RAMSEY MINIMAL GRAPHS FOR FORESTS

RALPH FAUDREE

Memphis State University

Abstract. Essentially all pairs of forests (F_1, F_2) are determined for which $\mathcal{R}(F_1, F_2)$ is finite, where $\mathcal{R}(F_1, F_2)$ is the class of minimal Ramsey graphs for the pair (F_1, F_2).

INTRODUCTION

For graphs F, G and H (finite with no loops or multiple edges), we write $F \to (G, H)$ (F "arrows" the pair G, H)) if whenever each edge of F is colored red or blue, then either the red subgraph of F, denoted $(F)_R$, contains a copy of G or the blue subgraph of F, denoted F_B, contains a copy of H. The graph F is called (G, H)−minimal if $F \to (G, H)$ but $F' \not\to (G, H)$ for each proper subgraph F' of F. The class of all (G, H)−minimal graphs will be denoted by $\mathcal{R}(G, H)$, and is called the Ramsey class of the pair (G, H).

There have been several papers dealing with the problem of determining for which pair of graphs (G, H) is $\mathcal{R}(G, H)$ infinite (or finite), see ([1]-[6],[8],[9]). For example, Nesetril and Rödl [9] proved that $\mathcal{R}(G, H)$ is infinite if both G and H are 3-connected or if both are 3-chromatic. The main result of the present paper extends the results in [2], where star forests without isolated edges were considered. We will deal with forests with isolated edges, and determine all pairs of forests (F_1, F_2) such that $\mathcal{R}(F_1, F_2)$ is finite. In the statement of the main theorem which follows and throughout the remainder of the paper, a star with n edges will be denoted by $S(n)$, instead of the usual notation $K_{1,n}$. Also for graphs G and H, $G \cup H$ will denote the graph with vertex-disjoint copies of G and H, and for any positive integer n, nG will denote n vertex-disjoint copies of G.

Theorem 1:. *If F_1 and F_2 are forests, then $\mathcal{R}(F_1, F_2)$ (or $\mathcal{R}(F_2, F_1)$) is finite if and only if F_1 and F_2 are both star forests such that,*

$$F_1 = \bigcup_{i=1}^{s} S(m_i) \cup mS(1),\ m_1 \geq \cdots \geq m_s \geq 2,\ m \geq 0,$$

$$F_2 = \bigcup_{i-1}^{t} S(n_i) \cup nS(1),\ n_1 \geq \cdots \geq n_t \geq 2,\ n \geq 0, s \geq t \geq 0,$$

ARS COMBINATORIA 31(1991), pp. 117-124

and one of the following conditions hold:

(1) $t = 0$, $n > 0$

(2) $s = t = 1$ *and* m_1, n_1 *are odd*

(3) $s \geq 2$, $t = 1$, m_1, n_1 *are odd,*
$m_1 \geq n_1 + m_2 - 1$ *and* $n \geq n_0 = n_0(F_1, F_2)$.

Some additional notation and terminology will be needed. The word "coloring" will always refer to coloring each edge of some graph either red or blue. A coloring of F with neither a red G nor a blue H will be called (G, H)–good, or just a good coloring when the meaning is clear. For a graph F, $V(F)$, $E(F)$ and $\Delta(F)$ will denote the vertex set, edge set, and maximum degree respectively. The degree of a vertex v in F will be written as $d_F(v)$ or just $d(v)$ if F is obvious. If F is colored, then $d_R(v)$ and $d_B(v)$ will be the degree of v in $(F)_R$ and $(F)_B$ respectively. Notation not specifically mentioned will follow [7].

Preliminary Results

The proof of Theorem 1 will be broken into several cases. Some of these cases have been considered in previous papers. The following list of results will completely handle some cases or will be used in the proof of other cases.

