# On the Vulnerability of Permutation Graphs of Complete and Complete Bipartite Graphs

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**Abstract.** In this paper the authors study the vulnerability parameters of integrity, toughness, and binding number for two classes of graphs. These two classes of graphs are permutation graphs of complete graphs and permutation graphs of complete bipartite graphs.

## Preliminaries

The vulnerability parameters studied in this paper are integrity, toughness, and binding number. These vulnerability properties are currently of growing interest among graph theorists and network designers. They have been studied recently by many authors on many different classes of graphs. Barefoot, Entringer, and Swart have found the integrity, toughness, and binding number of trees and powers of cycles [1,2]. Katerinis and Woodall have found relationships between the binding number and the existence of $k$-factors in [10] while Enomoto, Jackson, Katerinis, and Saito have similarly found relationships between toughness and the existence of $k$-factors [6,7]. Results on the binding number of different types of product graphs have been found by Guichard [8], Kane, Mohanty, and Hales [9], Liu and Tian [11], Luo [12], and Wang, Tian, and Liu [20,21].

The integrity of a graph $G$, $I(G)$, was defined in [2] and is defined as $I(G) = \min\{|S| + m(G-S)\}$, where the minimum is taken over all subsets $S$ of $V(G)$ and $m(G-S)$ is the number of vertices in a largest component of $G-S$. The toughness of a graph $G$, $t(G)$, as defined by Chvátal in [4], is defined to be $t(G) = \min\left\{\frac{|S|}{\omega(G-S)}\right\}$, where the minimum is taken over all disconnecting subsets $S$ of $V(G)$ and $\omega(G-S)$ is the number of components in $G-S$. The binding number of a graph $G$, $b(G)$, was defined by Woodall in [22] and is defined as $b(G) = \min\{\frac{|N(S)|}{|S|}\}$, where the minimum is taken over all nonempty subsets $S$ of $V(G)$ such that $N(S) \neq V(G)$, where $N(S)$ is the open neighborhood of $S$. The other measure of vulnerability discussed in this paper is the connectivity of a graph $G$, denoted $\kappa(G)$, which is defined to be the order of a smallest disconnecting set of vertices of $G$.

Given a graph $G$ with vertices labeled $1, 2, \cdots, n$ and a permutation $\alpha$ in $S_n$, the permutation graph $P_\alpha(G)$ is obtained by taking two copies of $G$, say $G_x$ with vertices $x_1, x_2, \cdots, x_n$ and $G_y$ with vertices $y_1, y_2, \cdots, y_n$, along with edges joining $x_i$ in $G_x$ to $y_{\alpha(i)}$ in $G_y$. It is known that permutation graphs have high connectivity properties, as is shown in "On the Cut Frequency Vector of Permutation Graphs" [17] and "Connectivity and Hamiltonian Properties of Permutation Graphs" [13]. Note also that permutation graphs are in some ways generalizations of the Cartesian product since if $\alpha = (1)$ then $P_\alpha(G) \cong G \times K_2$. Permutation graphs of cycles have been studied extensively as generalizations of the Petersen Graph in [3, 5, 14, 15, 18, 19] and their vulnerability properties were studied in [16]. In this paper the authors study the integrity of permutation graphs of $K_n$ and the toughness, integrity, and binding number of permutation graphs of $K_{m,n}$.

## Complete Graphs

It is clear that for any $\alpha$ in $S_n$, $P_\alpha(K_n) \cong K_2 \times K_n$. We begin by stating some known results which determine some of the vulnerability parameters for $P_\alpha(K_n)$.

**Theorem 1 [13].** *For $\alpha$ in $S_n$, $\kappa(P_\alpha(K_n)) = n$.*

**Theorem 2 [4].** $t(K_m \times K_n) = \frac{m+n}{2} - 1 \ (m, n \geq 2)$.

**Corollary 2.1.** *For $\alpha$ in $S_n$, $t(P_\alpha(K_n)) = \frac{n}{2}$.*

**Theorem 3 [9].** $b(K_m \times K_n) = \begin{cases} 1, & \text{if } m = n = 2 \\ \frac{mn-1}{1+(m-1)(n-1)}, & \text{otherwise.} \end{cases}$

**Corollary 3.1.** *For $\alpha$ in $S_n$, $b(P_\alpha(K_n)) = \begin{cases} 1, & \text{if } n \leq 2 \\ \frac{2n-1}{n}, & \text{otherwise.} \end{cases}$*

The final result on the permutation graphs of complete graphs gives the integrity. The notation $\lceil x \rceil$ shall denote the least integer greater than or equal to $x$.

