k-Saturated Graphs of Chromatic Number at Least k

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Abstract. In this paper we consider the structure of k-saturated graphs ($G \not\supset K_k$ but $G + e \supset K_k$ for all possible edges e) having chromatic number at least k.

Introduction

A (simple) graph G_n on n vertices, is said to be k- saturated if G does not contain a complete subgraph, K_k, on k vertices but does so whenever any new edge is added. These graphs have a long history dating back to Turan's celebrated paper in 1941 (see [6]). Turan's Theorem states that if a graph G_n has more than $\frac{k-2}{2(k-1)}(n^2 - r^2) + \binom{r}{2}$ edges where $n = t(k-1) + r, t \geq 0, 0 \leq r < k-1$, then G_n contains a K_k. Furthermore, the complete $(k-1)$-partite graph, $T_{k-1,n}$, where each of the $k-1$ parts has either t or $t+1$ vertices, is the unique extremal example. Turan's Theorem has been generalized in many ways. Among these results are the Erdös-Stone Theorem, [3], 1946, which states: given a positive integer t and $\epsilon > O$, if G_n has more than $\frac{1}{2}(\frac{k-2}{k-1} + \epsilon)n^2$ edges, $n > n_o(t, e)$, then G_n contains a $T_{k,kt}$. That is, if t and n are large enough, G_n contains any k-chromatic graph.

In this paper we consider the following problem. Suppose G_n is k-saturated and the chromatic number of G_n satisfies $\chi(G_n) \geq k$. How many edges, $e(G_n)$, can such a graph have and what are the extremal graphs? Our investigation of this is motivated in part by a result obtained by Erdös and Gallai and independently by Andrasfai (see [2] or exercise 7.3.3 [1]). They show the extremal graphs for $k = 3$ consist of a 5-cycle together with a complete bipartite graph on $n-5$ vertices, each of which is joined to exactly two vertices of the 5-cycle in such a way as to not create a triangle (see figure 1).

The problem we are considering has been studied in some generality by Simonovits ([4],[5]), the most relevant theorem being Simonovits' Chromatic Perturbation Theorem which in our case states: If $n > n_o(k), \chi(G_n) \geq k$, but G_n does not contain a K_k, then there exists a constant K such that $e(G_n) < e(T_{k-1,n}) - \frac{n}{k-1} + K$. Our purpose here then is to give a detailed description of the k-saturated graphs G_n with a maximum number of edges when $\chi(G_n) \geq k$.

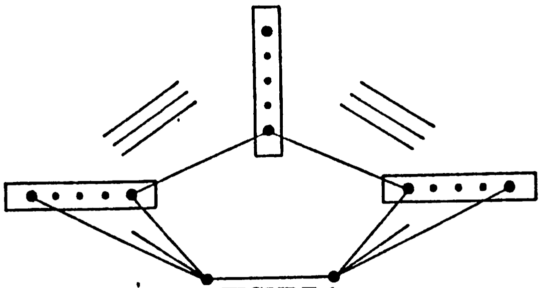

FIGURE 1

Statement of Results

For simplicity let us define the family of graphs $\mathcal{T}_n^k$ to be those graphs on n vertices consisting of a complete $(k-1)$-partite graph on $n-2$ vertices, with classes of independent points $C_1, C_2, \cdots, C_{k-1}$, together with two adjacent vertices x and y and where each vertex of C_1 is joined to precisely one of x or y, x and y each adjacent to at least one vertex of C_1, no vertex of C_2 is adjacent to either x or y and all vertices of $C_i, i > 2$ are adjacent to both x and y. It is easy to see that members of $\mathcal{T}_n^k$ are k-chromatic and k saturated. In fact we may think of such graphs for $k > 3$ as the join of a complete $(k-3)$ partite graph with the graph of figure 1, see figure 2.