Theorem A ([3] and [9]):. *If F_1 and F_2 are forests with a non-star component in at least one of the forests, then $\mathcal{R}(F_1, F_2)$ is infinite.*

Theorem B [4]:. *For an arbitrary graph G and any positive integer n, $\mathcal{R}(G, nS(1))$ is finite.*

Theorem C[2]:. *For fixed odd positive integers m_1 and n_1 and arbitrary non-negative intergers m and n, $\mathcal{R}(S(m_1) \cup mS(1), S(n_1) \cup nS(1))$ is finite.*

Theorem D [2]:. *If at least one of m_1 or n_1 is even, then $\mathcal{R}(S(m_1), S(n_1))$ is infinite.*

Two families of graphs will be used repeatedly in the proof. These graphs were introduced in [2], where a verification of the properties stated can be found.

Consider two star forests $F_1 = \bigcup_{i=1}^{s} S(m_i)$ and $F_2 = \bigcup_{i=1}^{t} S(n_i)$ where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$ and $n_1 \geq n_2 \cdots \geq n_t \geq 1$. Let $\ell_k = \max\{m_i + n_j - 1 : i + j = k + 1\}$ for $k = 1, 2, \ldots, s + t - 1$, and $H = \bigcup_{k=1}^{s+t-1} S(\ell_k)$. Then $H \rightarrow (F_1, F_2)$. In fact, it is easy to verify that $H \epsilon \mathcal{R}(F_1, F_2)$. We will denote the star forest H by $F(F_1, F_2)$. Although $F(F_1, F_2)$ will not be used explicity, the parameters $\ell_1, \ldots, \ell_{s+t-1}$ will be. We will always associate the parameters $\ell_1, \ldots, \ell_{\ell+t-1}$ with $F(F_1, F_2)$.

Let $\ell \geq 2$, $s \geq t \geq 2$ be fixed integers, and $k \geq 6$ be an even integer. Consider a family of disjoints sets $\{A_i\}_{i=1}^{k}$ such that $|A_1| = s+t-1$, $|A_2| = |A_{k-1}| = t$, $|A_k| = 1$ and $|A_i| = t(\ell-1)$ for $i = 3, \ldots, k-2$. For each k, let $G = G(k) = F(s, t, \ell, k)$ be a graph with vertex set $\bigcup_{i=1}^{k} A_i$ and edge set described as follows:

(1) The pairs (A_1, A_2) and (A_{k-1}, A_k) generate complete bipartite agraphs.
(2) The pair (A_i, A_{i+1}) generates a regular bipartite graph of degree $t+\ell-3$ when i is odd and of degree 1 when i is even $(3 \leq i \leq k-3)$.
(3) The pairs (A_2, A_3) and (A_{k-2}, A_{k-1}) generate bipartite graphs with the vertices of A_2 and A_{k-1} of degree $\ell - 1$ and the vertices of A_3 and A_{k-2} of degree 1.

The graph G has no other edges. Thus G is bipartite, each set A_i is an independent set, and each vertex in A_i $(3 \leq i \leq k-2)$ has degree $t+\ell-2$. It is easily verified that $G \to (S(s) \cup S(t), S(\ell))$, but more importantly that $(G - e) \not\to (S(s) \cup S(t), S(\ell))$ for any edge e not incident to a vertex in A_1. Therefore by deleting appropriate edges between A_1 and A_2, one obtains a diameter k subgraph $G'(k)$ of $G(k)$ such that $G'(k) \in \mathcal{R}(S(s) \cup S(t), S(\ell))$. This results in an infinite family of graphs in $\mathcal{R}(S(s) \cup S(t), S(\ell))$. The edges of $G'(k)$ will be called the critical edges of $G(k)$.

PROOFS

The proof of Theorem 1 will be a combination of a series of lemmas proved in this section and results from the previous section. We start with some lemmas which give conditions on forests F_1 and F_2 which insure that $\mathcal{R}(F_1, F_2)$ is infinite.

