39)$, $C_2(49) = B_2(49)$, $C_2(59) = B_2(59)$, $C_2(69) = B_2(69)$, $C_2(79) = B_2(79)$, $C_2(89) = B_2(89)$, and $C_2(109) = B_2(109)$.*

Proof. In Lemma 3.6 it is shown that there exists a 2-$RBIBD(6s+4,4,2)$ for many positive integers including $s = 3$ and $s = 13$. By adjoining $2s+1$ "new" points $\infty_1, \infty_2, \ldots, \infty_{2s+1}$ points to such a 2-$RBIBD$, with ∞_i being adjoined to each block of the ith class, an incomplete bicover $IB(8s+5, 2s+1)$ is obtained. In particular, there exist incomplete bicovers $IB(29,7)$ and $IB(109,27)$.

To create an incomplete bicover $IB(39,7)$, proceed as follows. Let $V = \mathbb{Z}_{32}$, the group of integers (mod 32), and let H_1 denote the subgroup of even integers and H_2 the coset of odd integers in $\mathbb{Z}_{32}$. Let $W = \{x_1, x_2, \ldots, x_7\}$ where every x_i, $i = 1, 2, \ldots, 7$ is invariant under the action of $\mathbb{Z}_{32}$. Let F denote the following set of blocks.

0	4	10	24	31	(mod 32)
x_1	0	10	13	25	(translated by H_1)
x_2	0	10	13	25	(translated by H_2)
x_3	0	5	6	29	(translated by H_1)
x_4	0	5	6	29	(translated by H_2)
x_5	0	2	11	15	(translated by H_1)
x_6	0	2	11	15	(translated by H_2)
x_7	0	2	16	18	(translated by $\{0, 1, 2, \cdots, 15\}$)

Then (V, W, F) is an incomplete bicover.

To construct an incomplete bicover $IB(49,7)$, proceed as follows. Let $V = \mathbb{Z}_6 \times \mathbb{Z}_7$, where $\mathbb{Z}_6$ and $\mathbb{Z}_7$ are the cyclic groups of order six and seven respectively. Let $W = \{x_1, x_2, \ldots, x_7\}$. Let

$$\begin{aligned} \textit{Let}\quad S_1 &= \{(0,j)\ \ (1,j)\ \ (2,j)\ \ :\ j \in \mathbb{Z}_7\}; \\ S_2 &= \{(2,j)\ \ (3,j)\ \ (4,j)\ \ :\ j \in \mathbb{Z}_7\}; \\ S_3 &= \{(4,j)\ \ (5,j)\ \ (0,j)\ \ :\ j \in \mathbb{Z}_7\}; \\ S_4 &= \{(1,j)\ \ (3,j)\ \ (5,j)\ \ :\ j \in \mathbb{Z}_7\}. \end{aligned}$$

On each set S_i, $i = 1,2,3,4$, form a copy of a $BIBD(21,5,1)$. Then the resulting 84 blocks of size 5 contain every pair of the form $\{(i,j),(i,k)\}$, $j \neq k$, exactly twice, and each pair of the form $\{(i,j),(h,k)\}$, $i \neq h$, $i - h \neq 3 \pmod 6$ exactly once, and no other pair. Now let $H = \{(0,j),(2,j),(4,j) : j \in \mathbb{Z}_7\}$. To the above blocks adjoin the following collection:

x_1	(0,0)	(1,1)	(3,2)	(4,3)	(translated by H)
x_2	(0,0)	(1,2)	(3,4)	(4,6)	(translated by H)
x_3	(0,0)	(1,3)	(3,6)	(4,2)	(translated by H)
x_4	(0,0)	(1,4)	(3,1)	(4,5)	(translated by H)
x_5	(0,0)	(1,5)	(3,3)	(4,1)	(translated by H)
x_6	(0,0)	(1,6)	(3,5)	(4,4)	(translated by H)
x_7	(0,0)	(1,0)	(3,0)	(4,0)	(translated by H)

These blocks contain every pair $\{(i,j),(h,k)\}$, $i \neq h$, exactly once, except for pairs in which $i - h \equiv 3 \pmod 6$ which occur twice. Further each pair $\{x_k,(i,j) : k = 1,2,\ldots,7,\ i \in \mathbb{Z}_6, j \in \mathbb{Z}_7\}$ occurs in precisely two blocks. Therefore these are the blocks of an incomplete bicover.

