A Turan-Type Problem on 3-Graphs

Bing Zhou

Department of Mathematics
University of Alberta
Edmonton, Canada T6G 2G1

Abstract. In this paper we study a problem related to one of the Turan problems: What is the maximum number of edges in a 3-graph without a complete subgraph on five vertices, the K_5? We prove that the exact bound Turan conjectured is true if we forbid a larger class of subgraphs including K_5.

A hypergraph H is an ordered couple (V, E) where $V = V(H)$ is called the vertex set of H and $E = E(H)$ is a set of subsets of V, called edges of H. A hypergraph in which every edge has k vertices is called a $k-graph$. A $k-graph$ on p vertices is complete if every k-subset of the vertex set is an edge and is denoted by K_p^k, or simply K_p if the size of the edges is clear. If a vertex u and a $(k-1)-set$ S form an edge of H, we say that u is adjacent to S in H.

Let $f^k(n; K_p^k) = \max\{|E(H)| : H$ is a $k-graph$ on n vertices and H does not contain a ${K_p}^k\}$.

Turan [7] determined the values of $f^2(n; K_p^2)$ for all n and p and he asked [8] for the determination of $f^k(n; K_p^k)$. We refer the reader to [3] for more complete references of this and other Turan type problems. One of Turan's conjecture is

$$f^3(n; K_5^3) = n^2(n-2)/8, \tag{1}$$

and the extreme graph H_n^* is 2-colorable and with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices in each color class respectively. It has been shown that Turan's conjecture is not true for all values of n[2, 5]. One example is the affine plane with 9 points. Although we can construct counter examples of Turan's conjecture for 3-graph on as many as 22 vertices, we do not know if it is possible to do so for arbitrarily large number of vertices.

The main aim of this paper is to show that if we forbid a larger class of subgraphs containing K_5 called t-triples,the 3-graph H_n^* is indeed the extreme graph thus the bound in Turan's conjecture is true for this class of forbidden graphs.

In a 3-graph H,we say that two vertices x,y are t-connected if there are three vertices a,b, and c such that both x and y are adjacent to all of the three 2-subsets of $\{a, b, c\}$. Three vertices x, y, and z are called a t-triple ,if they are pairwise t-connected and $(x, y, z) \in E(H)$. In a K_5, every two vertices are t-connected so every three vertices form a t-triple. Therefore,the fact that a 3-graph does not contain a t-triple will imply that the graph does not contain K_5. The 3-graph H_n^* has no t-triple; the easy proof is left to the reader.

Theorem. *A 3-graph with n vertices that does not contain a t-triple has at most e_n edges, where*

$$e_n = \begin{cases} (n+1)(n-1)(n-2)/8, & n \text{ is odd,} \\ n^2(n-2)/8, & n \text{ is even.} \end{cases}$$

Furthermore, the only graph that has e_n edges is the graph H_n^*.

Proof: We induct on n. It is easy to check that the result of the theorem is true for $n \leq 6$. So we assume that $n > 6$ and the result holds for smaller values of n.

Let H be a 3-graph on n vertices that does not contain a t-triple and with the maximum number of edges. Since H_n^* has e_n edges,H has at least e_n edges. For every vertex $v \in V(H)$, we define the degree of v:

$$d(v) = |\{e \in E(H) : v \in e\}|.$$

Let $\delta = \min\{d(v) : v \in V(H)\}$,and

$$\delta = e_n = \begin{cases} (n-1)(3n-5)/8, & n \text{ is odd,} \\ 3n(n-2)/8, & n \text{ is even.} \end{cases}$$

Claim 1. The result of the theorem is true if $\delta \leq \delta_n$.

If there is a vertex x such that $d(x) < \delta_n$, then the 3-graph $H - x$ has no t-triple and more than e_{n-1} edges. This contradicts our inductive hypothesis.

If there is a vertex x such that $d(x) = \delta_n$,then $H - x = H_{n-1}^*$ by the inductive hypothesis.Let V_1,V_2 be the two color classes of the vertex set of $H - x$.If H is also 2-colorable,it is clear that our claim is true. Suppose that H is not 2-colorable, then there are four vertices $a, b \in V_1, c, d \in V_2$ and x is adjacent to both $\{a, b\}$ amd $\{c, d\}$.

Let $V' = V(H)\backslash\{x, a, b, c, d\}$. We define a graph G as follows: $V(G) = V'$,$E(G) = \{(u, v) : u, v \in V' \text{ and } (x, u, v) \in E(H)\}$.We claim that there is no triangle in G. Suppose there is a triangle $\{u, v, w\}$ in G. Without loss of generality, we may assume that $u, v \in V_1$. Then x, c, d are all adjacent to all 2-sets of $\{u, v, w\}$ and (x, c, d) is a t-triple. This is a contradiction.Therefore according to Turan's theorem,x is adjacent to at most $\lceil (n-5)/2 \rceil \cdot \lfloor (n-5)/2 \rfloor$ 2-sets of V'. Also there does not exist a vertex u in V' such that both (x, a, u) and (x, b, u) are in $E(H)$. Otherwise, (x, c, d) is a t-triple. Similarly,there does not exist a vertex v such that both (x, c, v) and (x, d, v) are in $E(H)$. Therefore thre are at most $2 \cdot (n-5)$ edges in H in the form of (x, r, s), where $r \in \{a, b, c, d\}$ and $s \in V'$. Since H does contain K_5, x is adjacent to at most five 2-sets of $\{a, b, c, d\}$. Then we have

$$d(x) = \delta_n \leq \lceil (n-5)/2 \rceil \cdot \lfloor (n-5)/2 \rfloor + 2(n-5) + 5$$

which is a contradiction for all $n > 6$, and this concludes our proof of Claim 1.