Lemma 2:. *Let $F_1 = \bigcup_{i=1}^{s} S(m_i)$ and $F_2 = \bigcup_{i=1}^{t} S(n_i)$ be star forests with $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$. If n_1 or m_1 is even, then $\mathcal{R}(F_1, F_2)$ is infinite.*

Proof: Theorem D implies that $\mathcal{R}(S(m_1), S(n_1))$ contains an infinite family $\{G_i : i \geq 1\}$. We have already observed that $H = F(F_1, F_2) = \bigcup_{k=1}^{s+t-1} S(\ell_k)$ has the property that $H \to (F_1, F_2)$. If we replace $S(\ell_1) = S(m_1 + n_1 - 1)$ by G_j (for any j), we obtain a new graph $L_j = G_j \cup \bigcup_{k=2}^{s+t-1} S(\ell_k)$, and the same argument used to show $F(F_1, F_2) \to (F_1, F_2)$ implies that $L_j \to (F_1, F_2)$.

Let e_j be an edge of G_j and assume that $m_1 = \cdots = m_a > m_{a+1}$ and $n_1 = \cdots = n_b > n_{b+1}$. Then $\ell_1 = \cdots = \ell_{a+b-1} > \ell_{a+b}$. By assumption, $G_j - e_j$ can be colored such that there is no red $S(m_1)$ or blue $S(n_1)$.

Also $\bigcup_{k=2}^{s+t-1} S(\ell_k)$ can be colored such that there is no red $aS(m_1)$ or blue $bS(n_1)$. Hence $(L_j - e_j) \not\rightarrow (F_1, F_2)$, and it follows that $\mathcal{R}(F_1, F_2)$ is infinite, for although L_j may not be in $\mathcal{R}(F_1, F_2)$, any subgraph of L_j in $\mathcal{R}(F_1, F_2)$ must contain G_j. This completes the proof of Lemma 2. ■

Lemma 3:. *Let* $F_1 = \bigcup_{i=1}^{s} S(m_i)$ *with* $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$, *and* $F_2 = S(n_1) \cup nS(1)$. *If* $m_1 \leq n_1 + m_2 - 2$, *and* $m_1, m_2, n_1 \geq 2$, *then* $\mathcal{R}(F_1, F_2)$ *is infinite.*

Proof: Assume $m_2 = m_3 \cdots = m_r > m_{r+1}$. For k large and even, consider $G(k) = F(m_1, m_2, n_1, k)$. Recall that $G'(k) \rightarrow (S(m_1) \cup S(m_2), S(n_1))$, and for k large $G'(k)$ has many vertex disjoint stars $S(m_2 + n_1 - 2)$. Since $S(p) \rightarrow (S(a), S(b))$ if $a + b \leq p - 1$, it is easily seen that $L_k = G(k) \cup (r-2)S(m_2 + n_1 - 1) \rightarrow (F_1, F_2)$.

On the other hand, it can be shown that for any edge e_k of $G'(k)$, $L_k - e_k \not\rightarrow (S(m_1) \cup (r-1)S(m_2), S(n_1))$, and thus $L_k - e_k \not\rightarrow (F_1, F_2)$. Hence $\mathcal{R}(F_1, F_2)$ contains an infinite number of graphs, and this completes the proof of Lemma 3. ■

Lemma 4:. *Let* $F_1 = \bigcup_{i=1}^{s} S(m_i)$ *and* $F_2 = \bigcup_{i=1}^{t} S(n_i)$ *with* $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$ *and* $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$. *If* $m_2, n_2 \geq 2$, *then* $\mathcal{R}(F_1, F_2)$ *is infinite.*

Proof: Consider the graph

$$F(F_1, F_2) = \bigcup_{k=1}^{s+t-1} S(\ell_k),$$

and the graph $G(k) = G(m_1, m_2, n_1, k)$ for k even and large. We have the parameters $\ell_1, \ell_2, \ldots \ell_{s+t-1}$ from $F(F_1, F_2)$. The proof must be broken into several cases, but the argument in each case has the same pattern as the proof of Lemma 3. Therefore we will describe the appropriate graphs which arrow, but we will omit the details of the verification.

Case 1: $\ell_2 > \ell_3$

Without loss of generality we may assume $n_1 + m_2 \geq n_2 + m_1$, and if there is equality we may assume $n_2 \leq m_2$. It is straightforward to verify that if $n_1 + m_2 = n_2 + m_1$ and $n_2 \leq m_2$, then $G(k) \rightarrow (S(m_1), S(n_1) \cup S(n_2))$. Using this fact and the argument used in Lemma 3, one can prove that $G(k) \rightarrow (F_1, F_2)$ for k large, and $G(k) - e_k \not\rightarrow (F_1, F_2)$ for any critical edge e_k (in fact $G(k) - e_k \not\rightarrow (S(m_1) \cup S(m_2), S(n_1))$).

