On the Cordiality of Some Specific Graphs

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Abstract. Three types of graphs are investigated with respect to cordiality, namely, graphs which are the complete product of two cordial graphs, graphs which are the subdivision graphs of cordial graphs, and cactus graphs. We give sufficient conditions for the cordiality of graphs of the first two types and show that a cactus graph is cordial if and only if the cardinality of its edge set is not congruent to 2 (mod 4).

Introduction

Cordial labelings of graphs were introduced by Cahit [2] as a weakened version of the apparantly less tractable *graceful labelings* (see[1]) and *harmonious labelings* [6]. Cahit showed, among other things, that all trees are cordial (i.e., can be cordially labeled), that a length-n cycle, C_n, is cordial if and only if $n \not\equiv 2 \pmod 4$, that an Eulerian graph $\mathcal{G}$ is not cordial if $|E(\mathcal{G})| \equiv 2 \pmod 4$, that the complete graph on n vertices, K_n, is cordial if and only if $n \leq 3$, and that the complete bipartite graph on m and n vertices, $K_{m,n}$, is cordial for all $m, n \geq 1$. Since Cahit's paper, various particular types of graphs have been shown to be cordial (see, e.g., [8]). In [8] sufficient conditions were given for the direct product of two cordial graphs to be cordial. This result was generalized in [9] and [10] (using a different method from that of [8]) for a large class of graphs built from other cordial graphs using any type of what is called a NEPS operation. In parts 1 and 2 of this paper, we invesigate graphs built from cordial graphs using operations which do not fall in the category of NEPS operations, namely, the operations of *complete product* and *subdivision*. Subdivision graphs are a subclass of bipartite graphs; hence, investigation of this class of graphs extends knowlege about the cordiality of bipartite graphs. Cahit's result mentioned above was previously the only result known about this subject.

Cahit's result on cycles and Eulerian graphs raises the question of whether or not a planar Eulerian graph is cordial *if and only if* $|E(\mathcal{G})| \not\equiv 2 \pmod 4$. There is no apparant reason to think this might be true, yet there is no known counterexample. (Note that there are non-planar, non-cordial Eulerian graphs with edge sets whose cardinality is not congruent to 2 (mod 4); K_7, for example.) In part 3 of this paper we investigate a type of Eulerian planar graph, the *cactus graph*, and show that a cactus graph is cordial if and only if the cardinality of its edge set is not congruent to 2 (mod 4).

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Definitions for Cordial Graphs

All graphs in this paper are finite, loopless, and without multiple edges. A *labeling* of a graph $\mathcal{G} = (V, E)$ is a mapping $f: V \to \{0, 1\}$. The mapping f induces an edge-labeling f^* on $\mathcal{G}$, $f^*: E \to \{0, 1\}$, defined by $f^*((u, v)) = |f(u) - f(v)|$ for all $(u, v) \in E$. Let $v_f(0) = \{v \in V | f(v) = 0\}$, $v_f(1) = \{v \in V | f(v) = 1\}$, $e_{f^*}(0) = \{e \in E \mid f^*(e) = 0\}$, and $e_{f^*}(1) = \{e \in E | f^*(e) = 1\}$.

(We will omit the subscripts f and f^* when the context makes it clear.)

Cahit's original definition of a cordial labeling is the following:

Definition 1a. *A labeling f of a graph $\mathcal{G}$ is cordial if $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$.*

A graph $\mathcal{G}$ is cordial if it admits a cordial labeling.

Several equivalent defintions are possible. In particular, in this paper we will use the following one. Let *nearly equal* mean differing by at most one.

Definition 1b. *A graph $\mathcal{G}$ is cordial if there exists a partition of $V(\mathcal{G})$ into two nearly equal subsets, V_1 and V_2, such that the set of edges with both endpoints in V_1 or both endpoints in V_2 is nearly equal in size to the set of edges which have one endpoint in V_1 and one endpoint in V_2.*

(The proof of the equivalence of definitions 1a and 1b is left to the reader.)

