Steiner Heptagon Systems

Elisabetta Manduchi

Dipartimento di Matematica[1]
Università di Roma "La Sapienza"
I-00185 Roma, Italia

Abstract. Steiner Heptagon Systems (SHS) of type 1, 2, and 3 are defined and the spectrum of type 2 SHSs (SHS2) is studied. It is shown that the condition $n \equiv 1$ or 7 (mod 14) is not only necessary but also sufficient for the existence of an SHS2 of order n, with the possible exceptions of n=21 and 85. This gives an interesting algebraic result since the study of SHS2s is equivalent to the study of quasigroups satisfying the identities $x^2 = x$, $(yx)x = y$, and $(xy)(y(xy)) = (yx)(x(yx))$.

0. Introduction

In [1] C.C. Lindner and D.R. Stinson define and investigate Steiner Pentagon Systems. This paper extends their ideas by studying Steiner Heptagon Systems (SHSs). It turns out (Section 1) that there are three possible types of such systems and in each case a suitable quasigroup is associated with the system. The most interesting type appears to be type 2 and the aim of this paper is to investigate SHS2s. The necessary condition $n \equiv 1$ or 7 (mod 14) is shown to be sufficient with the possible exceptions of $n = 21$ and 85. The main constructions use certain types of orthogonal quasigroups (with holes). The cases not handled by the main constructions are taken care of by brute force examples, finite fields and direct products. The main constructions are inspired by those in [1]. From the algebraic point of view, the study of SHS2s is equivalent to the study of the class of quasigroups satisfying the 2-variable identities $x^2 = x, (yx)x = y$, and $(xy)(y(xy)) = (yx)(x(yx))$.

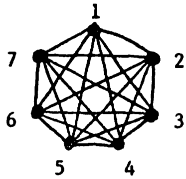
1. Heptagon Systems

A *Heptagon System* (HS) is a pair (K_n, E), where K_n is the complete (undirected) graph with $n \geq 7$ vertices and E is a collection of edge disjoint heptagons which partition the edges of K_n.

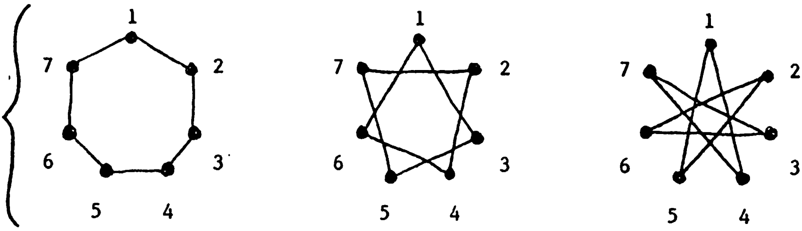
[1] Present Address: Department of Mathematics, University of Maryland, College Park, MD 20742, USA

Example 1.1 (K_7, E).

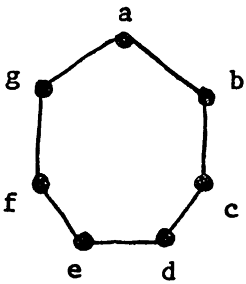
$K_7 =$



$E =$



So that the pictures do not get out of hand, from here on we will denote the heptagon (below) by any cyclic shift of (a, b, c, d, e, f, g) or (b, a, g, f, e, d, c).



Example 1.2.(K_{15}, E_1)

$$E_1 = \left\{ \begin{array}{l} (1,4,10,8,13,6,5) \\ (2,5,11,9,14,7,6) \\ (3,6,12,10,15,8,7) \\ (4,7,13,11,1,9,8) \\ (5,8,14,12,2,10,9) \\ (6,9,15,13,3,11,10) \\ (7,10,1,14,4,12,11) \\ (8,11,2,15,5,13,12) \\ (9,12,3,1,6,14,13) \\ (10,13,4,2,7,15,14) \\ (11,14,5,3,8,1,15) \\ (12,15,6,4,9,2,1) \\ (13,1,7,5,10,3,2) \\ (14,2,8,6,11,4,3) \\ (15,3,9,7,12,5,4) \end{array} \right.$$

If (K_n, E) is a HS, then the number n is called the *order* of the system and it is straightforward to see that the number of heptagons is $|E| = \frac{n(n-1)}{14}$. Moreover, the number of heptagons containing a vertex x is obviously $\frac{n-1}{2}$ so n must be odd. Hence: $n = 2m + 1, (2m + 1)2m = 14k, \Rightarrow (2m + 1)m = 7k, \Rightarrow 7|m$ or

$7|(2m+1) \Rightarrow n \equiv 1$ or 7 (mod 14). So a necessary condition for the existence of a HS of order n is $n \equiv 1$ or 7 (mod 14).