Case 2: $\ell_2 = \ell_3$

Again with no loss of generality we may assume $n_1 + m_2 \geq n_2 + m_1$. Let $m_2 = \cdots = m_a > m_{a+1}$ and $n_2 = \cdots = n_b > n_{b+1}$.

subcase 1: $n_1 + m_2 > n_2 + m_1$

For k large the graph $L_k = G(k) \cup (a-2)S(m_2 + n_1 - 1) \to (F_1, F_2)$, but for any critical edge e_k of $G(k)$, $L_k - e_k \not\to (\bigcup_{i=1}^{a} S(m_i), S(n_1))$.

subcase 2: $n_1 + m_2 = n_2 + m_1$ with $m_1 \neq m_2$

Without loss of generality we may assume that $a \geq b$. Just as in the previous cases, for k large, $L_k = G(k) \cup (a-2)S(m_2 + n_1 - 1) \to (F_1, F_2)$, but $L_k - e_k \not\to (\cup_{i=1}^{a} S(m_i), S(n_1))$ for any critical edge e_k of $G(k)$.

subcase 3: $m_1 = m_2$ and $n_1 = n_2$

We may again assume that $a \geq b$. Let $L_k = G(k) \cup (a+b-3)S(m_1+n_1-1)$. For k large, $L_k \to (F_1, F_2)$, but $L_k - e_k \not\to (aS(m_1), bS(n_1))$ for any critical edge e_k of $G(k)$.

This completes the proof of Lemma 4. ■

Lemma 5:. *Let $F_1 = \cup_{i=1}^{s} S(m_i)$ with $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$, $s \geq 2$, $m_2 \geq 2$ and $F_2 = S(n) \cup kS(1)$ with $n \geq 2$. If m_1 and n are odd and $m_1 \geq n + m_2 - 1$, then there is a $k_0 = k_0(F_1, n) \geq 0$ such that for $k \geq k_0$, $\mathcal{R}(F_1, F_2)$ is finite.*

Proof: We will verify that if $k \geq k_0 = s(m_1+1)(s(m_1+2)+n+1)$, then $\mathcal{R}(F_1, F_2)$ is finite. To prove this, it is sufficient to show that if $F \epsilon \mathcal{R}(F_1, F_2)$, then F has a bounded number of edges. We will assume that F has more than $(s(m_1+1)+2k+n+1)^3(k+s+1)$ edges and show that this leads to a contradiction.

First we make some observations about the degrees of vertices of F. If $\Delta(F) \leq m_1 + n - 2$, then F can be colored so that there is no red $S(m_1)$ or blue $S(n)$ by Petersen's Theorem [10] (any regular graph of even degree is 2-factorable). Thus $\Delta(F) \geq m_1 + n - 1$. On the other hand, $\Delta(F) \leq s(m_1+1)+n+2k+1$. If not, let v be a vertex of F which contradicts this, and let e be an edge incident to v. The graph $F - e$ has a (F_1, F_2)-good coloring. In this coloring either $d_R(v) \geq s(m_1+1)$ or $d_B(v) \geq n + 2k + 1$. Assume the first case occurs. If e is colored red, then F must contain a red F_1, and hence $F - v$ contains a red $H = \bigcup_{i \neq j} S(m_i)$ for some j. Since v satisfies $d_R(F-e) \geq s(m_1+1)$, there is a red $S(m_j)$ centered at v which is vertex disjoint from H. This implies that $F - e$ contains a red F_1, a contradiction. A similar argument yields a contradiction in the case when v satisfies $d_B(f-e) \geq n+2k+1$. This proves that $\Delta(F) \leq s(m_1+1)+n+2k+1$.