**Theorem 4.** *For $\alpha$ in $S_n$, $I(P_\alpha(K_n)) = \lceil \frac{3n}{2} \rceil$.*

**Proof:** Let $G_x$ and $G_y$ be the copies of $K_n$ in $P_\alpha(K_n)$. Clearly, if $S$ is not a disconnecting set then $|S| + m(P_\alpha(K_n) - S) = 2n$. So, assume $S$ is a disconnecting set. Let $S_x = V(G_x) \cap S$ and $S_y = V(G_y) \cap S$. Since $S$ is a disconnecting set it follows that $1 \leq |S_x|, |S_y| \leq n - 1$. Furthermore, a vertex in $V(G_x) - S_x$ cannot be adjacent to any vertex in $V(G_y) - S_y$; otherwise $P_\alpha(G) - S$ would be connected. So, in $P_\alpha(G)$, the vertices in $G_y$ adjacent to the vertices of $V(G_x) - S_x$ are in $S_y$ and $|S_y| \geq |V(G_x) - S_x|$. Let $k = |S_x|$, where $1 \leq k \leq n-1$. Then $|S_y| = n - k + q$, where $0 \leq q \leq k$. Hence $|S| = n + q$ and $m(P_\alpha(K_n) - S) = \max\{n - k, k - q\}$. Thus, $\min\{|S| + m(P_\alpha(K_n) - S)\}$, taken over all disconnecting subsets of vertices is equal to

$$
\begin{aligned}
\min_{k,q}\{n+q+\max\{n-k,k-q\}\} &= n+\min_{k,q}\{q+\max\{n-k,k-q\}\} \\
&= n+\min_{k,q}\{\max\{n-k+q,k\}\} \\
&= n+\min_{k}\{\min_{q\le k}\{\max\{n-k+q,k\}\}\} \\
&= n+\min_{k}\{\max\{n-k,k\}\} \\
&= n+\lceil\frac{n}{2}\rceil \\
&= \lceil\frac{3n}{2}\rceil.
\end{aligned}
$$

Now, for $n \geq 1$, $\lceil\frac{3n}{2}\rceil \leq 2n$ and so $I(P_\alpha(K_n)) = \lceil\frac{3n}{2}\rceil$. ∎

**Complete Bipartite Graphs**

We now turn our attention to permutation graphs of the complete bipartite graphs, $K_{m,n}$. We will assume a standard labelling of the vertices of $K_{m,n}$. So, assume that $m \leq n$ and that $M$ and $N$ are the sets of the partition of size $m$ and $n$ respectively. Furthermore, assume that the vertices of $M$ are labelled $1, 2, \cdots, m$ and that vertices of $N$ are labelled $m+1, m+2, \cdots, m+n$. For the permutation graph $P_\alpha(K_{m,n})$ let $q$ denote the number of vertices in $M_x$ that are joined by permutation edges to vertices in $M_y$. The connectivity, toughness, integrity, and binding number of $P_\alpha(K_{m,n})$ can be expressed in terms of the parameters $m$, $n$, and/or $q$ as shown in the following theorems.

**Theorem 5 [13].** *For $\alpha$ in $S_{m+n}$, $\kappa(P_\alpha(K_{m,n})) = m+1$.*

In the proofs of the remaining theorems we will use the following definitions and observations. Let $M_x'$ be the set of vertices in $M_x$ that are joined by permutation edges to vertices in $M_y$ and let $M_y'$ be these vertices in $M_y$. So $|M_x'| = |M_y'| = q$. Let $M_x'' = M_x - M_x'$ and $M_y'' = M_y - M_y'$ and thus $|M_x''| = |M_y''| = m - q$. Now the vertices in $M_x''$ are adjacent to vertices in $N_y$ by permutation edges. Call this set of vertices $N_y''$. Similarly define $N_x''$ to be the set of vertices in $N_x$ adjacent to the vertices in $M_y''$ by permutation edges. Thus $|N_x''| = |N_y''| = m - q$. Finally let $N_x' = N_x - N_x''$ and $N_y' = N_y - N_y''$. Clearly the vertices in $N_x'$ are adjacent to the vertices in $N_Y'$ by permutation edges and $|N_x'| = |N_y'| = n - m + q$. Note that since $0 \leq q \leq m$ some of these sets may be empty. Let $K = \{M_x', M_x'', M_y', M_y'', N_x', N_x'', N_y', N_y''\}$. The relationship among these sets is shown in Figure 1.