Given a HS (K_n, E), we can define a binary operation on the set $Q = \{1, 2, \cdots, n\}$ in three reasonable ways:

(i) $\forall x \in Q, x \circ_1 x = x$, and $\forall x, y \in Q, x \neq y, x \circ_1 y = y \circ_1 x = z$ iff $(x, y, a, b, z, c, d) \in E$. The groupoid $(Q, \circ_1)$ is commutative and idempotent.

(ii) $\forall x \in Q, x \circ_2 x = x$, and $\forall x, y \in Q, x \neq y, x \circ_2 y = z$ and $y \circ_2 x = v$ iff $(x, y, z, a, b, c, v) \in E$. The groupoid $(Q, \circ_2)$ is idempotent but not commutative.

(iii) $\forall x \in Q, x \circ_3 x = x$, and $\forall x, y \in Q, x \neq y, x \circ_3 y = z$ and $y \circ_3 x = v$ iff $(x, y, a, z, b, v, c) \in E$. Again the groupoid $(Q, \circ_3)$ is idempotent but not commutative.

Example 1.3. The tables of (K_7, E) (Example 1.1) for $\circ_1, \circ_2$, and $\circ_3$ are the following:

$\circ_1$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

$\circ_2$	1	2	3	4	5	6	7
1	1	3	5	7	2	4	6
2	7	2	4	6	1	3	5
3	6	1	3	5	7	2	4
4	5	7	2	4	6	1	3
5	4	6	1	3	5	7	2
6	3	5	7	2	4	6	1
7	2	4	6	1	3	5	7

$\circ_3$	1	2	3	4	5	6	7
1	1	4	7	3	6	2	5
2	6	2	5	1	4	7	3
3	4	7	3	6	2	5	1
4	2	5	1	4	7	3	6
5	7	3	6	2	5	1	4
6	5	1	4	7	3	6	2
7	3	6	2	5	1	4	7

In this example the groupoids $(Q, \circ_1)$, $(Q, \circ_2)$, and $(Q, \circ_3)$ are quasigroups. That is to say, each element of Q occurs exactly once in each row and column in each table.

Proposition 1.4. *If (K_n, E) is a HS and $(Q, \circ_1)$, $(Q, \circ_2)$, and $(Q, \circ_3)$ are the associated groupoids, then:*

(i) *$(Q, \circ_1)$ is a quasigroup iff every pair of vertices are joined by a path of length 3 in exactly one heptagon of E.*

(ii) *$(Q, \circ_2)$ is a quasigroup iff every pair of vertices are joined by a path of length 2 in exactly one heptagon of E.*

(iii) $(Q, \circ_3)$ *is a quasigroup iff every pair of vertices are joined by a path of length 2 in exactly one heptagon of* E *and by a path of length 3 in exactly one heptagon of* E.

Proof: It immediately follows from the given definitions of the binary operations. ∎

Definition 1.5 A *Steiner Heptagon System* of type 1 (SHS1) is a HS (K_n, E) such that the groupoid $(Q, \circ_1)$ associated with it is a quasigroup. Analogously we define a *Steiner Heptagon System* of type 2 (SHS2) and of type 3 (SHS3).

By the above proposition, every SHS3 is a SHS1 and a SHS2. The HS (K_7, E) of Example 1.1 is a SHS3. The HS (K_{15}, E_1) of Example 1.2 is SHS2 but it is not an SHS1 (and consequently not a SHS3) because $1 \circ_1 3 = 13 = 1 \circ_1 4$.

2. SHS2s

If (K_n, E) is a SHS2, then the quasigroup $(Q, \circ_2)$ associated with it satisfies the following identities:

(2.1) $\forall x \in Q, x^2 = x$,

(2.2) $\forall x, y \in Q, (yx)x = y$, and

(2.3) $\forall x, y \in Q, (xy)(y(xy)) = (yx)(x(yx))$.

From (2.2) and (2.3), by replacing x with y and y with xy we obtain:

(2.4) $\forall x, y \in Q, (y(xy))((xy)(y(xy))) = x(yx)$.