Each edge e of F is incident to a vertex of degree at least m_s. Again note that $F - e$ has a (F_1, F_2)-good coloring. Thus, if e is colored red, F must contain a red F_1 with e in F_1. Therefore one end vertex of e must have degree at least m_s.

Before giving the final argument, we will make two observations about the number of vertices of F of degree at least m_1. If all the edges of F incident to a vertex of degree at least m_1 are colored blue and the remaining edges are colored red, then clearly there is no red F_1. Since there must be a blue F_2, there are at least $k+1$ vertices of degree $\geq m_1$. However, there are at most $(k+s)(s(m_1+1)+2k+n+1)^2$ vertices of degree $\geq m_1$. The following argument verifies this. Note that $S(m_1+n-1) \cup (s+k-1)S(m_1) \to (F_1, F_2)$ (in fact it arrows $(sS(m_1), F_2)$). Select the maximum r such that $F \supseteq S(m_1+n-1) \cup rS(m_1) = L$. Then $r \leq s+k-1$ and every vertex of degree at least m_1 must be adjacent to a vertex in L. Hence the number of vertices of degree at least m_1 is at most $|V(L)| \cdot \Delta(F) \leq (k+s)(s(m_1+1)+2k+n+1)^2$.

Let L' be the subgraph of F induced by those edges which are incident to a vertex of degree at least m_1. Then $L' \supseteq L$, and L' has at most $(k+s)(s(m_1+1)+2k+n+1)^3$ edges. By assumption there is an edge in F not in L'. Select such an edge, say $e = xy$, such that $\ell = max\{d(x), d(y)\}$ is a minimum. Hence $m_s \leq \ell < m_1$ and there is a p such that $m_p > \ell \geq m_{p+1}$.

Consider the graph $F - e$ and fix a good coloring of this graph. We claim that $F - e$ contains a red $\bigcup_{i=1}^{p} S(m_i)$ and a blue $S(n)$. The first assertion follows immediately. If e is colored red, then F contains a red F_1 using e. But e is not in any $S(m_i)$ for $i \leq p$. Thus $F-e$ contains a red $\bigcup_{i=1}^{p} S(m_i)$. To verify the second assertion, assume that $F - e$ contains no blue $S(n)$. Select all vertices $v_1, v_2 \ldots, v_a$ (a could possibly be 0) in F of degree at least $(m_1+2)s+n$, and then select pairwise vertex disjoint stars $S_{a+1}, \ldots, S_b$ each with m_1 edges which are disjoint from $\{v_1, v_2, \ldots, v_a\}$. Assume that b is maximum with respect to these properties. Since $\Delta(F) \geq m_1+n-1$, $b \geq 1$. Also, since there is no blue $S(n)$ in $F-e$, there is a red $\bigcup_{i=1}^{b} S(m_i)$ (or $\bigcup_{i=1}^{s} S(m_i)$ if $s \leq b$). Therefore $b > s$. Consider all vertices w in F and not in $\{v_1, v_2, \ldots, v_a\} \cup \bigcup_{j=a+1}^{b} S_j$ which are of degree at least m_1. There are at least $(k+1-(m_1+1)s)$ such vertices, and the maximality of b implies that each w is adjacent to vertex of some S_j $(a+1 \leq j \leq b)$. Therefore, some S_j contains a vertex of degree at least

$$(k+1-m_1+1)s)/(m_1+1)s \geq s(m_1+2)+n.$$

This contradicts the choice of $v_1, v_2, \ldots v_a$ and hence, $F-e$ contains a blue $S(n)$. Thus, we have verified that $F-e$ contains a red $\bigcup_{i=1}^{p} S(m_1)$ and a blue $S(n)$. Denote the union of these two graphs by H.

Recall that each edge of F is incident to a vertex of degree at least ℓ, and $\Delta(F) \leq s(m_1+1)+2k+n+1$. Because of the large number of edges in $F-e$, we can find a $(k+s-p)S(\ell)$ which is vertex-disjoint from H. Since $(k+s-p)S(\ell) \to ((s-p)S(\ell), kS(1))$ and $\ell \geq n_j$ for $j > p$, $F-e$ contains either a red F_1 of blue F_2. This contradiction completes the proof. ∎

Lemma 5 and the remark that followed it, along with Lemmas 2, 3 and 4 and Theorems A, B and C, complete the proof of Theorem 1. Lemma 5 is a surprising result, since it was conjectured that if $\mathcal{R}(F_1, F_2)$ is infinite, then the addition of isolated edges to either F_1, or F_2 would leave the Ramsey class infinite. Lemma 5 proves that this is not true in general.