**Lemma 6.** *There exists a disconnecting set $S'$ of $P_\alpha(K_{m,n})$ with $t(P_\alpha(K_{m,n})) = \frac{|S'|}{\omega(P_\alpha(K_{m,n})-S')}$ such that for all $Z$ in $K$, if $Z \cap S'$ is nonempty then $Z \subseteq S'$.*

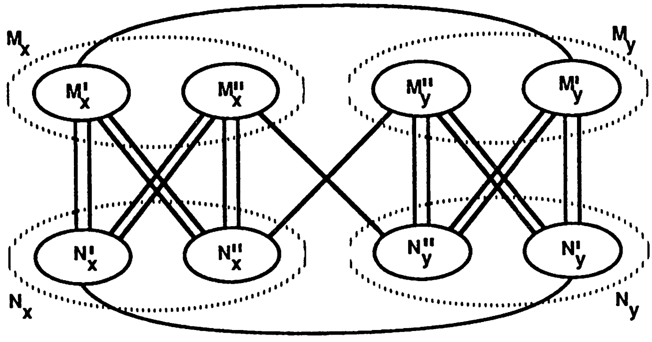


Figure 1. Relationship among the sets in K .

**Proof:** We do the case when $Z = M'_x$. Let $S$ be a disconnecting set of $P_\alpha(K_{m,n})$ with $t(G) = \frac{|S'|}{\omega(P_\alpha(K_{m,n})-S)}$ and let $A_x = S \cap M'_x$ and $B_x = M'_x - A_x$. Suppose that $A_x$ and $B_x$ are both nonempty. The proof proceeds in four cases.

**Case 1:** If $N_x$ and $N_y$ are not contained in $S$ then all of the vertices in $N_x - S$ are in the same component since $B_x$ is nonempty. Let $x_i$ be in $A_x$. Then $y_{\alpha(i)}$ is in $A_y$. If $y_{\alpha(i)}$ is in $S$ let $T = S - \{x_i\}$. Then $|T| = |S| - 1$ and $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S)$, since $x_i$ is in the same component as the vertices in $N_x$ in $P_\alpha(K_{m,n}) - T$. Thus $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, which is a contradiction. Hence $v$ is not in $S$ for all $v$ in $A_y$ and so $A_y \cap S$ is empty. Also, if there exists $v$ in $B_y$ such that $v$ is not in $S$ then the component of $P_\alpha(K_{m,n}) - S$ containing $v$ contains a vertex in $B_x$ and so contains all vertices in $N_y - S$ and $N_x - S$. This implies that this component contains all of the vertices in $P_\alpha(K_{m,n}) - S$ and so $S$ is not a disconnecting set, a contradiction. Thus $B_y$ is contained in $S$. Define $T = S - B_y \cup B_x$. Clearly $|T| = |S|$. If $M''_x$ is not contained in $S$ then $\omega(P_\alpha(K_{m,n})-T) = \omega(P_\alpha(K_{m,n})-S)$ and so $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$ and $M'_x$ is contained in $T$. If $M''_x$ is in $S$ then $\omega(P_\alpha(K_{m,n})-T) = \omega(P_\alpha(K_{m,n})-S) + |N_x - S| - 1$ and so $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$. Thus $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$ and $M'_x$ is contained in $T$.

**Case 2:** If $N_x$ is contained in $S$ and $N_y$ is not contained in $S$ then let $x_i$ be an element of $A_x$ and so $y_{\alpha(i)}$ is in $A_y$. Let $T = S - \{x_i\}$ and so $|T| = |S| - 1$. If $y_{\alpha(i)}$ is not in $S$ then $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S)$ and if $y_{\alpha(i)}$ is in $S$ we have $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S) + 1$. So $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, a contradiction. Therefore this case yields a contradiction.