To construct an incomplete bicover IB(59,7), proceed as follows. Let $V = \mathbb{Z}_{52}$ and $W = \{x_1, x_2, \ldots, x_7\}$ and let H_1 denote the subgroup of even integers and H_2 the coset of odd integers in $\mathbb{Z}_{52}$. Let F denote the following set of blocks.

0	4	16	20	28	(mod 52)
0	13	14	19	44	(mod 52)
0	3	9	14	32	(mod 52)
x_1	0	2	35	45	(translated by H_1)
x_2	0	2	35	45	(translated by H_2)
x_3	0	2	15	37	(translated by H_1)
x_4	0	2	15	37	(translated by H_2)
x_5	0	11	42	45	(translated by H_1)
x_6	0	11	42	45	(translated by H_2)
x_7	0	1	26	27	(translated by $\{0,1,2,\ldots,25\}$)

Then (V,W,F) is an incomplete bicover.

To construct an incomplete bicover (69, 17) proceed as follows. Let D be a resolvable $BIBD(52,4,1)$ and let $C_1, C_2, \ldots, C_{17}$ be its resolution classes. Let $W = \{x_1, x_2, \ldots, x_{17}\}$ be disjoint from the point-set of D. By adjoining x_i to each block of C_i, and taking each block twice, the blocks of an incomplete bicover are formed.

To construct an incomplete bicover (79, 7), proceed as follows. Let $V = \mathbb{Z}_{72}$ and $W = \{x_1, x_2, \ldots x_7\}$ and let H_1 denote the subgroup of even integers and H_2 the coset of odd integers in $\mathbb{Z}_{72}$. Let F denote the following set of blocks.

0	1	3	5	9	(mod 72)
0	3	13	18	38	(mod 72)
0	6	26	27	55	(mod 72)
0	8	18	32	53	(mod 72)
0	12	24	40	53	(mod 72)
x_1	0	11	42	49	(translated by H_1)
x_2	0	11	42	49	(translated by H_2)
x_3	0	9	25	42	(translated by H_1)
x_4	0	9	25	42	(translated by H_2)
x_5	0	7	22	33	(translated by H_1)
x_6	0	7	22	33	(translated by H_2)
x_7	0	14	36	50	(translated by $\{0,1,2,\ldots,35\}$)

Then (V, W, F) is an incomplete bicover.

To construct an $IB(89,17)$, proceed as follows. Let $V = \{(i,j) : i \in Z_{36}, j \in Z_2\}$, and let $W = \{[h,j], : h \in Z_8, j \in Z_2\} \cup \{x\}$. Let σ and τ be the mappings given by $\sigma x = \tau x = x$, $\sigma[h,j] = [h,j]$, $\sigma(i,j) = (i+1,j)$, $\tau[h,j] = [h,j+1]$, $\tau(i,j) = (i,j+1)$. Let G be the group of order 72 generated by σ and τ.

There are 18 distinct blocks obtained by applying the elements of G to the base block

$$x \quad (0,0) \quad (9,0) \quad (18,0) \quad (27,0).$$

We take each of these 18 blocks twice. In addition we take the 720 blocks obtained by applying the elements of G to the following 10 base blocks:

[0,0]	(0,0)	(1,0)	(3,1)	(5,1)
[1,0]	(0,0)	(4,0)	(11,1)	(16,1)
[2,0]	(0,0)	(6,0)	(0,1)	(10,1)
[3,0]	(0,0)	(7,0)	(15,1)	(28,1)
[4,0]	(0,0)	(8,0)	(13,1)	(25,1)
[5,0]	(0,0)	(10,0)	(1,1)	(22,1)
[6,0]	(0,0)	(11,0)	(13,1)	(29,1)
[7,0]	(0,0)	(13,0)	(3,1)	(22,1)
(0,0)	(1,0)	(3,0)	(7,0)	(15,0)
(1,0)	(6,0)	(17,0)	(20,0)	(0,1).

If F denotes the set of blocks above, then (V, W, F) is an incomplete bicover.

The relation $C_2(v) = B_2(v)$ is obtained for $v \in \{29, 39, 49, 59, 69, 79, 89, 109\}$ by adjoining the blocks of an appropriate bicover on the set W in each case.
□

Lemma 4.4 *Let* $v \equiv 9 (\mathrm{mod}\ 50)$. *If* $v \neq 9$, *then* $C_2(v) = B_2(v)$. *Further* $C_2(9) = B_2(9) + 1 = 9$.