Remark: If $\mathcal{R}(F_1, F_2)$ is finite, then $\mathcal{R}(F_1, (F_2 \cup S(1)))$ is also finite.

This may be verified as follows. Let $F_2' = F_2 \cup S(1)$ and assume that $\mathcal{R}(F_1, F_2)$ is finite but that $\mathcal{R}(F_1, F_2')$ is not. Let $F' \in \mathcal{R}(F_1, F_2')$, and thus $F' \supset F \epsilon \mathcal{R}(F_1, F_2)$. Without loss of generality we can assume that F' has many more vertices than F. Denote by A the vertices of F' which are not adjacent in F' to any vertex of F. Recall that any vertex in $\mathcal{R}(F_1, F_2')$ has bounded degree, implying that there are many vertices in A. Select an edge $e = xy$ with $x, y \epsilon A$ subject to the condition that $t = \max\{d(x), d(y)\}$ is a minimum for all such edges.

The graph $F' - e$ has a good (F_1, F_2') coloring. In this fixed good coloring there must be a blue F_2, in fact a blue $F_2 \subseteq F$. Since there is no blue F_2', all of the edges with both end vertices in A must be red. If the good coloring of $F' - e$ is extended to a coloring of F' by coloring e red, then there is a red copy of F_1, where the edge e is in some $S(m_j)$ with $t \geq m_j$. Since A is very large there is a star with $t \geq m_j$ edges whose vertices are contained in A and which is vertex disjoint from the red copy F_2. This implies that $F' - e$ contains a red F_1, a contradiction.

If one could determine the smallest k_0 such that $\mathcal{R}(F_1, S(n) \cup k_0 S(1))$ is finite, then $\mathcal{R}(F_1, S(n) \cup kS(1))$ is finite if and only if $k \geq k_0$. However, the precise determination of k_0 appears to be very difficult ($k_0 > 1$ by a result in [2]).

REFERENCES

1. S.A.Burr, *A survey of noncomplete Ramsey theory for graphs*, Topics in Graph Theory (New York Academy of Scienes (1979), 58-75.
2. S.A. Burr, P. Erdös, R.J. Faudree, C.C. Rousseau and R.H. Schelp, *Ramsey-minimal graphs for star-forests*, Discr. Math. **33** (1981), 227-237.
3. S.A. Burr, P. Erdös, R.J. Faudree, C.C. Rousseau and R.H. Schelp, *Ramsey minimal graphs for forests*, Discr. Math. **38** (1982), 23-32.
4. S.A. Burr, P. Erdös, R.J. Faudree, R.H. Schelp, *A class of Ramsey-finite graphs*, Proc. 9th S.E. Conf. Comb., Graph Theory and Computing (1978), 171-180.
5. S.A. Burr, P. Erdös, R.J. Faudree, C.C. Rousseau and R.H. Schelp, *Ramsey minimal graphs for the pair star-connected graphs*, Studia Scient. Math. Hungar. **15** (1980), 265-273.
6. S.A. Burr, R.J. Faudree and R.H. Schelp, *On Ramsey-minimal graphs*, Proc. 8th S.E. Conf. Comb. Graph Theory and Computing (1977), 115-124.

7. F. Harary, "Graph Theory," Addison-Wesley, Reading, MA, 1969.
8. J. Nesetril and V. Rödl, *On Ramsey minimal graphs*, Colloq. Internationaux C.N.R.S **260** (1978), 307-308.
9. J. Nesetril and V. Rödl, *The structure of critical Ramsey graphs*, Acta Math. Acad. Sci. Hugar. **32** (1978), 295-300.
10. J. Petersen, *Die theorie der regulären graphen*, Acta. Math **15** (1981), 193-220.