**Case 3:** If $N_x$ is not contained in $S$ and $N_y$ is contained in $S$ then let $x_i$ be in $A_x$

and so $y_{\alpha(i)}$ is in $A_y$. If $y_{\alpha(i)}$ is in $S$ let $T = S - \{x_i\}$. Thus $|T| = |S| - 1$ and $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S)$, implying that $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, a contradiction. Hence $A_y \cap S$ is empty. Now, let $x_i$ be in $B_x$. Then $y_{\alpha(i)}$ is in $B_y$. If $y_{\alpha(i)}$ is in $S$ then let $T = S - \{y_{\alpha(i)}\} \cup \{x_i\}$. Hence $|T| = |S|$ and $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S) + 1$, again a contradiction. So $M'_y \cap S$ is empty. Now, if $t(P_\alpha(K_{m,n})) < 1$ then let $T = S - \{x_i\}$ for some $x_i$ in $A_x$. Hence $|T| = |S| - 1$ and $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S) - 1$. So $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, a contradiction. If $t(P_\alpha(K_{m,n})) > 1$ then let $T = S \cup \{x_i\}$ for $x_i$ in $B_x$. Then $|T| = |S| + 1$ and $\omega(P_\alpha(K_{m,n}) - T) = \omega(P_\alpha(K_{m,n}) - S) + 1$. Hence $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, another contradiction. Finally if $t(P_\alpha(K_{m,n})) = 1$ then let $T = S \cup B_x$. Thus $|T| = |S| + |B_x|$ and $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S) + |B_x|$. Hence $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} \leq \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$ so $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} = \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$ and $M_x$ is in $T$.
**Case 4:** If $N_x$ and $N_y$ are contained in $S$ then let $T = S - A_x$. Hence $|T| = |S| - |A_x| < |S|$ and $\omega(P_\alpha(K_{m,n}) - T) \geq \omega(P_\alpha(K_{m,n}) - S)$. So $\frac{|T|}{\omega(P_\alpha(K_{m,n})-T)} < \frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$, a contradiction. Therefore this case yields a contradiction.

The remaining cases progressively update the set under consideration to obtain the set $S$ satisfying the conditions of the lemma. Notice that when the set $M'_y$ is considered the case similar to case 1 is not applicable since either $M'_x$ is contained in $S$ or $M'_x$ and $S$ are disjoint. Thus the status of $M'_x$ remains unchanged. Similarly, we can see that at any stage all previously considered sets remain unchanged. ∎

**Theorem 7.** *For $\alpha$ in $S_{m+n}$ and $m \leq n$,*

$$t(P_\alpha(K_{m,n})) = \begin{cases} \frac{2m}{n+m-q}; & \text{if } q < \frac{n^2+m^2}{n+3m} \\ \frac{n+m}{n+q}; & \text{if } q \geq \frac{n^2+m^2}{n+3m}. \end{cases}$$

**Proof:** By Lemma 6 we need only consider disconnecting sets of $P_\alpha(K_{m,n})$ which are obtained by taking unions of elements of $K$. It is easy to find the 55 disconnecting sets of this type. Most of these sets trivially do not give disconnecting sets of $P_\alpha(K_{m,n})$ that yield $t(P_\alpha(K_{m,n}))$. There are four sets which are not trivial. These sets are $S_1 = M_x \cup M''_y$, $S_2 = M_x \cup M_y$, $S_3 = M_x \cup N_y$, and $S_4 = M_x \cup N'_y$. The values for $\frac{|S|}{\omega(P_\alpha(K_{m,n})-S)}$ given by these sets are $v_1 = \frac{2m-q}{m-q+1}$, $v_2 = \frac{2m}{n+m-q}$, $v_3 = \frac{n+m}{n+q}$, and $v_4 = \frac{n+q}{n-m+q+1}$. For fixed $m$ and $n$ it is clear that as $q$ increases the following occur. The value $v_1$ increases, so the minimum value for $v_1$ is $\frac{2m-1}{m}$, $v_4$ decreases, so the minimum value for $v_4$ is $\frac{n+m-1}{n}$, and $v_2$ increases, so the maximum value for $v_2$ is $\frac{2m}{n+1}$. It is easy to see that the minimum values for $v_1$ and $v_4$ are larger than the maximum value for $v_2$ and so $S_1$ and $S_4$ may be discarded.

Now, $v_3$ decreases as $q$ increases and the intersection point for $v_2$ and $v_3$ occurs where $q = \frac{n^2+m^2}{3m+n}$ and the theorem follows. ∎

Next, we will find the integrity of $P_\alpha(K_{m,n})$. Again in order to find the integrity we will prove a lemma similar to Lemma 6 which will allow us to consider only unions of sets in $K$.