Thus, if $\mathcal{G}$ is a cordial graph, such a partition exists. We will call edges with both endpoints in V_1 or both in V_2 *internal edges* and edges with one endpoint in V_1 and the other in V_2 *spanning edges*. For a cordial graph $\mathcal{G}$, (with implied the partition of $V(\mathcal{G})$ into V_1 and V_2) $N(\mathcal{G})$ will denote the number of internal edges in $\mathcal{G}$ and $S(\mathcal{G})$ will denote the number of spanning edges in $\mathcal{G}$.

Note that $e(0)$ in definition 1a corresponds to *internal edges* in definition 1b and that $e(1)$ in definition 1a corresponds to *spanning edges* in definition 1b.

Throughout this paper we will assume that $|V_1| \geq |V_2|$. We note that there are six different "types" of cordial graphs, in the following sense. Let an (i, j)*-cordial graph* be a cordial graph such that, under some cordial labeling, $|V_1| - |V_2| = i$ and $e(0) - e(1) = j$. Note that $i \in \{0, 1\}$, and $j \in \{-1, 0, 1\}$. Note also that a graph with an odd number of edges may be both an $(i, 1)$- and an $(i, -1)$-cordial graph.

More generally, we will call a graph $\mathcal{G}$ an (i, j) graph if there exists a partition of the vertex set $V(\mathcal{G})$ into two sets V_1 and V_2 such that $|V_1| - |V_2| = i$ and $e(0) - e(1) = j$. In this more general case, i and j need not be in the sets $\{0, 1\}$ and $\{-1, 0, 1\}$.

Part 1. The Complete Sum of Two Cordial Graphs

Definition 2. *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (W, F)$ be two graphs. The complete product of $\mathcal{G}$ and $\mathcal{H}$, $\mathcal{G}\nabla\mathcal{H}$, is a graph $\overline{\mathcal{G}} = (\overline{V}, \overline{E})$, where $\overline{V} = V \cup W$ and $\overline{E} = E \cup F \cup \{(v, w): v \in V$ and $w \in w\}$.*

In other words, $\mathcal{G}\nabla\mathcal{H}$ is obtained from taking a copy of $\mathcal{G}$, a copy of $\mathcal{H}$, and connecting every vertex of $\mathcal{G}$ with every vertex of $\mathcal{H}$. (Note that '∇' is commutative.)

The complete product has been studied extensively in the literature, most often with respect to the question of charactarizing those graphs which are not the complete product of smaller graphs. (See [4] where such indecomposable graphs are called *elementary*.)

Let $\mathcal{G}$ and H be cordial on n and m vertices, respectively. Let V_1 and V_2 and W_1 and W_2 be the implied partitions of V and W. Let $\overline{V}_1 = V_1 \cup W_2$ and $\overline{V}_2 = V_2 \cup W_1$ be a partition of $\overline{V}$. We may now speak of internal and spanning edges in $\mathcal{G}\nabla\mathcal{H}$.

Lemma 1. $\left||\overline{V}_1| - |\overline{V}_2|\right| \leq 1$.

Proof: $|\overline{V}_1| = \lceil\frac{n}{2}\rceil + \lfloor\frac{m}{2}\rfloor$. $|\overline{V}_2| = \lfloor\frac{n}{2}\rfloor + \lceil\frac{m}{2}\rceil$. The result follows from an examination of the four cases of n or m even or odd. ∎

Lemma 2.

$$N(\mathcal{G}\nabla\mathcal{H}) = \left\lfloor\frac{nm}{2}\right\rfloor + N(\mathcal{G}) + N(\mathcal{H}) \tag{1}$$

and

$$S(\mathcal{G}\nabla\mathcal{H}) = \left\lceil\frac{nm}{2}\right\rceil + S(\mathcal{G}) + S(\mathcal{H}) \tag{2}$$

Proof: Edges internal to $\overline{V}_1$ consist of all edges internal to V_1, all edges internal to W_2, and the $|V_1| \cdot |W_2| = \lceil\frac{n}{2}\rceil \cdot \lfloor\frac{m}{2}\rfloor$ edges connecting vertices in V_1 to vertices in W_2. Edges internal to $\overline{V}_2$ consist of all edges internal to V_2, all edges internal to W_1, and the $|V_2| \cdot |W_1| = \lfloor\frac{n}{2}\rfloor \cdot \lceil\frac{m}{2}\rceil$ edges connection vertices of V_2 to vertices of W_1. Since $\lceil\frac{n}{2}\rceil \cdot \lfloor\frac{m}{2}\rfloor + \lfloor\frac{n}{2}\rfloor \cdot \lceil\frac{m}{2}\rceil = \lfloor\frac{nm}{2}\rfloor$, expression (1) follows.

