# A Lattice Generated by (0, 1)-Matrices

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**Abstract.** We study the lattice generated by the class of $m$ by $n$ matrices of 0's and 1's with a fixed row sum vector and a fixed column sum vector.

## 1. Introduction

Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be positive integral vectors such that $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Let $\mathcal{A}(R, S)$ denote the class of all $m$ by $n$ (0, 1)-matrices having row sum vector $R$ and column sum vector $S$. We assume without loss of generality that $r_1 \geq \cdots \geq r_m$ and that $s_1 \geq \cdots \geq s_n$. There are well known necessary and sufficient conditions in order that the class $\mathcal{A}(R, S)$ be nonempty (see e.g. **[Rys]**) and we assume throughout that they are satisfied. The class $\mathcal{A}(R, S)$ has been well studied (see the survey paper **[Bru]**), and as is well known it can be identified with the set of all labelled bipartite graphs such that the degree sequence for the vertices in one set of the vertex-bipartition is $R$ and in the other is $S$. Recently there has been some interest in studying the (integer) lattices generated by combinatorially defined sets of vectors. The reason is threefold: (i) Some interesting lattices (and questions concerning them) may result. (ii) The lattices may shed some light on the combinatorially defined sets. (iii) The role of lattices in combinatorics may become clearer. For example, in **[Lov1]** and **[Lov2]** the lattice generated by the perfect matchings of a graph is described, in **[JunLec1]** the lattice generated by the $n$-tuples of 0's and 1's with a fixed number of 1's is studied while in **[JunLec2]** the lattice generated by the incidence vectors of perfect 2-matchings of a graph is studied, and in **[Rie]** the lattice generated by the incidence vectors of the bases of a matroid is investigated. In this note we study the lattice generated by the matrices in $\mathcal{A}(R, S)$. There are a number of results in the literature which directly or indirectly have some consequences for this lattice, and one of our aims is to point out these results.

Consider a class $\mathcal{A}(R, S)$, which as pointed out above is always assumed to be nonempty. We make one assumption about this class which is without any

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essential loss of generality. A position is called an *invariant 1-position* of $\mathcal{A}(R, S)$ provided every matrix in the class has a 1 in that position. An *invariant 0-position* is defined in a similar way and we refer to invariant 1-positions and invariant 0-positions collectively as *invariant positions*. A theorem of Ryser (see [Rys]) asserts that the class $\mathcal{A}(R, S)$ has an invariant 1-position if and only if there exist positive integers $e$ and $f$ such that every matrix in the class has a decomposition of the form

$$\begin{bmatrix} J_{ef} & A_1 \\ A_2 & O_{m-e,n-f} \end{bmatrix} \tag{1}$$

where $J_{ef}$ denotes an $e$ by $f$ matrix of all 1's and $O_{m-e,n-f}$ denotes a $m-e$ by $n-f$ zero matrix. When (1) holds the $ef$ positions in the upper left are invariant 1-positions and the $(m-e)(n-f)$ positions in the lower right are invariant 0-positions. Because we are assuming that the vectors $R$ and $S$ are positive vectors the class has invariant 1-positions whenever it has invariant 0-positions. The matrices $A_1$ and $A_2$ in (1) each determine a class of matrices with fixed row sum vector and fixed column sum vector, and the study of $\mathcal{A}(R, S)$ essentially reduces to the study of these two classes.

Let $t$ be the number of matrices in the class $\mathcal{A}(R, S)$, and let $A_1, \cdots, A_t$ be a listing of the matrices in this class. We define the *lattice*

$$\mathcal{L}(R, S) := \left\{ \sum_{i=1}^{t} c_i A_i : c_i \in Z \right\}, \tag{2}$$

the set of all integral linear combinations of the matrices in $\mathcal{A}(\mathcal{R}, S)$. If $X = [x_{ij}]$ and $Y = [y_{ij}]$ are $m$ by $n$ matrices, then their *inner product* is

$$X \circ Y := \sum_{i=1, j=1}^{m,n} x_{ij} y_{ij}.$$

The *dual lattice* of $\mathcal{L}(R, S)$ is the lattice

$$\mathcal{L}^*(R, S) := \{X \in Q^{m,n} : X \circ B \in Z \text{ for all } B \in \mathcal{L}(R, S)\}, \tag{3}$$

the set of all rational $m$ by $n$ matrices whose inner product with each matrix in $\mathcal{L}(R, S)$ is an integer. We also define the *linear space*

$$lin(R, S) := \left\{ \sum_{i=1}^{t} c_i A_i : c_i \text{ real} \right\}, \tag{4}$$

the set of all real linear combinations of matrices in $\mathcal{A}(R, S)$. Finally we define

$$lin_Z(R, S) := lin(R, S) \cap Z^{m,n}, \tag{5}$$

the set of all integral matrices in $lin(R, S)$.