Recall that for any graph $G$, if $S$ is not a disconnecting set of $G$ then $|S| + m(G-S) = p$, where $p = |V(G)|$. Hence, in order to have $|S|+m(G-S) < p$, $S$ must be a disconnecting set. Also, if $S = M_x \cup M_y$ then $|S|+m(P_\alpha(K_{m,n})-S) = 2m+2$ and so we know that $I(P_\alpha(K_{m,n})) \leq 2m+2$.

**Lemma 8.** *There exists a disconnecting set $S'$ of $P_\alpha(K_{m,n})$ with $I(P_\alpha(K_{m,n})) = |S'| + m(P_\alpha(K_{m,n}) - S')$ such that for all $Z$ in $K$, if $Z \cap S'$ is nonempty then $Z \subseteq S'$.*

**Proof:** We again do the case when $Z = M'_x$. Let $S$ be a disconnecting set with $I(P_\alpha(K_{m,n})) = |S|+m(P_\alpha(K_{m,n})-S)$. Let $A_x = S \cap M'_x$ and $B_x = M'_x - A_x$. Suppose $A_x$ and $B_x$ are both nonempty. The proof proceeds in three cases.

**Case 1:** If $N_x$ is contained in $S$ then let $T = S - A_x$. Then $|T| = |S| - |A_x|$ and $m(P_\alpha(K_{m,n})-T) \leq m(P_\alpha(K_{m,n})-S)+|A_x|$. Hence $|T|+m(P_\alpha(K_{m,n})-T) \leq |S| + m(P_\alpha(K_{m,n})-S)$. Therefore $T$ is a disconnecting set with $|T|+m(P_\alpha(K_{m,n}) - T) = I(P_\alpha(K_{m,n}))$ and $M'_x \cap T$ is empty.

**Case 2:** If $N_x$ is not contained in $S$ and $N_y$ is contained in $S$ then let $T = S \cup B_x$. Then $|T| = |S| + |B_x|$ and, since $B_x$ must be contained in the largest component of $P_\alpha(K_{m,n}) - S$ and all others are isolated vertices, $m(P_\alpha(K_{m,n})-T) \leq m(P_\alpha(K_{m,n})-S)-|B_x|$. Thus $|T|+m(P_\alpha(K_{m,n})-T) \leq |S|+m(P_\alpha(K_{m,n})-S)$, so $T$ is a disconnecting set with $|T| + m(P_\alpha(K_{m,n}) - T) = I(P_\alpha(K_{m,n}))$ and $M'_x$ is contained in $T$.

**Case 3:** If neither $N_x$ nor $N_y$ are contained in $S$ then consider $M_y$. If $M_y$ is contained in $S$ then the proof is similar to the proof of case 2 . Now, if $M_y$ is not contained in $S$ then $P_\alpha(K_{m,n}) - S$ contains exactly 2 components, one in each copy of $K_{m,n}$. If neither $x_i$ nor $y_{\alpha(i)}$ is in $S$ then $S$ is not a disconnecting set. Hence at least one of $x_i$ and $y_{\alpha(i)}$ is in $S$ for all $i = 1, \cdots, n+m$. Thus $|S| \geq m+n$. Let $C$ be the component of $P_\alpha(K_{m,n}) - S$ containing $B_x$. Then

$$\begin{aligned} I(P_\alpha(K_{m,n})) &= |S| + m(P_\alpha(K_{m,n}) - S) \\ &\geq |S| + |V(C)| \\ &\geq n+m+|B_x|+|N_x - S| \\ &\geq n+m+2. \end{aligned}$$

But by the previous remark, $I(P_\alpha(K_{m,n})) \leq 2m+2$. Hence, in this case $I(P_\alpha(K_{m,n})) = 2m+2$. Let $T = M_x \cup M_y$. Then $T$ is a disconnecting set with $|T| + m(P_\alpha(K_{m,n}) - T) = I(P_\alpha(K_{m,n}))$ and $M'_x$ is contained in $T$.