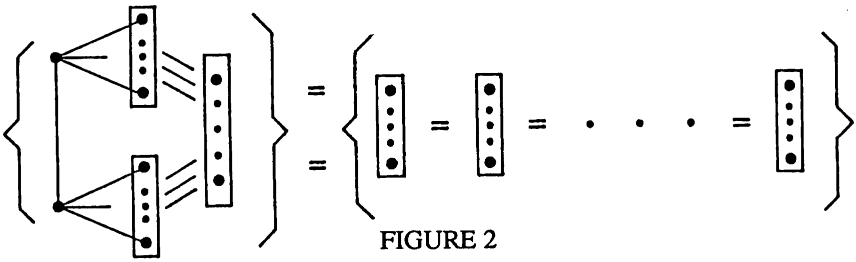

FIGURE 2

Define, for $k \geq 3$, $T'_{k-1,n}$ to be graphs in $\mathcal{T}_n^k$ for which $|C_1|+1, |C_2|+2, |C_3|, \cdots, |C_{k-1}|$ are equal or as equal as possible. For $n \geq 3k-4$ we can describe $T'_{k-1,n}$ as follows: let $n+1 = t(k-1)+r, 0 \leq r < k-1$ and let G_{n_o} denote a member of $\mathcal{T}_{n_o}^k$ on $n_o = n-r$ vertices and $e_o = e(T_{k-1,n-r}) - (t-2)$ edges where the classes C_i satisfy $|C_1| = t-1, |C_2| = t-2$ and $|C_i| = t, i > 2$. (G_{n_o} is unique up to adjacencies of x and y to class C_1). Define $T'_{k-1,n}$ to be a graph G_{n_o} with one vertex added to precisely r of the classes $C_1, \cdots, C_{k-1'}$. It is not too difficult to see that the graphs, $T'_{k-1,n}$, are maximal, with respect to the number of edges, in the family $\mathcal{T}_n^k$.

We have the following:

Theorem. *Let G_n be a maximal k-saturated graph on $n \geq k+2 \geq 5$ vertices with $\chi(G_n) \geq k$, then G_n is a $T'_{k-1,n}$ graph.*

The proof of the theorem consists of a number of steps, showing that G_n is in fact k-chromatic and has the correct form. This latter step involves showing that G_n contains a vertex x whose removal implies $\chi(G_n - x) \leq k - 1$. One technique we use in the proof is based on Zykov's proof, [7], of Turan's Theorem, called symmetrization by Simonovits (see [5]). Two vertices x and y in a graph are called symmetric if they have the same neighbours, i.e. $N(x) = N(y)$. We symmetrize a vertex u to a vertex v by removing the edges incident with u and joining u to $N(v)$. The important observation about symmetrization is that it preserves the property that the graph has no K_k.

Proof of the theorem: Let $G = G_n$ satisfy the conditions of the theorem and let

$$X = \{x \in V(G) | \quad \chi(G - x) \leq k - 1\}$$

We will show that $G \in \mathcal{T}_n^k$ Denote $E(G)$ by E. We have by symmetrization and the maximality of G, the following properties:

i) If $(x, y) \notin E$ where $x \notin X$ then $d(x) \geq d(y)$.
ii) If $(x, y) \in E$ where $x \notin X$ then $d(x) \geq d(y) - 1$.
iii) If $(x, y) \in E, (x, z), (y, z) \notin E$ and $\{x, y, z\} \cap X = \emptyset$,

then there exists a graph G' satisfying the conditions of the theorem with $X' \supseteq X \cup \{x\}$ where $X' = \{x \in V(G') | \chi(G' - x) \leq k - 1\}$.

We elaborate only on property iii). It follows from i) that $d(x) = d(y) = d(z)$. Symmetrize y to z. The resulting graph G' has the same number of edges as G, no K_k has been created and $\chi(G') \geq k$. Suppose that in G' we have that x is not critical i.e. $x \notin X'$. Symmetrizing x to z now results in a graph that is k-saturated and of chromatic number at least k but with one more edge, contradicting the maximality of G.

It follows from iii) that there exists graphs G satisfying the condition of the theorem with $X \neq \emptyset$. We first concentrate on these and then show that all graphs of the desired type have, in fact, $X \neq \emptyset$. For (x,y)$\notin$ E denote by $K(x, y)$ the complete k graph formed in G by the addition of the edge (x, y).