Theorem 2.5. *If* $(Q, \circ)$ *is a quasigroup of order* n *which satisfies (2.1), (2.2), and (2.3), then there exists a SHS2* (K_n, E) *of order* n.

Proof: For each pair $\{a, b\} \subseteq Q$ with $a \neq b$ we put in E the heptagon

$$(a, b, ab, b(ab), (ab)(b(ab)) = (ba)(a(ba)), a(ba), ba).$$

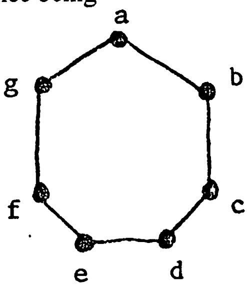
The points $a, b, ab, b(ab), (ab)(b(ab)), a(ba)$, and ba are distinct and each of the seven edges $\{a, b\}$, $\{b, ab\}$, $\{ab, b(ab)\}$, $\{b(ab), (ab)(b(ab))\}$, $\{(ab)(b(ab)), a(ba)\}$, $\{a(ba), ba\}$, and $\{ba, a\}$ defines the same heptagon. ∎

Thus that study of the *spectrum* of SHS2s is equivalent to the study of the spectrum of quasigroups satisfying (2.1), (2.2), and (2.3). In what follows we shall show that the condition of $n \equiv 1$ or 7 (mod 14) is also sufficient for the existence of a SHS2 of order n, with two possible exceptions.

Definition 2.6. We define a *cyclic semi-perpendicular array* (CSPA) of order n and strength k to be an $\binom{n}{2} \times k$ array A such that each cell is occupied with one of the symbols $1, 2, \cdots, n$ and such that if we run our fingers down any two columns that are adjacent (the first and the last columns are considered adjacent) or that are at distance 2 (the second and the last columns and the first and the

$(k-1)^{\text{th}}$ columns are considered at distance 2) we obtain each of the $\binom{n}{2}$ 2 element subsets of $\{1,2,\cdots,n\}$ at least (therefore exactly) once. Moreover we require that A is invariant with respect to cyclically permuting the columns according to the permutation $\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. That is to say, when the columns are permuted according to α the resulting array contains exactly the same rows (but not necessarily at the same level, or course).

Observation. A CSPA(n, 7) A is equivalent to a SHS2 (K_n, E); the equivalence being



ε E iff (a,b,c,d,e,f,g),

iff (a,b,c,d,e,f,g),(b,c,d,e,f,g,a), (c,d,e,f,g,a,b), (d,e,f,g,a,b,c), (e,f,g,a,b,c,d),(f,g,a,b,c,d,e), and (g,a,b,c,d,e,f) are rows of A.

In what follows it is a good deal easier to describe the construction of SHS2s in terms of CSPAs than in terms of graph theory, and so we switch over to CSPA vernacular.

3. $n \equiv 7$ (mod 14) $n > 21$.

Theorem 3.1. *Let $n \equiv 7$ (mod 14) and $n > 21$. Then there exists a CSPA(n, 7).*

Proof: Write $n = 7(2k+1)$. Let $(Q,\circ)$ and $(Q,\otimes)$ be a pair of *orthogonal* idempotent quasigroups of order $m = 2k+1$. Further let $(Q,\otimes)$ be commutative. Such a pair of quasigroups exists for every $m = 2k+1 \geq 5$ [2], [3]. Denote by A the $(m^2-m)\times 7$ array with rows $(a,b,a\circ b,a,a\otimes b,b,b\circ a)$, all $a \neq b \in Q$. Further, denote by T the $\frac{m(m-1)}{2}$ rows of A with $a < b$, and by B the $\frac{m(m-1)}{2}$ rows of A with $a > b$. Note that $(a,b,c,d,e,f,g) \in T$ iff $(b,a,g,f,e,d,c) \in B$. Finally, let $C = \{(1,1,2,4,7,4,2),(2,2,3,5,1,5,3),(3,3,4,6,2,6,4),(4,4,5,7,3,7,5),(5,5,6,1,4,1,6),(6,6,7,2,5,2,7),(7,7,1,3,6,3,1)\}$ and set $X = Q\times\{1,2,3,4,5,6,7\}$.