As in Lemma 6 we can progressively update the set under consideration to obtain the set $S'$ satisfying the conditions of the lemma. ∎

**Theorem 9.** *For $\alpha$ in $S_{m+n}$ and $m \leq n$,*

$$I(P_\alpha(K_{m,n})) = \begin{cases} 2m+1; & \text{if } n = m \text{ and } q \in \{0, m\} \\ 2m+2; & \text{otherwise.} \end{cases}$$

**Proof:** By the previous construction we know there exists a disconnecting set $S$ such that $|S| + m(P_\alpha(K_{m,n}) - S) = I(P_\alpha(K_{m,n}))$. The proof now proceeds in a manner similar to the proof of Theorem 7 . By Lemma 8 we need only consider the 55 disconnecting sets of $P_\alpha(K_{m,n})$ obtained by taking unions of elements of $K$. Again, all but 5 of these sets may easily be discarded as giving too large a value for $|S| + m(P_\alpha(K_{m,n}) - S)$. The remaining sets are $S_1 = M_x \cup M_y$, $S_2 = M_x \cup M_y \cup N'_x$, $S_3 = M_x \cup M''_y \cup N_y$, $S_4 = M'_x \cup M''_y \cup N'_x \cup Ny''$, and $S_5 = M_x \cup M''_y \cup N'_y$ with the associated values of $|S| + m(P_\alpha(K_{m,n}) - S)$ given by $v_1 = 2m+2$, $v_2 = n+m+q+1$, $v_3 = n+2m-q+1$, $v_4 = \max\{n+3m-2q, 2n+2q\}$, and $v_5 = n+2m$.

If $q \neq 0$ and $q \neq m$ then $n \geq m \geq 2$ and clearly $\min\{v_1, v_2, v_3, v_4, v_5\} = v_1$. Hence $I(P_\alpha(K_{m,n})) = 2m+2$. If $q = 0$ then it is easy to see that $v_3 \geq v_2$, $v_4 \geq v_2$, and $v_5 \geq v_2$. Also, if $m = n$ then $v_2 = 2m+1 < 2m+2$ but otherwise $v_1 \leq v_2$. Finally, if $q = m$ then $v_2 \geq v_1$, $v_4 \geq v_1$, and $v_5 \geq v_3$. Again, if $n = m$ then $v_3 = 2m+1 < 2m+2$ but otherwise $v_1 \leq v_3$. ∎

Finally, we determine the binding number of $P_\alpha(K_{m,n})$. Note that it is impossible to prove a result similar to Lemmas 6 and 8 since if $m = 4$, $n = 4$, and $q = 2$ then all the sets in $K$ have order 2 while $b(P_\alpha(K_{m,n})) = \frac{55}{11}$. Thus, $|S| = 11$, and $S$ cannot be the union of sets in $K$. Hence, a different approach must be taken.

**Theorem 10.** *For $\alpha$ in $S_{m+n}$, and $m \leq n$,*

$$b(P_\alpha(K_{m,n})) = \begin{cases} \frac{n+q}{n}, & q < \frac{nm}{2n+m-1} \\ \frac{2m+2n-1}{m+2n-1}, & \frac{nm}{2n+m-1} \leq q < \frac{m^2+3mn-2m}{4n+2m-2} \\ \frac{n+3m-2q}{n+m}, & \frac{m^2+3mn-2m}{4n+2m-2} \leq q. \end{cases}$$

**Proof:** First, we compute $\min\left\{\frac{|N(S)|}{|S|}\right\}$ under a variety of assumptions about $S$, then note which of these is smallest in various circumstances. In all cases we assume $n \geq 2$ and that any set $S$ such that $N(S) = V(P_\alpha(K_{m,n}))$ is not considered when the minimum is computed.

**Case 1:** Suppose $S \subseteq M_x$ (or $M_y$). Then $|N(S)| = |S| + n$. The minimum is $b_{11} = \frac{m+n}{m}$ and this minimum occurs when $S = M_x$ (or $M_y$). Similarly, if $S \subseteq N_x$ (or $N_y$), the minimum is $b_{12} = \frac{m+n}{n}$ and this occurs when $S = N_x$ (or $N_y$).

**Case 2:** Suppose $S \subseteq M_x \cup N_x$, $S \cap M_x \neq \emptyset$, and $S \cap N_x \neq \emptyset$. Then $|N(S)| = |S| + n + m$. In this case, the minimum is $b_{21} = \frac{2n+2m-1}{n+m-1}$ and this occurs when $S = M_x \cup N_x$ minus a vertex.