In the next sections we study the four sets in (2)–(5).

## 2. The lattice $\mathcal{L}(R, S)$

We shall make use of the following result from [**BruHarHwa**, Theorem 2.1] and [**LewLiuLiu**, Theorem 1].

**Proposition 1.** *An $m$ by $n$ matrix $X = [x_{ij}]$ with real entries can be expressed as a linear combination of the matrices in $\mathcal{A}(R, S)$ with nonnegative coefficients if and only if there is a nonnegative number $q$ such that the row sum vector of $X$ is $qR$, the column sum vector of $X$ is $qS$, and $0 \leq x_{ij} \leq q$ ($1 \leq i \leq m$, $1 \leq j \leq n$).* ∎

**Corollary 1.** *An $m$ by $n$ matrix $X$ with real entries can be expressed as a linear combination of matrices in $\mathcal{A}(R, S)$ with nonnegative integer coefficients if and only if $X$ is an nonnegative integral matrix satisfying the properties in Proposition 1.*

**Proof:** Suppose that $X$ is a nonnegative integral matrix satisfying the properties in Proposition 1. It follows from Proposition 1 that there is a matrix $A \in \mathcal{A}(R, S)$ such that $X$ has positive entries in those positions in which $A$ has 1's and $A$ has 1's in those positions in which $X$ has $q$'s. The matrix $X' = X - A$ is a nonnegative integral matrix satisfying the properties in Proposition 1 with $q$ replaced with $q-1$. It now follows by induction on $q$ that $X$ is a nonnegative integer linear combination of the matrices in $\mathcal{A}(R, S)$. The converse is trivial. ∎

We define

$$J_{R,S} := A_1 + \cdots + A_t, \tag{6}$$

the sum of all matrices in $\mathcal{A}(R, S)$. The row sum and column sum vectors of $J_{R,S}$ are, respectively, $tR$ and $tS$. If $\mathcal{A}(R, S)$ has no invariant positions, then it follows that each entry of $J_{R,S}$ is an integer between 1 and $t - 1$.

We use the above proposition to characterize $lin(R, S)$.

**Proposition 2.** *Assume that the class $\mathcal{A}(R, S)$ has no invariant positions. An $m$ by $n$ real matrix belongs to $lin(R, S)$ if and only if there is a real number $q$ such that its row sum vector is $qR$ and its column sum vector is $qS$.*

**Proof:** Let $X$ be an $m$ by $n$ real matrix with row sum vector $qR$ and column sum vector $qS$ for some $q$. Since there are no invariant 0-positions, it is possible to choose an integer $k$ large enough so that the matrix $X' := X + kJ_{R,S}$ has all of its entries nonnegative, and row sum vector equal to $q'R$ and column sum vector equal to $q'S$ where $q' := q + k$ is positive. By Proposition 1, $X'$ is a nonnegative linear combination of the matrices in $\mathcal{A}(R, S)$, and it follows that that $X \in lin(R, S)$. The converse is trivial. ∎

We now characterize the lattice of all integer linear combinations of the matrices in $\mathcal{A}(R, S)$.

**Theorem 1.** *Assume that the class $\mathcal{A}(R,S)$ has no invariant positions. Then $\mathcal{L}(R,S)$ consists of all $m$ by $n$ integral matrices $X$ for which there exists an integer $q$ such that the row sum vector of $X$ is $qR$ and the column sum vector of $X$ is $qS$.*

**Proof:** We first observe that each matrix $X$ in $\mathcal{L}(R,S)$ satisfies the condition in the theorem. Now suppose that $X = [x_{ij}]$ is an $m$ by $n$ integral matrix with row sum vector $qR$ and column sum vector $qS$ for some integer $q$. Let $p$ be an integer. Since $pJ_{R,S}$ is in $\mathcal{L}(R,S)$, it follows that $X$ is in $\mathcal{L}(R,S)$ if and only if $X + pJ_{R,S}$ is in $\mathcal{L}(R,S)$. Since $J_{R,S}$ is a positive matrix, $X + J_{R,S}$ is a nonnegative matrix for $p$ large enough. Hence it suffices to assume that $X$ is a nonnegative matrix. If $x_{ij} \leq q$ for all $i,j$, then $X \in \mathcal{L}(R,S)$ by Corollary 1. Since $X = cJ_{R,S} - (cJ_{R,S} - X)$ for each integer $c$, it suffices to show that there exists a positive integer $c$ such that $cJ_{R,S} - X \in \mathcal{L}(R,S)$. Since $\mathcal{A}(R,S)$ has no invariant positions each entry of $J_{R,S} = [f_{ij}]$ is less than $t$ (the number of matrices in $\mathcal{A}(R,S)$), and we may choose $c$ large enough so that the matrix $G = [g_{ij}] := cJ_{R,S} - X$ satisfies

$$\begin{cases} g_{ij} & \geq 0 \\ \sum_j g_{ij} & = (ct - q)r_i \\ \sum_i g_{ij} & = (ct - q)s_j. \end{cases} \tag{7}$$