Now define a $\binom{7m}{2}\times 7$ array P based on $X = Q\times\{1,2,3,4,5,6,7\}$ by:

(1) For each $a \in Q$, define a CSPA(7,7) on $\{(a,1),(a,2),(a,3),(a,4),(a,5),(a,6),(a,7)\}$ (which exists because of the existence of a SHS2 of order 7) and place these 21 rows in P, and

(2) for each row $(a,b,a\circ b,a,a\otimes b,b,b\circ a) \in T$ and each $(i,i,j,k,t,k,j) \in C$ place the 7 rows

$((a,i),(b,i),(a\circ b,j),(a,k),(a\otimes b,t),(b,k),(b\circ a,j))$,
$((b,i),(a\circ b,j),(a,k),(a\otimes b,t),(b,k),(b\circ a,j),(a,i))$,
$((a\circ b,j),(a,k),(a\otimes b,t),(b,k),(b\circ a,j),(a,i),(b,i))$,
$((a,k),(a\otimes b,t),(b,k),(b\circ a,j),(a,i),(b,i),(a\circ b,j)$,
$((a\otimes b,t),(b,k),(b\circ a,j),(a,i),(b,i),(a\circ b,j),(a,k))$,
$((b,k),(b\circ a,j),(a,i),(b,i),(a\circ b,j),(a,k),(a\otimes b,t))$, and
$((b\circ a,j),(a,i),(b,i),(a\circ b,j),(a,k),(a\otimes b,t),(b,k))$ in P.

Claim: P is a CSPA($7m,7$). It is clear that P is cyclic and has $\frac{7m(7m-1)}{2}$ rows. It remains to show that if we run our fingers down any two columns of P that are at a distance 1 or 2, we obtain each 2-element subset of X at least once. So, consider the 2-element subset $\{(a,i),(b,j)\}$ of X. If $a = b$, since the rows of a CSPA($7,7$) defined on $\{(a,1),(a,2),(a,3),(a,4),(a,5),(a,6),(a,7)\}$ belong to P we are done.

If $a \neq b$ there are two cases.

Case 1. $a \neq b$, $i = j$. It suffices to show that P contains a row of the form $((a,i),(b,i),.,.,.,.,.)$ or $((b,i),(a,i),.,.,.,.,.)$ and a row of the form $(.,.,.,(a,k),.,(b,k),.)$ or $(.,.,.,(b,k),.,(a,k),.)$. A must contain a row of the form
(a,b,c,a,d,b,e). If this row is in T then

$$((a,i),(b,i),(c,j),(a,k),(d,t),(b,k),(e,j)) \in P;$$

otherwise $(b,a,e,b,d,a,c) \in T$ and

$$((b,i),(a,i),(e,j),(b,k),(d,t),(a,k),(c,j)) \in P.$$

Case 2. $a \neq b, i \neq j$. It suffices to consider the cases $i = 1, j = 2; i = 1, j = 4; i = 2, j = 7$.

Case 2a. $i = 1, j = 2$. It suffices to show that P contains a row of the form $(.,(a,1),(b,2),.,.,.,.)$ or $((a,1),.,.,.,.,.,(b,2))$ and a row of the form $((a,1),.,(b,2),.,.,.,.)$ or $(.,(a,1),.,.,.,.,(b,2))$. Since $(Q,\circ)$ is a quasigroup, there exists in A a row of the form (x,a,b,x,d,a,c). If this row belongs to T then

$$((x,1),(a,1),(b,2),(x,4),(d,7),(a,4),(c,2)) \in P; \text{ otherwise}$$

$(a,x,c,a,d,x,b) \in T$ and

$$((a,1),(x,1),(c,2),(a,4),(d,7),(x,4),(b,2)) \in P.$$

Since $(Q, \circ)$ is a quasigroup there exists in A a row of the form (a, x, b, a, d, x, c). If this row is in T then

$$((a,1),(x,1),(b,2),(a,4),(d,7),(x,4),(c,2)) \in P;$$

otherwise $(x, a, c, x, d, a, b) \in T$ and

$$((x,1),(a,1),(c,2),(x,4),(d,7),(a,4),(b,2)) \in P.$$

Case 2b. $i = 1, j = 4$. It suffices to show that P contains a row of the form $(.,.,.,(a,1),(b,4),.,.)$ or $(.,.,.,.,(b,4),(a,1),.)$ and a row of the form $(.,(a,1),.,(b,4),.,.,.)$ or $((a,1),.,.,.,.,(b,4),.)$. Since $(Q, \otimes)$ is a quasigroup, there exists in A a row of the form (a, x, c, a, b, x, d). If this row is in T then