Spanning edges in $\mathcal{G}\nabla\mathcal{H}$ consist of the spanning edges in $\mathcal{G}$, the spanning edges in $\mathcal{H}$, the $|v_1| \cdot |W_2| = \lceil\frac{n}{2}\rceil \cdot \lceil\frac{m}{2}\rceil$ edges connecting vertices in $V - 1$ to vertices in W_1, and the $|V_2| \cdot |W_2| = \lfloor\frac{n}{2}\rfloor \cdot \lfloor\frac{m}{2}\rfloor$ edges connecting vertices in V_2 to vertices in W_2. Since $\lceil\frac{n}{2}\rceil \cdot \lceil\frac{m}{2}\rceil + \lfloor\frac{n}{2}\rfloor \cdot \lfloor\frac{m}{2}\rfloor = \lceil\frac{nm}{2}\rceil$, expression (2) follows. ∎

From these lemmas, we get Theorem 1, the main result of this section. Let '$\oplus$' indicate exclusive-or.

Theorem 1. *Let $\mathcal{G}$ and $\mathcal{H}$ be (a,b) and (c,d) cordial graphs, respectively. Then $\mathcal{G}\nabla\mathcal{H}$ is an $(e,f) = ((a\oplus c),(-ac+b+d))$ graph.*

Proof:

$$\begin{aligned}
e &= ||V_1| - |V_2|| \\
&= \left|\left\lceil\frac{n}{2}\right\rceil - \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{m}{2}\right\rfloor - \left\lceil\frac{m}{2}\right\rceil\right| \\
&= \begin{cases} 0, & \text{if } n \text{ and } m \text{ are both even or both odd;} \\ 1, & \text{if one of } n \text{ and } m \text{ is even and one is odd.} \end{cases} \\
&= a \oplus c
\end{aligned}$$

$$\begin{aligned}
f &= N(\mathcal{G}\nabla\mathcal{H}) - S(\mathcal{G}\nabla\mathcal{H}) \\
&= \left\lfloor\frac{nm}{2}\right\rfloor + N(\mathcal{G}) + N(\mathcal{H}) - \left\lceil\frac{nm}{2}\right\rceil - S(\mathcal{G}) - S(\mathcal{H}) \\
&= \left\lfloor\frac{nm}{2}\right\rfloor - \left\lceil\frac{nm}{2}\right\rceil + b + d \\
&= \begin{cases} -1 + b + d, & \text{if } n \text{ and } m \text{ are both odd} \\ 0 + b + d, & \text{if } n \text{ or } m \text{ is even.} \end{cases} \\
&= -ac + b + d
\end{aligned}$$

∎

Theorem 1 (as summarized in Table 1) yields the following corollaries.

Corollary 1. *Let $\mathcal{G}$ be a $(0,0)$ cordial graph and $\mathcal{H}$ be an (i,j) cordial graph. Then $\mathcal{G}\nabla\mathcal{H}$ and $\mathcal{H}\nabla\mathcal{G}$ are both (i,j) cordial graphs.* ∎

Corollary 2. *The class of $(0,0)$ cordial graphs is closed under '∇'.* ∎

Since $K_{m,n}$ (the complete bipartite graph on m and n vertices) is $\overline{K}_m\nabla\overline{K}_n$ (that is, m isolated vertices and n isolated vertices) the following corollary, first proved by Cahit [2], follows.

Corollary 3. *$K_{m,n}$ is cordial for all integers m and n.* ∎

For the next theorem, we need three results of Cahit [2].