Proof. It is readily verified that $C_2(9) = B_2(9) + 1 = 9$. (Such a cover can be obtained from the blocks of 0 1 3 4 6 (mod 9).) For $v \geq 409$, the result follows from Lemma 3.2 (the case $m \equiv 0$ (mod 5) and $t = 29$), and the fact that $C_2(59) = B_2(59)$. Also, by Lemma 4.3, $C_2(109) = B_2(109)$. The cases $v = 309$ and 359 follow from Lemma 3.2 and the fact that $C_2(39) = B_2(39)$ by taking $t = 19$ and $m = 27$ and 32 respectively. This leaves only the cases of $v \in \{159, 209, 259\}$. Cases 159 and 209 follow from the existence of resolvable *BIBDs* with parameters $v = 65, k = 5, \lambda = 1$ and $v = 85, k = 5, \lambda = 1$ and the equations $159 = 2.65 + 29$ and $209 = 2.85 + 39$ respectively, using Lemma 3.3.

For $v = 259$, use Lemma 4.2 and the fact that there exists an incomplete bicover $IB(49, 7)$ and a $TD(6, 21)$. Since $259 = 10.21 + 49$, it follows from Lemma 4.2 that $C_2(259) = B_2(259)$. □

Lemma 4.5 *Let* $v \equiv 19 (\mathrm{mod}\ 50)$. *Then* $C_2(v) = B_2(v)$.

Proof. If $v \geq 19$ and $v \equiv 19$ (mod 100) the result follows from Corollary 4.1.1.

For $v = 69$, the result follows from Lemma 4.3. For $v \equiv 69 (\mathrm{mod}\ 100)$ and $v \geq 169$, the result follows from Lemma 3.2 (the case $m \equiv 0 (\mathrm{mod}\ 5)$ and $t = 9$). □

Lemma 4.6 *Let* $v \equiv 29 (\mathrm{mod}\ 50)$. *Then* $C_2(v) = B_2(v)$.

Proof. For $v \geq 179$, $v \neq 329$, the result follows from Lemma 3.2 (the case $m \equiv 0 (\mathrm{mod}\ 5)$ and $t = 14$). For $v = 329$, begin with a $TD(6, 15)$. Since there exists a $BIBD(21, 5, 1)$ and $BIBD(25, 5, 1)$, there exists a $GDD(4^5, \{5\}, 1)$ and a $GDD(4^6, \{5\}, 1)$ obtained by deleting a point from each of these designs. Doubling each block of these group divisible designs produces a $GDD(4^5, \{5\}, 2)$ and $GDD(4^6, \{5\}, 2)$. Deleting 8 points from one group of the transversal design, and inflating the remaining points by a factor of four yields a $GDD(60^5 28^1, \{5\}, 2)$, say D'. Let ∞ be a point not occurring in this group divisible design. For each group G of size 60 of D', adjoin the blocks of a $BIBD(61, 5, 2)$ defined on $G \cup \{\infty\}$ to the blocks of D', then adjoin the blocks of a $B_2(29)$ bicover of $G * \cup \{\infty\}$, where $G*$ is the group of size 28. The resulting set of blocks is a bicover of 329 points with $B_2(329)$ blocks, showing that $C_2(329) = B_2(329)$.

For $v = 29$ and 79, the result follows from Lemma 4.3.

The case $v = 129$ can be treated as follows. A pairwise balanced design D of index $\lambda = 1$ of order 129 with precisely one block of size 29 and all other blocks of size 5 was constructed in [12]. If each block of size 5 of D is replaced by two copies of itself, and if the block of size 29 is replaced by a bicover of its underlying set which contains $B_2(29)$ blocks, the resulting set of blocks of size 5 has excess frequency 8, and is therefore a bicover of order 129 which contains $B_2(129)$ blocks, hence $C_2(129) = B_2(129)$.

Lemma 4.7 *Let* $v \equiv 39 (\bmod\ 50)$. *Then* $C_2(v) = B_2(v)$.

Proof. For $v \geq 239$, $v \neq 339$, the result follows from Lemma 3.2 (the case $m \equiv 0 (\bmod\ 5)$) and $t = 19$) and the fact that $C_2(39) = B_2(39)$.