Since $f_{ij} < t$, we have $c(f_{ij} - t) < 0$. Hence for $c$ large enough,

$$q + c(f_{ij} - t) \leq 0 \leq x_{ij};$$

Thus

$$0 \leq g_{ij} = cf_{ij} - x_{ij} \leq ct - q$$

for all $i,j$. We now apply Corollary 1 to $G$ and conclude that $G$ and hence $X$ belongs to $\mathcal{L}(R,S)$. ∎

As a corollary we obtain the main result in [JunLec1].

**Corollary 2.** *Let $n$ and $k$ be integers with $1 \leq k \leq n-1$, and let $\mathcal{L}_{n,k}$ be the lattice generated by all $n$-tuples of 0's and 1's with exactly $k$ 1's. Then $\mathcal{L}_{n,k}$ consists of all integral $n$-tuples the sum of whose components is divisible by $k$.*

**Proof:** Let $R = (k, n-k)$ and let $S = (1,1,\cdots,1)$, the $n$-tuple of all 1's. Since $1 \leq k \leq n-1$, the class $\mathcal{A}(R,S)$ has no invariant positions. Hence by Theorem 1, $\mathcal{A}(R,S)$ consists of all integral matrices with row sum vector $(qk, q(n-k))$ and column sum vector $(q,q,\cdots,q)$ for some integer $q$. The lattice $\mathcal{L}_{n,k}$ consists of all the first rows of the matrices in the lattice $\mathcal{A}_{R,S}$ and the result follows. ∎

**Corollary 3.** *Assume that the class $\mathcal{A}(R, S)$ has no invariant positions, and also assume that the integers $r_1, \cdots, r_m$ are relatively prime. Then*

$$\mathcal{L}(R, S) = lin_Z(R, S).$$

**Proof:** Suppose that $X \in lin_Z(R, S)$. Since $X \in lin(R, S)$, $X$ has row sum vector $qR$ and column sum vector $qS$ for some $q$. Since $X$ is an integral matrix, $qR$ and $qS$ are integral vectors. Because $r_1, \cdots, r_m$ are relatively prime integers and $qR$ is an integral vector, it follows that $q$ is an integer. Hence by Theorem 1, $X \in \mathcal{L}(R, S)$. The reverse implication is obvious. ∎

We remark that the conclusion in the preceding corollary does not hold if the assumption of the relative primeness of the components of $R$ is dropped. For instance, let $R = S = (2, 2, 2)$. By Theorem 1, the identity matrix $I_3$ of order 3 does not belong to the lattice $\mathcal{L}(R, S)$. But $I_3 \in lin_Z(R, S)$ since

$$I_3 = \frac{1}{2}A_1 + \frac{1}{2}A_2 - \frac{1}{2}A_3$$

where

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We now show that Theorem 1 is not valid without the assumption that the class has no invariant positions. Indeed we show that if $m$ and $n$ are integers with $m, n \geq 2$, and $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ are any positive integral vectors such that the class $\mathcal{A}(R, S)$ is nonempty and has invariant 1-positions, then there is an integral matrix with row sum vector $R$ and column sum vector $S$ which does not belong to $\mathcal{L}(R, S)$. Assume that $\mathcal{A}(R, S)$ has invariant 1-positions. By the theorem of Ryser there are positive integers $e$ and $f$ such that every matrix in $\mathcal{A}(R, S)$ has the form (1). It is well known (see e.g. **[JurRys]**) that there is a nonnegative integral matrix $X$ with row sum vector $R$ and column sum vector $S$ such that the entry in the lower right corner of $X$ is the minimum of $r_m$ and $s_n$. Since this minimum is not zero, $X \notin \mathcal{L}(R, S)$ unless perhaps if $e = m$ or $f = n$. Suppose that $e = m$. Then every matrix in $\mathcal{L}(R, S)$ has a constant first column. But it is easy to construct a matrix $X$ of the desired type without a constant first column when $n \geq 2$. The case $f = n$ is similar.