Lemma 3 [Cahit]. *C_n (the cycle on n vertices) is cordial if and only if $n \not\equiv 2 \pmod 4$.*

Lemma 4 [Cahit]. *An Eulerian graph on n vertices is not cordial if $|E| \equiv 2 \pmod 4$.*

Lemma 5 [Cahit]. *K_n (the complete graph on n vertices) is cordial if and only if $n \leq 3$.*

Type of $\mathcal{G}$ (a,b)	Type of $\mathcal{H}$ (c,d)	Type of $\mathcal{G}\nabla\mathcal{H}$ (e,f)
(0,-1)	(0,-1)	(0,-2) **
(0,-1)	(0,0)	(0,-1)
(0,-1)	(0,1)	(0,0)
(0,-1)	(1,-1)	(1,-2) **
(0,-1)	(1,0)	(1,-1)
(0,-1)	(1,1)	(1,0)
(0,0)	(0,0)	(0,0)
(0,0)	(0,1)	(0,1)
(0,0)	(1,-1)	(1,-1)
(0,0)	(1,0)	(1,0)
(0,0)	(1,1)	(1,1)
(0,1)	(0,1)	(0,2) **
(0,1)	(1,-1)	(1,0)
(0,1)	(1,0)	(1,1)
(0,1)	(1,1)	(1,2) **
(1,-1)	(1,-1)	(0,-3) **
(1,-1)	(1,0)	(0,-2) **
(1,-1)	(1,1)	(0,-1)
(1,0)	(1,0)	(0,-1)
(1,0)	(1,1)	(0,0)
(1,1)	(1,1)	(0,1)

Table 1. The type of $\mathcal{G}\nabla\mathcal{H}$

Theorem 2. *Let A denote the class of (i,j) cordial graphs and B denote the class of (k,l) cordial graphs for*

$$((i,j),(k,l)) \in \{ ((0,-1),(0,-1)) , ((0,-1),(1,-1)) , ((0,1),(1,1)) , ((1,-1),(1,-1)) , ((1,-1),(1,0))\} .$$

Then there exist $\mathcal{G} \in A$ and $\mathcal{H} \in B$ such that $\mathcal{G}\nabla\mathcal{H}$ is not cordial.

Proof: We offer the following table of counter-examples:

Type of $\mathcal{G}$	Type of $\mathcal{H}$	$\mathcal{G}$	$\mathcal{H}$	$\mathcal{G}\nabla\mathcal{H}$	Reason $\mathcal{G}\nabla\mathcal{H}$ is not cordial
$(0,-1)$	$(0,-1)$	K_2	K_2	K_4	Lemma 5
$(0,-1)$	$(1,-1)$	K_2	K_3	K_5	Lemma 5
$(0,1)$	$(0,1)$	$C_5 \cup K_1$	$C_5 \cup K_1$	(see below)	Lemma 4
$(1,-1)$	$(1,-1)$	K_3	K_3	K_6	Lemma 5
$(1,-1)$	$(1,0)$	K_3	K_1	K_4	Lemma 5

Table 2. Counterexamples

Note that $(C_5 \cup K_1)\nabla(C_5 \cup K_1)$ yields a connected graph with every vertex of even degree (6 or 8) and with 46 edges. Hence, it is Eulerian with $|E| \equiv 2 \pmod 4$. (The cordiality of $C_5 \cup K_1$ is obvious in light of Lemma 3.)

Thus it is open whether or not for all $(0,1)$ cordial graphs $\mathcal{G}$ and for all $(1,1)$ graphs $\mathcal{H}$, $\mathcal{G}\nabla\mathcal{H}$ is cordial. Note that such a graph is never Eulerian, nor is it ever a complete graph.

Part 2. The Subdivision Graph of a Cordial Graph

Let $\mathcal{G}$ be a graph. The *subdivision graph* of $\mathcal{G}$, $S(\mathcal{G})$, is the bipartite graph obtained from $\mathcal{G}$ by replacing each edge of $\mathcal{G}$ with a path of length 2, or, equivalently, by inserting an additional vertex into each edge of $\mathcal{G}$. (Subdivision graphs are discussed in [4].)

Let $\mathcal{G}$ be a cordial graph with $V(\mathcal{G})$ partitioned into V_1 and V_2 under a cordial labeling. Let E_1 denote the edges internal to V_1, E_2 denote the edges internal to V_2, and E_S denote the spanning edges. The following theorem offers conditions for when the subdivision graph of a cordial graph is cordial.