For $v = 39$ and 89, the result follows from Lemma 4.3.

For $v = 139$, 189, and 339, the result follows from Lemma 3.2 with $t = 9$ and $m = 12$, 17, and 32 respectively.

This establishes the lemma. □

Lemma 4.8 *Let* $v \equiv 49 (\bmod\ 50)$. *Then* $C_2(v) = B_2(v)$.

Proof. For $v \equiv 99\ (\bmod\ 100)$, the result follows from Corollary 4.1.1.

For $v \equiv 49\ (\bmod\ 100)$, the result follows from Lemma 3.2 (the case $m \equiv 0\ (\bmod\ 5)$) and the fact that $C_2(49) = B_2(49)$, except for $v \in \{149, 249, 349\}$. For $v = 149$, the result follows from the fact that there is a $2-RBIBD(112, 4, 2)$ (obtainable by doubling the $RBIBD(112, 4, 1)$), and therefore there exists an incomplete bicover $IB(149, 37)$, whose "hole" can be filled. For $v = 249$, use Lemma 4.2 and the facts that there exists an $IB(49, 7)$ and a $TD(6, 21)$. Since $249 = 10.21 + 39$, it follows that $C_2(249) = B_2(249)$. For $v = 349$, use Lemma 3.2 with $m = 32$ and $t = 14$. □

The foregoing can be summarized as follows.

Theorem 4.9 Let v be a positive integer congruent to 9 (mod 10). Then $C_2(9) = B_2(9) + 1$, and if $v > 9$, then $C_2(v) = B_2(v)$.

5 Odd bicovers for odd v.

Since there exists a $BIBD(v, 5, 2)$ for all $v \equiv 1$ or $5\ (\bmod\ 10)$, $v \geq 5$, $v \neq 15$, it follows that $C_2(v) = B_2(v)$ for these values.

Lemma 5.1 $C_2(15) = 22 = B_2(15) + 1$.

Proof. It is well-known that there does not exist a $BIBD(15, 21, 7, 5, 2)$. Hence $C_2(15) > 21$. To show that $C(15) = 22$, let the set of points be $\{(i, j) : i \in Z_5, j \in Z_3\}$. Then the set of 22 blocks exhibited below form a bicover.

$$\begin{array}{llllllll}
(0,h) & (1,h) & (2,h) & (3,h) & (4,h) & h=0,1 & \\
(0,0) & (g,0) & (0,1) & (2g,1) & (3g,2) & g=1,2 & \mathrm{mod}(5,-) \\
(0,0) & (3g,1) & (0,2) & (2g,2) & (3g,2) & g=1,2 & \mathrm{mod}(5,-).
\end{array}$$

This establishes the lemma. □

The foregoing can be summarized as follows.

Theorem 5.2 Let v be an integer satisfying $v \geq 5$, $v \equiv 1 \pmod 2$, $v \not\equiv 3 \pmod{10}$, and $v \notin \{9, 15\}$. Then $C_2(v) = B_2(v)$. If $v = 9$ or 15, then $C_2(v) = B_2(v) + 1$.

If we analyze our constructions carefully we obtain the following result.

Theorem 5.3 There is an $IB(v, 7)$ in the following cases:

(i) $v \equiv 7$ or $29 \pmod{50}$;

(ii) $v \equiv 9 \pmod{50}$, $v \neq 9, 109$;

(iii) $v \equiv 39 \pmod{50}$, $v \neq 89, 139, 189, 339$;

(iv) $v \equiv 49 \pmod{100}$, $v \neq 149$;

(v) $v = 67, 87, 127, 327$.

Furthermore, there is an $IB(v, 17)$ if $v = 69, 89$, or if $v \equiv 37 \pmod{50}$, $v \neq 37, 87, 237, 437, 537$. There is an $IB(v, 19)$ if $v = 77, 139, 189, 339$, or if $v \equiv 69 \pmod{100}$, $v \neq 69$. There is an $IB(v, 27)$ if $v = 109$ or if $v \equiv 27 \pmod{50}$, $v \neq 77, 127, 327$. There is an $IB(v, 37)$ if $v = 149$ or if $v \equiv 37 \pmod{50}$, $v \neq 87, 137, 187, 337$.

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