We now exhibit a basis for the lattice $\mathcal{L}(R, S)$. By a theorem of Ryser (see [Rys] or [Bru]) any matrix in $\mathcal{A}(R, S)$ can be transformed into any other by a sequence of *interchanges*, where an *interchange* is the replacement of one of the following two submatrices by the other:

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\ ,\quad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

Let $\mathcal{C}_{m,n}$ denote the set of all $m$ by $n$ matrices all of whose entries are 0 except for those in a 2 by 2 submatrix $X$ where $X$ is equal to the difference (in some order) of the two matrices above. Then $\mathcal{C}_{m,n} \subseteq \mathcal{L}(R, S)$ and it follows from Ryser's theorem that for any matrix $A \in \mathcal{A}(R, S)$, $\{A\} \cup \mathcal{C}_{m,n}$ spans $\mathcal{L}(R, S)$ over the integers $Z$. Now it is well known that there are $mn - m - n + 1$ matrices in $\mathcal{C}_{m,n}$ which are linearly independent such that every other matrix in $\mathcal{C}_{m,n}$ can be obtained as an integer linear combination. We conclude that the dimension of the lattice $\mathcal{L}(R, S)$ satisfies

$$\dim \mathcal{L}(R, S) = mn - m - n + 2, \tag{8}$$

and that we have many nice choices for a basis. For example, if $A$ is any matrix in $\mathcal{A}(R, S)$ (or any matrix in $\mathcal{L}(R, S)$ with nonzero row sums and column sums), then $A$ and the set $\mathcal{C}'_{m,n}$ of $mn - m - n + 1$ matrices which have a 1 in the (1,1) and $(i, j)$ positions, and a $-1$ in the $(1, j)$ and $(i, 1)$ positions ($1 \leq i \leq m$, $1 \leq j \leq n$) form a basis of $\mathcal{L}(R, S)$. We note that the basis for $\mathcal{L}_{n,k}$ (see Corollary 2) given in [JunLec1] results in this way by choosing the matrix $A$ to be

$$\begin{bmatrix} k & 0 & \cdots & 0 \\ -(k-1) & 1 & \cdots & 1 \end{bmatrix}.$$

We now consider the dual lattice $\mathcal{L}^*(R, S)$. We define the *lattice of* $\mathcal{L}(R, S)$-*orthogonal matrices* to be

$$\mathcal{L}^\circ(R, S) := \{X \in Q^{m,n} : X \circ B = 0 \text{ for all } B \in \mathcal{L}(R, S)\}.$$

Clearly $\mathcal{L}^\circ(R, S) \subseteq \mathcal{L}^*(R, S)$, and in addition we have $Z^{m,n} \subseteq \mathcal{L}^*(R, S)$.

**Theorem 2.** *Let* $R = (r_1, \cdots, r_m)$ *and* $S = (s_1, \cdots, s_n)$ *be integral vectors and let* $d$ *be the greatest common divisor of* $r_1, \cdots, r_m, s_1, \cdots, s_n$. *Then*

$$\mathcal{L}^*(R, S) = \mathcal{L}^\circ(R, S) + d^{-1} Z^{m,n},$$

*that is, every matrix in* $\mathcal{L}^*(R,S)$ *is the sum of a* $\mathcal{L}(R,S)$*-orthogonal matrix and* $1/d$ *times an integral matrix.*

**Proof:** Let $X$ be a matrix in $\mathcal{L}^*(R,S)$. Since $X \circ B \in Z$ for all $B \in \mathcal{L}(R,S)$, it follows that $X \circ C \in Z$ for all $C \in \mathcal{C}_{m,n}$. There exists an integral matrix $G$ such that $(X - G) \circ C = 0$ for all $C \in \mathcal{C}'_{m,n}$ (the first row and first column of $G$ can be chosen to consist of all 0's and then $G$ is uniquely determined). Let $Y = X - G = [y_{ij}]$. Since every matrix in $\mathcal{C}_{m,n}$ is an integer linear combination of the matrices in $\mathcal{C}'_{m,n}$ it follows that $Y \circ C = 0$ for all $C \in \mathcal{C}_{m,n}$. It is well known (see e.g. **[Ber]**) that there exist rational numbers $a_1, \cdots, a_m, b_1, \cdots, b_n$ such that $y_{ij} = a_i + b_j$ for all $i$ and $j$. Hence for all $A \in \mathcal{A}(R,S)$ we have $q := A \circ Y$ where $q$ is the integer given by

$$q = a_1 r_1 + \cdots + a_m r_m + b_1 s_1 + \cdots + b_n s_n.$$

Since $r_1, \cdots, r_m, s_1, \cdots, s_n$ have greatest common divisor $d$, there exist *integers* $e_1, \cdots, e_m, f_1, \cdots, f_n$ such that