Now that we have established Claim 1,for the rest of our proof,we will assume that $d(v) > \delta_n$ for all $v \in V(H)$. We define the graph $G_H : V(G_H) = V(H)$ and (x, y) is an edge in G_H if x and y are t-connected in H.

Claim 2. There is no triangle in the complement of G_H.

Suppose this claim is not true and there is a triangle $\{x, y, z\}$ in the complement of G_H, i.e., any two vertices in $\{x, y, z\}$ are not t-connected. Denote by P the set of vertices that are adjacent to all three 2-sets of $\{x, y, z\}$ in H and let $|P| = p$. Since every two vertices in P are t-connected, there is no edge among the vertices in P. Therefore $p < \lceil n/2 \rceil$, otherwise $d(u) \leq \delta_n$ for every $u \in P$.

For every vertex $w \in \{x, y, z\}$, we define the "neighbor-graph" $G_w : V(G_w) = V(H)\backslash\{x, y, z\}, E(G_w) = \{(u, v) : (w, u, v) \in E(H)\}$. Consider the three neighbor-graphs G_x, G_y, and G_z. Let e_x, e_y, e_z be the numbers of edges and t_x, t_y, t_z be the numbers of triangles in G_x, G_y, G_z respectively. By a result of Moon and Moser [6, also 1, p.297],we have

$$t_x + t_y + t_z \geq \frac{1}{3(n-3)} \{e_x[4e_x - (n-3)^2] + e_y[4e_y - (n-3)^2] + e_z[4e_z - (n-3)^2]\}$$

$$\geq \frac{\bar{e}[4\bar{e} - (n-3)^2]}{n-3}$$

where $\bar{e}$ is the average of e_x, e_y and e_z.

Case 1. n is even.

There are at most $(n-2)/2$ vertices adjacent to all 2-sets of $\{x, y, z\}$. So

$$\bar{e} \geq \delta_n - 2(n-3) + \frac{2}{3} \cdot \frac{n-4}{2} = (9n^2 - 58n + 12)/24,$$

and

$$t_x + t_y + t_z \geq \frac{1}{24(n-3)}(9n^2 - 58n + 96)\left[\frac{1}{6}(9n^2 - 58n + 96) - (n-3)^2\right]$$

$$> \binom{n-3}{3}, \text{ for all } n > 6.$$

Therefore there is at least one triangle appearing in two of the graphs, say G_x and G_y.Then x and y are t-connected. This contradicts to our assumption.

Case 2. n is odd and $p \leq (n-3)/2$.

In this case we can get a contradiction by computation similar to the one in Case 1.

Case 3. n is odd and $p = (n-1)/2$.

In this case, every vertex in $\{x, y, z\}$ is adjacent to at least one 2-set of P. For example, if x is not adjacent to all 2-sets of P, then

$$d(x) \leq \binom{n-1}{2} - \binom{(n-1)/2}{2} = \delta_n.$$

Suppose that x and y are both adjacent to $\{u, v\} \subset P$. Then x, y are adjacent to all 2-sets of $\{z, u, v\}$. So x and y are t-connected. This contradiction shows that there are no two vertices in $\{x, y, z\}$ adjacent to the same 2-set of P.

We can 4-colour the 2-sets of P according to whether they are adjacent to x, y, z or none of the three. It is easy to check that for any such 4-colouring, there are at least $n-3$ 3-sets of P such that they contain different coloured 2-sets. These 3-sets cannot appear as triangles in any of the graphs G_x, G_y, G_z.

Therfore,as before,since $\bar{e} \geq \delta_n - 2(n-3) + \frac{2}{3} \cdot \frac{n-5}{2}$,

$$\begin{aligned} t_x + t_y + t_z &\geq \frac{9n^2 - 64n + 119}{24(n-3)} \left[\frac{9n^2 - 64n + 119}{6} - (n-3)^2\right] \\ &> \binom{n-3}{3} - (n-3) \end{aligned}$$

for all $n > 6$.There is at least one triangle appearing in two of the graphs. This contradiction concludes our proof of Claim 2.

A result of Lorden [4] states that if the complement of the graph G_H does not contain any triangles, G_H contains at least $\binom{n}{3} - e_n$ triangles and the unique extreme graph is the graph formed by two disconnected complete subgraphs with sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ respectively.

If $\{x, y, z\}$ is a triangle in G_H, then it is not an edge in H, otherwise it would be a t-triple. So combining this observation and Claim 2,we conclude that there are at least $\binom{n}{3} - e_n$ 3-subsets of $V(H)$ that are not edges of H, so $|E(H)| \leq e_n$ and $|E(H)| = e_n$ can only occur when H is the 3-graph H_n^*. This completes the proof of the theorem. ∎

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