Theorem 3. *Let $\mathcal{G}$ be a cordial graph. The $S(\mathcal{G})$ is cordial if both $|E_1|$ and $|E_2|$ are even or both are odd.*

Proof: Let $\lceil m \rfloor$ mean '$\lceil m \rceil$ or $\lfloor m \rfloor$'. (In other words, there is a choice involved.) Let $\lfloor m \rceil$ indicate $\lfloor m \rfloor$ if $\lceil m \rfloor$ has been chosen to be $\lceil m \rceil$ and $\lceil m \rceil$ if $\lceil m \rfloor$ has been chosen to be $\lfloor m \rfloor$. (This notation is taken from the usual $\pm, \mp$ notation.)

Note that $S(\mathcal{G})$ has $|V(\mathcal{G})| + |E(\mathcal{G})|$ many vertices and $2 \cdot |E(\mathcal{G})|$ many edges. In forming $S(\mathcal{G})$, let V_{E_1} refer to those vertices which were inserted along an edge of E_1, V_{E_2} to those which were inserted along an edge of E_2, and V_{E_S} to those which were inserted along an edge of E_S.

We now describe a partition of $\overline{V} = V(S(\mathcal{G}))$ into $\overline{V}_1$ and $\overline{V}_2$ and the induced edge sets, $\overline{E}_1$, $\overline{E}_2$, and $\overline{E}_S$. We first assume that $|E_1| \geq |E_2|$. In this case, $\overline{V}_1$ consists of V_1, $\left\lceil \frac{|E_S|}{2} \right\rfloor$ vertices of V_{E_S} (arbitrarily chosen), $\left\lceil \frac{|E_2|}{2} \right\rfloor$ vertices of

V_{E_2} (arbitrarily chosen), and $\left\lceil \frac{|E_2|}{2} \right\rceil + \left\lceil \frac{|E_1|-|E_2|}{2} \right\rceil$ vertices of V_{E_1} (arbitrarily chosen). $\overline{V}_2 = \overline{V} - \overline{V}_1$. Hence,

$$|\overline{V}_1| - |\overline{V}_2| = (|V_1| - |V_2|) + \left(\left\lceil \frac{|E_S|}{2} \right\rceil - \left\lfloor \frac{|E_S|}{2} \right\rfloor\right) + \left(\left\lceil \frac{|E_2|}{2} \right\rceil - \left\lfloor \frac{|E_2|}{2} \right\rfloor\right)$$
$$= \left(\left\lceil \frac{|E_1| - |E_2|}{2} \right\rceil - \left\lfloor \frac{|E_1| - |E_2|}{2} \right\rfloor\right).$$

This quantity can always be made an element of $\{1, 0, -1\}$ by correct choices of ceilings and floors. The case when $|E_2| > |E_1|$ is similar.

Let the vertices of V_1 and $\overline{V}_1$ be called 1-vertices and the vertices of V_2 and $\overline{V}_2$ be called 0-vertices. Note that introducing a new 1-vertex along an edge in E_1 or introducing a new 0-vertex along an edge in E_2 yields 2 new 0-edges, and introducing a new 0-vertex along an edge in E_1 or introducing a new 1-vertex along an edge in E_2 introduces 2 new 1-edges. Introducing a new 0-vertex along an edge in E_S introduces 1 new 0-edge and 1 new 1-edge, as does introducing a new 1-vertex along an edge in E_S. With this in mind, it is not hard to see that

$$|\overline{E}_1| + |\overline{E}_2| = |E_S| + 2\left(\left\lceil \frac{|E_2|}{2} \right\rceil + \left\lceil \frac{|E_1| - |E_2|}{2} \right\rceil\right) + 2\left\lceil \frac{|E_2|}{2} \right\rceil,$$
$$|\overline{E}_S| = |E_S| + 2\left(\left\lfloor \frac{|E_2|}{2} \right\rfloor + \left\lfloor \frac{|E_1| - |E_2|}{2} \right\rfloor\right) + 2\left\lfloor \frac{|E_2|}{2} \right\rfloor,$$

and

$$|(|\overline{E}_1| + |\overline{E}_2|) - |\overline{E}_S|| = 2\left(\left\lceil \frac{|E_2|}{2} \right\rceil - \left\lfloor \frac{|E_2|}{2} \right\rfloor\right)$$
$$+ 2\left(\left\lceil \frac{|E_1| - |E_2|}{2} \right\rceil - \left\lfloor \frac{|E_1| - |E_2|}{2} \right\rfloor\right)$$
$$+ 2\left(\left\lceil \frac{|E_2|}{2} \right\rceil - \left\lfloor \frac{|E_2|}{2} \right\rfloor\right).$$