$$qd = e_1 r_1 + \cdots + e_m r_m + f_1 s_1 + \cdots + f_n s_n.$$

Let $H = [h_{ij}]$ be the $m$ by $n$ integral matrix defined by $h_{ij} = e_i + f_j$ for all $i$ and $j$. Then $A \circ d^{-1} H = q$ for all $A \in \mathcal{A}(R,S)$ and hence

$$A \circ (X - G - d^{-1} H) = A \circ (Y - d^{-1} H) = 0$$

for all $A \in \mathcal{A}(R,S)$. Thus $X - G - d^{-1} H \in \mathcal{L}^\circ(R,S)$. Since $X = (X - G - d^{-1} H) + (G + d^{-1} H)$ where $G + d^{-1} H \in d^{-1} Z^{m,n}$, the theorem now follows. ∎

**Corollary 4.** *Let* $R = (r_1, \cdots, r_m)$ *and* $S = (s_1, \cdots, s_n)$ *be integral vectors such that* $r_1, \cdots, r_m, s_1, \cdots, s_n$ *are relatively prime. Then every matrix in* $\mathcal{L}^*(R,S)$ *is the sum of an* $\mathcal{L}(R,S)$*-orthogonal matrix and an integral matrix.* ∎

We remark that the preceding corollary does not hold if we remove the assumption of relative primeness. For example, let $R = (2,2,2)$ and $S = (2,2,2)$. Then the matrix $X$ defined by

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is in $\mathcal{L}^*(R,S)$, but it is easy to verify that $X$ is not the sum of an $\mathcal{L}(R,S)$-orthogonal matrix and an integral matrix.

We conclude with some remarks. The results reported here hold under more general circumstances. Let $P = [p_{ij}]$ be an $m$ by $n$ matrix of 0's and 1's. The class $\mathcal{A}(R,S)$ can be replaced by the class $\mathcal{A}_p(R,S)$ of all $m$ by $n$ matrices of

0's and 1's with row sum vector $R$ and column sum vector $S$ having 0's in those positions where $P$ has 0's (and possibly elsewhere). *Invariant positions* now refer only to those positions of $P$ occupied by a 1, that is, to those positions which are not automatically zero in $A_p(R, S)$. Provided we assume that $P$ is *connected* in the sense that its associated bipartite graph is connected, then the dimension formula becomes dim $\mathcal{L}_p(R, S) = \sigma(P) - m - n + 2$, where $\sigma(P)$ equals the number of 1's of $P$. Thus the results apply, in particular, to the matching lattice of a matching covered connected bipartite graph (see[**Lov1**] and [**Lov2**]).

## References


[Ber] C. Berge, "Graphs and Hypergraphs", North-Holland, 1973.

[Bru] R. A. Brualdi, *Matrices of zeros and ones with fixed row and column sum vectors*, Linear Alg. Applics. **33** (1980), 159–231.

[BruHarHwa] R. A. Brualdi, D. J. Hartfiel, and S.-K. Hwang, *On assignment functions*, Linear Multilinear Alg. **19** (1986), 203–219.

[JunLec1] D. Jungnickel and M. Leclerc, *A class of lattices*, Ars Combinatoria **26** (1988), 243–248.

[JunLec2] D. Jungnickel and M. Leclerc, *The 2-matching lattice of a graph*, J. Combinatorial Theory, Ser. B **46** (1989), 246–248.

[JurRys] W. B. Jurkat and H. J. Ryser, *Term ranks and permanents of nonnegative matrices*, J. Algebra **5** (1967), 342–357.

[LewLiuLiu] J. L. Lewandowski, C. L. Liu, and J.W.S. Liu, *An algorithmic proof of a generalization of the Birkhoff-Von Neumann theorem*, J.Algorithms **7** (1986), 323–330.

[Lov1] L. Lovász, *Some algorithmic problems on lattices*, in "Theory of Algorithms", L. Lovász and E. Szemeredi, eds., Coll. Math. Soc. J. Bolyai **44**, Bolyai, North Holland, Amsterdam, 1986, pp. 323–337.

[Lov2] L. Lovász, *Matching structure and the matching lattice*, J. Combinatorial Theory, Ser. B **43** (1987), 187–222.

[Rie] J. Rieder, *The lattices of matroid bases and exact matroid bases*, submitted.

[Rys] H. J. Ryser, "Combinatorial Mathematics", (Carus Math. Monograph No.14, Math. Assoc. Amer.), Wiley, 1963.