**Case 3:** Suppose $S \subseteq M_x \cup M_y$, $S \cap M_x \neq \emptyset$, and $S \cap M_y \neq \emptyset$. Then $|S| = |S \cap M'_x| + |S \cap M'_y| + |S \cap M''_x| + |S \cap M''_y|$ and $|N(S)| = 2n + |S \cap M'_x| + |S \cap M'_y|$. If $q < m$, then the minimum is $b_{31} = \frac{n+q}{m}$ and this occurs when $S = M_x \cup M_y$. If $m = q$, then the minimum is $b_{32} = \frac{2n+2m-1}{2m-1}$ and this occurs when $S = M_x \cup M_y$ minus a vertex.

Similarly, if $S \subseteq N_x \cup N_y$, $S \cap N_x \neq \emptyset$, and $S \cap N_y \neq \emptyset$, then there are two possibilities depending upon the relationship between $q$ and $m$. If $q < m$, then the minimum is $b_{33} = \frac{n+q}{n}$ and this occurs when $S = N_x \cup N_y$. If $m = q$, then the minimum is $b_{34} = \frac{2n+2m-1}{2n-1}$ and this occurs when $S = N_x \cup N_y$ minus a vertex.

**Case 4:** Suppose $S \subseteq M_x \cup N_y$, $S \cap M_x \neq \emptyset$, and $S \cap N_y \neq \emptyset$. Then $|S| = |S \cap M'_x| + |S \cap N'_y| + |S \cap M''_x| + |S \cap N''_y|$ and $|N(S)| = m + n + |S \cap M''_x| + |S \cap N''_y|$. If $m < n$ or $q > 0$, then the minimum is $b_{41} = \frac{n+3m-2q}{n+m}$ and this occurs when $S = M_x \cup N_y$. If $m = n$ and $q = 0$, then the minimum is $b_{42} = \frac{4n-1}{2n-1}$ and this occurs when $S = M_x \cup N_y$ minus a vertex.

**Case 5:** Suppose $S \subseteq N_x \cup N_y \cup M_y$, $S \cap N_x \neq \emptyset$, $S \cap N_y \neq \emptyset$, and $S \cap M_y \neq \emptyset$. Then $|S| = |S \cap N_x| + |S \cap M'_y| + |S \cap M''_y| + |S \cap N'_y| + |S \cap N''_y|$ and $|N(S)| = 2m + n + |S \cap M''_y| + |S \cap N'_y|$. The minimum is $b_{51} = \frac{2m+2n-1}{m+2n-1}$ and this occurs when $S = N_x \cup N_y \cup M_y$ minus a vertex of $M''_y$ or $N'_y$.

Similarly, if $S \subseteq M_x \cup M_y \cup N_y$, $S \cap M_x \neq \emptyset$, $S \cap M_y \neq \emptyset$, and $S \cap N_y \neq \emptyset$, then the minimum is $b_{52} = \frac{2m+2n-1}{n+2m-1}$ and this occurs when $S = M_x \cup M_y \cup N_y$ minus a vertex of $M'_y$ or $N''_y$.

It is easy to see that $b_{11} \geq b_{12}$, $b_{21} \geq b_{12}$, $b_{31} \geq b_{33}$, $b_{32} \geq b_{34} \geq b_{41}$, $b_{42} \geq b_{33}$, $b_{52} \geq b_{51}$, $b_{12} \geq b_{33}$ when $q \leq m$, and $b_{12} \geq b_{41}$ when $q = m$. So, in all cases, $b(P_\alpha(K_{m,n}))$ is equal to $b_{33}$, $b_{41}$, or $b_{51}$. It is easy to check that $b_{33} < b_{51}$ iff $q < \frac{mn}{2n+m-1}$, $b_{51} < b_{41}$ iff $q < \frac{m^2+3mn-2m}{4n+2m-2}$, and $b_{33} < b_{41}$ iff $q < \frac{2mn}{3n+m}$. The theorem follows since $\frac{mn}{2n+m-1} \leq \frac{2mn}{3n+m} \leq \frac{m^2+3mn-2m}{4n+2m-2}$. ∎

Note that the binding number of $P_\alpha(K_{m,n})$ is always at least 1 and is equal to 1 if and only if $q \in \{0, m\}$. Also, the three cases of interest in the theorem are illustrated by choosing $n = 10$, $m = 8$, and $q = 2, 4$, and 6, respectively.

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