$S(\mathcal{G})$ is cordial if the above expression, through correct choice of ceilings and floors, can be made to be an element of $\{-1, 0, 1\}$. It can be if $|E_1|$ and $|E_2|$ are both even or both odd. The case when $|E_2| > |E_1|$ is similar. ∎

If one of $|E_1|$ and $|E_2|$ is even and the other is odd, $S(\mathcal{G})$ cannot be guaranteed cordial. For the case of $|E_1|$ odd and $|E_2|$ even, this follows from the fact that $S(C_3) = C_6$ and Lemma 3 above. For the case of $|E_1|$ even and $|E_2|$ odd, we have no counterexample.

Part 3. Cactus Graphs

Definition 3. *A k-cycle cactus graph, M_k, is a connected graph composed of k edge-disjoint cycles, $c_1, \ldots, c_k$, such that for all pairs if cycles c_i and c_j, $i \neq j$, $c_i \cap c_j$ is either empty or consists of a single vertex only. Furthermore, if $c_i \cap c_j = v$ and $c_i \cap c_l = w$, $i \neq j \neq l$, then $c_j \cap c_l \neq$ implies $v = w$.*

A cactus graph may be thought of in the general form in figure 2 below, where eight cycles are pictured.

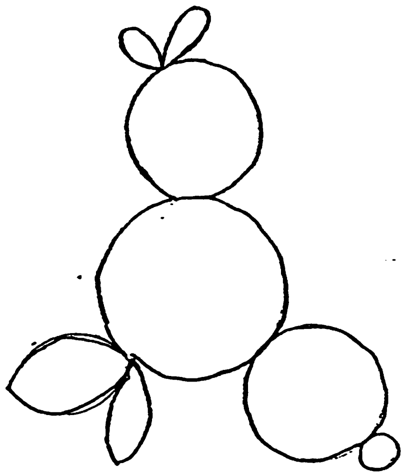

Figure 2. An eight-cycle catcus graph.

A catcus graph is necessarily planar and Eulerian. We will call a vertex contained in more than one cycle an *articulation point*. A cycle containing only one articulation point will be called an *outer cycle*. There must be at least one outer cycle in every cactus graph.

We will show that a k-cycle cactus graph, M_k, is cordial if and only if $|E(M_k)| \not\equiv 2 \pmod 4$. This in turn will imply that M_k is cordial if and only if $k + |V(M_k)| \not\equiv 3 \pmod 4$.

We first need three lemmas.

Lemma 6. *A cycle of length n, C_n, for $n \equiv 2 \pmod 4$, is both a $(0, 2)$ and a $(0, -2)$ graph.*

Proof: The two labelings are given in figure 3 below.

In the labeling on the left each spanning edge is matched by the interior edge preceeding it, then two spanning edges complete the cycle for a $(0, -2)$ labeling. (Vertices in V_1 are black and vertices in V_2 are white.) The labeling on the right is derived from the labeling on the left by switching vertex v_{4k+1} from V_1 to V_2

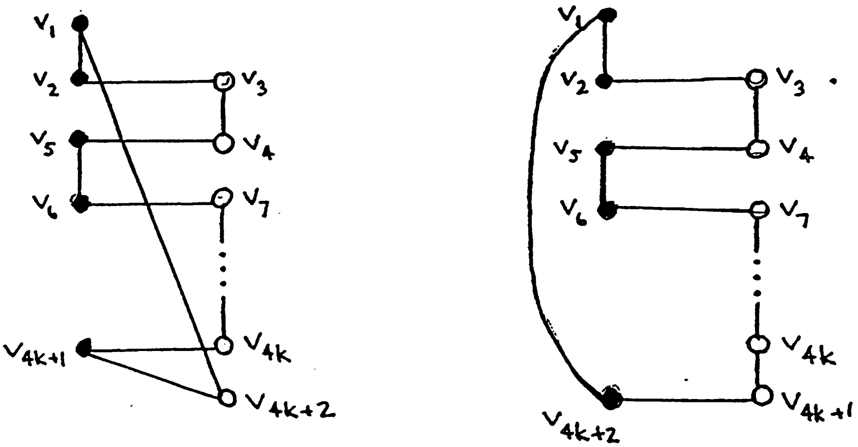

Figure 3. A $(0,-2)$ (left) and a $(0,2)$ (right) labeling of C_n for $n \equiv 2 \pmod 4$