$$((a,5),(x,5),(c,6),(a,1),(b,4),(x,1),(d,6)) \in P;$$

otherwise $(x, a, d, x, b, a, c) \in P$ and

$$((x,5),(a,5),(d,6),(x,1),(b,4),(a,1),(c,6)) \in P.$$

There exists in A a row of the form (a, b, c, a, d, b, e). If this row is in T, then

$$((a,1),(b,1),(c,2),(a,4),(d,7),(b,4),(e,2)) \in P;$$

otherwise $(b, a, e, b, d, a, c) \in T$ and

$$((b,1),(a,1),(e,2),(b,4),(d,7),(a,4),(c,2)) \in P.$$

Case 2c. $i = 2, j = 7$. It suffices to show that P contains a row of the form $(.,.,(b,7),(a,2),.,.,.)$ or $(.,.,.,.,.,(a,2),(b,7))$ and a row of the form $(.,.,(a,2),.,(b,7),.,.)$ or $(.,.,.,.,(b,7),.,(a,2))$. Since $(Q, \circ)$ is a quasigroup, there exists in A a row of the form (a, x, b, a, d, x, c). If this row belongs to T then

$$((a,6),(x,6),(b,7),(a,2),(d,5),(x,2),(c,7)) \in P;$$

otherwise $(x, a, c, x, d, a, b) \in T$ and

$$((x,6),(a,6),(c,7),(x,2),(d,5),(a,2),(b,7)) \in P.$$

Since $(Q, \circ)$ and $(Q, \otimes)$ are orthogonal, there exists in A a row of the form (x, y, a, x, b, y, c). If this row is in T, then

$$((x,1),(y,1),(a,2),(x,4),(b,7),(y,4),(c,2)) \in P;$$

otherwise $(y, x, c, y, b, x, a) \in T$ and

$$((y,1),(x,1),(c,2),(y,4),(b,7),(x,4),(a,2)) \in P. \quad \blacksquare$$

4. $n \equiv 1$ **(mod 14)**

Write $n = 7(2k) + 1$. Let Q be a set of the size $m = 2k$ and $H = \{h_1, \cdots, h_t\}$ a partition of Q with the property that $7|h_i| + 1$ is in the spectrum of SHS2s. Let $(Q, \circ)$ and $(Q, \otimes)$ be a pair of orthogonal quasigroups with holes H and further let $(Q, \otimes)$ be commutative. Denote by A the $(m^2 - 2m) \times 7$ array with rows $(a, b, a \circ b, a, a \otimes b, b, b \circ a)$, all $a, b \in Q$ and a and b do not belong to the same hole. Denote by $T(B)$ the $\frac{m(m-2)}{2}$ rows of A with $a < b (a > b)$. Set $X = \{\infty\} \cup (Q \times \{1,2,3,4,5,6,7\})$ and define a CSPA(n, 7) whose rows are elements of X by:

(1) For each hole $h_i \in H$ construct a CSPA($7|h_i| + 1, 7$) on $\{\infty\} \cup (h_i \times \{1,2,3,4,5,6,7\})$ and place these rows in P, and

(2) for $x \neq y$ in different holes of H and each $(i, i, j, k, t, k, j) \in C$ (see the previous section) we proceed in the same way as in the $14k + 7$ Construction.

Since there exists a pair of quasigroups with the above properties with holes of size two of order $m = 2k$ for every m except when

$$m \in S = \{4,6,8,12,16,20,24,28,30,32,36,40, 44,48,52,56,60,66,68,72,76,80,84,88,92,96,100, 104,108,116,124,136,144,148,152,160,168,176,216,228\}$$

[2] ,[3] we have a SHS2 of order $7m + 1$ for every even $m \in S$. ∎

What follows provides a solution for $m = 2k \in S$ and $m \neq 12$. We split the solution into three self-explanatory parts.

5. The remaining cases

$m = 8, n = 57; 57 = 7 \cdot 8 + 1$ We exhibit two orthogonal quasigroups of order 8, with holes of size two, one of which is commutative:

1	2	7	8	4	3	6	5
2	1	5	6	8	7	4	3
8	6	3	4	7	2	5	1
7	5	4	3	1	8	2	6
3	7	8	2	5	6	1	4
4	8	1	7	6	5	3	7
5	3	6	1	2	4	7	8
6	4	2	5	3	1	8	7

1	2	5	6	7	8	3	4
2	1	8	7	3	4	6	5
5	8	3	4	1	7	2	6
6	7	4	3	8	2	5	1
7	3	1	8	5	6	4	2
8	4	7	2	6	5	1	3
3	6	2	5	4	1	7	8
4	5	6	1	2	3	8	7

