Addendum to: On The Construction of Color-Critical Linear Hypergraphs

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Abstract. It is shown that there exists a 4-critical 3-uniform linear hypergraph of order m for every $m \geq 56$

Let n and r be positive integers, $n \geq 3$, $r \geq 3$. Denote by $M^*(n,r)$ the least integer such that for every $m \geq M^*(n,r)$ there exists a linear (m,n,r)-graph; that is, an r-color critical n-uniform hypergraph of order m any two edges of which have at most one vertex in common. That $M^*(n,r)$ exists was shown in [1]. Only one value of $M^*(n,r)$ is known, namely, $M^*(3,3) = 9$. In [3] we showed that $M^*(4,3) \leq 51$ and $M^*(3,4) \leq 94$. In this note we show that $M^*(3,4) \leq 56$.

Let $S = \{m$: there exists a linear $(m,3,4)$-graph$\}$. Rosa [6] proved that the 3-graph whose edges are the lines in $PG(4,2)$ is 4-chromatic and Liu [5] verified that it is vertex-color-critical, so that $31 \in S$. In [1], [2] and [3] various constructions of color-critical linear hypergraphs are given and from these and the fact that $31 \in S$ it is deduced that $m \in S$ for all $m \geq 94$. Our improved bound is obtained from these general constructions and the following facts:

Fact 1. The cyclic Steiner triple system of order 25 with base triples $\{1,2,4\}$, $\{1,3,17\}$, $\{1,6,12\}$, $\{1,8,18\}$ is 4-chromatic and vertex-color-critical. It thus contains a linear (25,3,4)-graph, so that $25 \in S$.

Fact 2. The cyclic Steiner triple system of order 33 with base triples $\{1,2,4\}$, $\{1,5,15\}$, $\{1,6,14\}$, $\{1,7,19\}$, $\{1,8,17\}$, $\{1,12,23\}$ is 4-chromatic and vertex-color-critical. It thus contains a linear (33,3,4)-graph, so that $33 \in S$.

That the graph described in Fact 2 is 4-chromatic was decided by de Brandes, Phelps and Rödl [4]. We also verified this by computer and now show that it is vertex-color-critical. Since the system is cyclic it suffices to exhibit a 3-coloring of the subgraph obtained by deleting the edges containing the vertex 1.

The following are color classes of such a coloring: {2,3,4,9,13,16,19,27,29,33}, {5,6,7,14,15,18,21,22,25, 26}, {8,10,11,12,17,20,23,24,28,30,31,32}.

We verified, also by computer, that the graph described in Fact 1 has no 3-coloring. The following are color-classes of a 4-coloring: {1,2,3,6,7,8,11}, {4,20,23,24,25}, {5,9,10,13,14,15,19}, {12,16,17,18,21,22}. Thus the graph is 4-chromatic. We also verified that the graph is vertex-color-critical. Since the system is cyclic it suffices to give a 3-coloring of the subgraph obtained by deleting the edges containing the vertex 1. Such a 3-coloring is given by: {2,3,4,9,10,11,22,25}, {5,12,14,16,18, 19,20,23}, {6,7,8,13,15,17,21,24}. That the graph in Fact 1 is 4-chromatic is also given in [4], but there is an error in the description of the graph,

We now show how to construct such graphs of order m for $56 \leq m \leq 93$. The general constructions referred to are those given in the proof of Theorem 2 of [3]. In Construction 1 take $q = 3$, $m_1 = m_2 = m_3 = 25$ and $t \in \{7,9,10,11,\ldots,20\}$. This shows that $80,82,83,84,\ldots,93 \in S$. If we take $m_1 = m_2 = m_3 = 25$ in Constructions 4,5,6,7,8 we find, respectively, that $76,77,78,79,81 \in S$. In Construction 2 take $m_1 = 33$, $m_2 = 31$ and $r = 11$. This shows that $75 \in S$. If, in Construction 2, we take $m_1, m_2 \in \{25,31\}$ and $r \in \{7,10,11,12\}$, we find that $57,60,61,62,63,66,67,68,69,72,73,74 \in S$. Finally, in Construction 1 take $q = 2$, $m_1, m_2 \in \{25,31\}$ and $t \in \{7,9,10\}$. This shows that $56,58,59,64,65,70,71 \in S$. Thus $M^*(3,4) \leq 56$. Note that Fact 2 is used only once; namely, to show that $75 \in S$.

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