and by switching v_{4k+2} from V_2 to V_1, resulting in completing the cycle with two extra interior edges for a $(0,2)$ labeling. ∎

Lemma 7. *C_n for $n \equiv 1 \pmod 4$ is both a $(1,1)$ and a $(1,-3)$ graph.*

Proof: The labelings are given in figure 4.

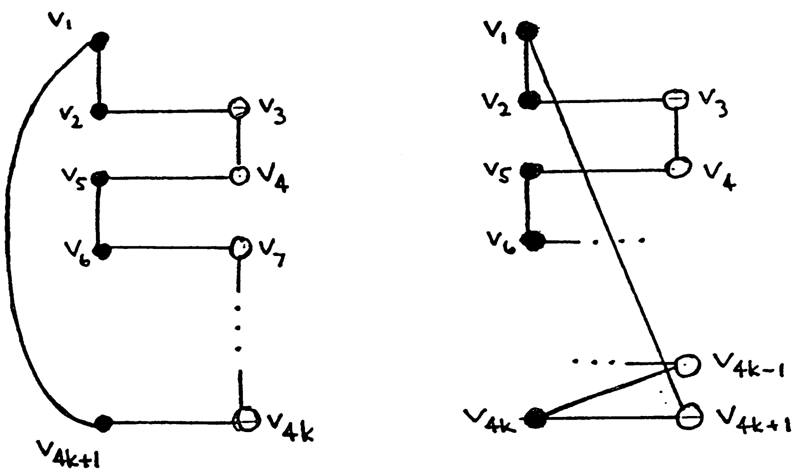

Figure 4. A $(1,1)$ (left) and a $(1,-3)$ (right) labeling of C_n for $n \equiv 1 \pmod 4$.

∎

Lemma 8. *C_n for $n \equiv 3 \pmod 4$ is both a $(1,-1)$ and a $(1,3)$ graph.*

Proof: See figure 5.

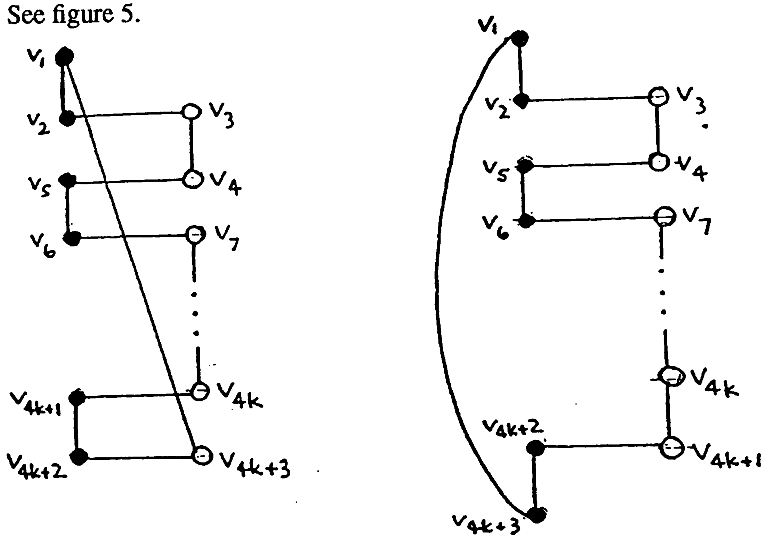

Figure 5. A $(1,-1)$ (left) and a $(1,3)$ labeling for C_n for $n \equiv 3 \pmod 4$.

∎

These lemmas lead to the following theorem. Let Γ_k denote the class of all k-cycle cactus graphs.