Finite field construction

This construction provides a solution for $m \in \{4,6,30,48,60,66,96\}$. Consequently $n \in \{29,43,211,337,421,463,673\}$ and $n = 14k + 1$. Now, let $(F,+,\cdot)$ be finite field of order $n = 14k + 1$ and let x be a primitive element of F. Set

$$B = \{(x^i, x^{2k+i}, x^{4k+i}, x^{6k+i}, x^{8k+i}, x^{10k+i}, x^{12k+i}) \,|\, i = 0,1,\cdots,k-1\}$$

and define an $\binom{n}{2} \times 7$ array A by: for each $\alpha \in F$ and each $(x,y,z,u,w,t,v) \in B$, place the 7 rows

$$(x+\alpha, y+\alpha, z+\alpha, u+\alpha, w+\alpha, t+\alpha, v+\alpha),$$
$$(y+\alpha, z+\alpha, u+\alpha, w+\alpha, t+\alpha, v+\alpha, x+\alpha),$$
$$(z+\alpha, u+\alpha, w+\alpha, t+\alpha, v+\alpha, x+\alpha, y+\alpha),$$
$$(u+\alpha, w+\alpha, t+\alpha, v+\alpha, x+\alpha, y+\alpha, z+\alpha),$$
$$(w+\alpha, t+\alpha, v+\alpha, x+\alpha, y+\alpha, z+\alpha, u+\alpha),$$
$$(t+\alpha, v+\alpha, x+\alpha, y+\alpha, z+\alpha, u+\alpha, w+\alpha), \text{and}$$
$$(v+\alpha, x+\alpha, y+\alpha, z+\alpha, u+\alpha, w+\alpha, t+\alpha) \text{in } A.$$

Then A is a CSPA$(n,7)$. ∎

Direct product construction

This construction provides a solution for

$$m \in \{20,24,28,32,36,40,44,52, 56,64,68,72,76,80,84,88,92,100,104,108, 116,124,136,144,148,152,160,168,176,216,228\}.$$

We give the construction for $m = 20$ and 24. For the other cases see Table 5.1.

$m = 20, n = 141 : 141 = 7 \cdot 20 + 1$. Take a pair of (idempotent) orthogonal quasigroups of order 5 say A and B. Further let B be commutative. Take the pair of orthogonal quasigroups C and D defined as follows.

C

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

D

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

The direct products $A \times C$ and $B \times D$ are orthogonal quasigroups of order 20, with holes of size 4. Moreover, $B \times D$ is commutative. ∎

$m = 24, n = 169 : 169 = 7 \cdot 24 + 1$. Let A and B be orthogonal quasigroups of order 8 (see case $m = 8$). Let

$$C = \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \quad \text{and} \quad D = \begin{matrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{matrix}.$$

Then $A \times C$ and $B \times D$ are orthogonal quasigroups of order 24, with holes of size 6 and $B \times D$ is commutative. ∎

Table 5.1

m	$n = 7 \cdot m + 1$	orders of quasigroups used for the direct products	size of the holes of the orthogonal quasigroups
28	197	4, 7	4
32	225	4, 8	8
36	253	4, 9	4
40	281	5, 8	8
44	304	4, 11	4
52	365	4, 13	4
56	393	7, 8	8
64	449	8, 8	16
68	477	4, 17	4
72	505	8, 9	8
76	533	4, 19	4
80	561	5, 16	16
84	589	4, 21	4
88	617	8, 11	8
92	645	4, 23	4
100	701	4, 25	4
104	729	8, 13	8
108	757	4, 27	4
116	813	4, 29	4
124	869	4, 31	4
136	953	8, 17	8
144	1009	9, 16	16
148	1037	4, 37	4
152	1065	8, 19	8
160	1121	5, 32	32
168	1176	7, 24	24
176	1233	11, 16	16
216	1513	9, 24	24

Theorem 5.1. *There exists a CSPA*$(n,7)$ *for every* $n \equiv 1$ *(mod 14), except possibly* $n = 85$.

Combining Theorems 3.1 and 5.1 gives the following theorem, which is the main result of this paper.

Theorem 5.2. *The spectrum for CSPA*$(n,7)$*s is precisely the set of all* $n \equiv 1$ *or 7 (mod 14), except possibly* $n = 21$ *and 85.*

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