Case I: $X \neq \emptyset$. Assume for the moment that the theorem fails for some least k and that in an extremal graph G, $X \neq \emptyset$. (From our previous remarks we must have $k \geq 4$). If G contains a vertex $x \in X$ of degree $n - 1$ then $G - x$ is $(k - 1)$-saturated, of the correct form and hence G is k-chromatic and of the correct form. If no such vertex exists, consider an $x \in X$, then G has the form shown in figure 3 where x is adjacent to proper subsets A_1 and A_2 of classes C_1 and C_2 respectively and where G contains all edges of the form (u, w) where $u, w \notin N(x)$ and of the form (v, w) where $v \in N(x), w \notin N(x)$ (i.e. $G - x$ is a complete $(k - 1)$-partite graph with the exception of some edges of the form (s, t) where s and t are both neighbours of x). That both A_1 and A_2 exist follows from the observation that for $v \in A_1, u \in C_1 - A_1, K(u, v)$ does not contain x implying the existence of a vertex $w \in C_2 - N(x)$. The existence of the $(k-1)$-partite edges follows from the k-saturated property. We observe for $v \in A_1$, there is an edge missing to some other class, else $K(u,x)$ implies a K_k in G, namely $K(u,x) - \{u\} \cup \{v\}$. Since u is adjacent to all other vertices in all other classes, we have that $d(v) \leq d(u)$. The vertices of A_2 have similar properties. We look at two subcases, whether or not there is a $v \in A_1$ belonging to the set of critical vertices X.

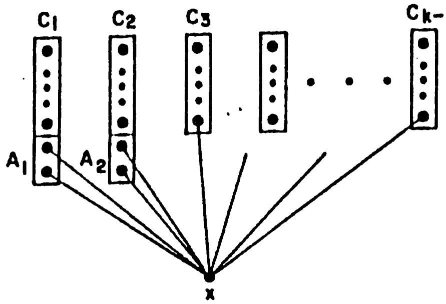

FIGURE 3

Case I.1: Suppose $v \in X$. Consider a $(k-1)$-colouring, $C(v)$, of $G - v$ and consider $K(u,x)$ where $u \in C_1 - A_1$. Since $|K(u,x) \cap C_i| = 1$ for each i, we may assume that the vertex of $K(u,x)$ in C_i is of colour i under $C(v)$ and that x is of colour 1. Let $w \in C_2 - A_2$, then every vertex of $K(w,x)$ other than those in C_1 or x itself is adjacent to u and therefore coloured with the colours $2,3,\cdots,k-1$ under $C(v)$. Thus $K(w,x) \cap C_1$ has colour 1 as does x. This impossibility implies that $K(w,x) \cap C_1 = \{v\}$ and that no other vertex of A_1 is critical. For $a \in A_1, a \neq v$, we have that $d(a) \leq d(u)$ by i) and therefore $d(a) = d(u)$. It follows that for such $a \in A_1$ there is exactly one edge missing from a to the classes $C_2, \cdots, C_{k-1}$. Consider such an $a \in A_1, a \neq v$. Let $K(u,x) \cap C_2 = \{b\}$. Under the colouring $C(v)$, b has colour 2 as does w (w is adjacent to all vertices of $K(u,x) - \{b,x\}$). If $(a,b) \notin E$, a is adjacent to w and all of $K(u,x) - \{u,b\}$, i.e. to vertices of colour $1,2,3,\cdots k-1$ under $C(v)$, a contradiction. If on the other hand we had that $(a,d) \notin E$ where for example $d = K(u,x) \cap C_{k-1}$, under $C(v)$ a is adjacent to vertices of colour $1,2,\cdots,k-2$, namely all of $K(u,x) - \{u,d\}$, so that a must have colour $k-1$. However $K(u,a) - (u,a)$ is coloured by $1,2,\cdots k-1$ under $C(v)$ hence u and a have the same colour, a contradiction. These contradictions together imply that if v is a critical vertex in A_1 then $A_1 = \{v\}$. Similarly if A_2 contain a critical vertex, $|A_2| = 1$.