Theorem 4. *For all $k \geq 1$ the following is true of all $\mathcal{G} \in \Gamma_k$ for some $i \in \{0,1\}$:*

1) *if $|E(\mathcal{G})| \equiv 0 \pmod 4$ then $\mathcal{G}$ admits an $(i,0)$ cordial labeling.*
2) *if $|E(\mathcal{G})| \equiv 1 \pmod 4$ then $\mathcal{G}$ admits an $(i,1)$ cordial labeling and an $(i,-3)$ labeling.*
3) *if $|E(\mathcal{G})| \equiv 2 \pmod 4$ then $\mathcal{G}$ admits an $(i,2)$ labeling and an $(i,-2)$ labeling.*
4) *if $|E(\mathcal{G})| \equiv 3 \pmod 4$ then $\mathcal{G}$ admits an $(i,-1)$ cordial labeling and an $(i,3)$ labeling.*

Proof: The proof is by induction on k.

$k = 1$: This is Cahit's theorem [2] along with Lemmas 5, 6, and 7 above.

$k \mapsto k+1$: Let $\mathcal{G} \in \Gamma_{k+1}$. Delete one outer cycle, C_n, of $\mathcal{G}$ to form $\overline{\mathcal{G}} \in \Gamma_k$. We now consider the sixteen cases which are the possible values of the 2-tuple, $(n \pmod 4, |E(\mathcal{G})| \pmod 4)$. The results are summarized in Table 3 below.

The idea is that if C_n is an (i, j) graph and $\overline{\mathcal{G}}$ is a (k, l) graph, then C_n can be re-attached to $\overline{\mathcal{G}}$ to re-form $\mathcal{G}$ as an $(i + k, j + l)$ graph.

The inductive hypothesis gives appropriate labelings for C_n and $\overline{\mathcal{G}}$.

Note that we need not consider $|V(\mathcal{G})|$, since $i + k$ can always be made an element of $\{0, 1\}$, if need be by reversing the roles of V_1 and V_2 in C_n or $\overline{\mathcal{G}}$.

n (mod 4)	$e(0) - e(1)$ for C_n	$\lvert E(\overline{\mathcal{G}})\rvert$ (mod 4)	$e(0) - e(1)$ for $\overline{\mathcal{G}}$	$\lvert E(\mathcal{G})\rvert$ (mod 4)	$e(0) - e(1)$ for $\mathcal{G}$
0	0	0	0	0	0
0	0	1	1 or −3	1	1 or −3
0	0	2	2 or −2	2	2 or −2
0	0	3	3 or −1	3	3 or −1
1	1 or −3	0	0	1	1 or −3
1	1 or −3	1	1 or −3	2	2 or −2
1	1 or −3	2	2 or −2	3	3 or −1
1	1 or −3	3	3 or −1	0	0
2	2 or −2	0	0	2	2 or −2
2	2 or −2	1	1 or −3	3	3 or −1
2	2 or −2	2	2 or −2	0	0
2	2 or −2	3	3 or −1	1	1 or −3
3	3 or −1	0	0	3	3 or −1
3	3 or −1	1	1 or −3	0	0
3	3 or −1	3	2 or −2	1	1 or −3
3	3 or −1	3	3 or −1	2	2 or −2

Table 3. Labelings for C_n, $\overline{\mathcal{G}}$, and $\mathcal{G}$

∎

Recall that Cahit [2] showed that for an Eulerian graph, $\mathcal{G}$, $|E(\mathcal{G})| \equiv 2 \pmod 4$ implies that $\mathcal{G}$ is not cordial. Combining this with Theorem 4 above, we get,

Corollary 5. *A cactus graph, M, is cordial if and only if $|E(M)| \not\equiv 2 \pmod 4$.* ∎

The following corolary relates the cordiality of a k-cycle cactus graph, M_k, to k.

Corolary 6. *The k-cycle cactus graph, M_k, is cordial if and only if $k + |V(M_k)| \not\equiv 3 \pmod 4$.*

Proof: Let $|V(M_k)| = n$ and $|E(M_k)| = m$. Euler's formula says that $m = n + f - 2$, where f is the number of faces in M_k. In M_k, $f = k + 1$. Hence,

$m = n + k - 1$. Combining this with corolary 5 we get that M_k is cordial if and only if $n + k - 1 \not\equiv 2 \pmod 4$. ∎

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