In the event that A_1 contains such a critical vertex v a number of possibilities arise. First suppose $z \in A_2$ is also critical, then using the fact that our graph has a maximum number of edges and that each of v and z has at least one edge missing to another class, it is not to difficult to show using the maximality of E that $(v,z) \notin E$ and that $G \in \mathcal{T}_n^k$ and the theorem holds for this k also. Alternatively we may have that $v \in A_1$ is critical but no such vertex exists in A_2. Let the vertices of A_2 be $z_1, z_2, \cdots, z_p, \cdots, z_{p+q}$ where $(v, z_i) \in E$ iff $i \leq p$. Suppose $p \geq 1$. We may assume that x is adjacent to all vertices of $\cup_{i>2} C_i$ (otherwise there exists a graph G' contradicting the maximality of E, namely with v not adjacent to any vertex in A_2, x adjacent to $\cup_{i>2} C_i$ and all other $(k-1)$-partite edges present). However $K(w, z_1), w \in C_2 - A_2$, now implies the existence of a $K_k = K(w, z_1) - \{w\} \cup \{x\}$ in G, contradicting the fact that $p \geq 1$. Thus in this case also we must have $G \in \mathcal{T}_n^k$.

Case I. 2: We now consider the cases where no such classes A_i contain a critical vertex. Suppose $v \in A_1$. By (i) we have $d(v) \geq d(u)$ implying $d(v) = d(u)$ for $u \in C_1 - A_1$ and v has exactly one missing edge to the other classes. If for some pair of vertices $v \in A_1, z \in A_2$ the missing edges were to $\cup_{i>2} C_i$ then the graph G' mentioned at the end of Case I.1 would contradict the maximality of E. Hence we may assume that the missing edges associated with vertices in A_1 are all to vertices of A_2. Choose a vertex $u \in C_1 - A_1$ and consider $K(u, x)$. Let $K(u, x) \cap A_2 = \{a\}$. Then there exists a vertex $b \in A_1$ such that $(b, a) \in E$, (note $|A_1| > 1$), implying $K(u, x) - \{u\} \cup \{b\}$ is a K_k in G, a contradiction. It follows that we must have in fact that $A_1 = \{v\}$ and v is critical.

In all cases where $X \neq \emptyset$ we have, when G is maximal, $A_1 = \{v\}$, v is not adjacent to any vertices of A_2, x is adjacent to $\cup_{i>2} C_i$ and all other $(k-1)$-partite edges are present. In other words $G \in \mathcal{T}_n^k$ and the theorem holds for this case also.

Case II: $X = \emptyset$. In order to complete the proof of the theorem we must finally consider the case where for some extremal G, $X = \emptyset$. As we have mentioned, by iii) (from the symmetrization process) there exists a graph G' with $X' \neq \emptyset$. We may assume that G can be changed to G' by symmetrizing two vertices y and z of G leaving $x \in X'$ in G'. From our discussion of the cases where $X \neq \emptyset$ we may assume G' has the form exhibited in figure 3 with x adjacent to all of $\cup_{i>2} C_i, A_1 = \{v\}$, v non-adjacent to A_2 and all other $(k-1)$-partite edges being present. It should also be clear that G must have been of chromatic number k. If in G, $N(y) \cap C_1 = \emptyset$ then x was critical in G, a contradiction, therefore y is adjacent to some member of C_1 in G. We consider now different possibilities for G by removing the adjacencies of y in G' and replacing them by its adjacencies in G. G has the form illustrated in figure 4.

If y is not adjacent to some class, say C_j, we can simply add y to C_j implying x was in fact critical in G. Therefore we have that y is adjacent to some $u_i \in C_i$, $i = 1, \cdots, k-1$. These u_i's cannot form a K_{k-1}. This implies $u_1 \in A_1, u_2 \in A_2$ but then $\{y, x, u_1, u_3, \cdots, u_{k-1}\}$ is a K_k in G, a contradiction. Thus G cannot satisfy $X = \emptyset$ and the proof of the theorem is complete.

Conclusion

The extremal k-saturated graphs of chromatic number at least k are obtained by

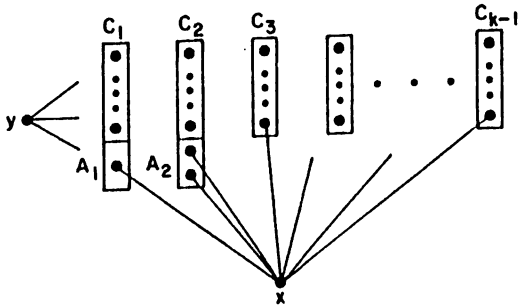

FIGURE 4

a very simple and seemingly obvious construction although it is difficult to see a short proof of it. The question of what the extremal graphs would look like if we insist that G be k-saturated and of fixed chromatic number $\ell > k$ remains open.

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