CONTRACTIBLE EDGES IN 4-CONNECTED MAXIMAL PLANAR GRAPHS

William D. McCuaig[1]
Department of Mathematics,
University of Victoria,
Victora, British Columbia
Canada V8W 2Y2

David J. Haglin, Shankar M. Venkatesan
Department of Computer Science,
University of Minnesota,
Minneapolis, Minnesota,
55455 U.S.A.

Abstract. The fact that any n-vertex 4-connected maximal planar graph admits at least $\frac{3n+6}{5}$ 4-contractible edges readily follows from the general results of W.D. McCuaig [9],[10],[11] and of L.Andersen, H.Fleischner, and B.Jackson [1].

Here we prove a lower bound of $\lceil \frac{3n}{4} \rceil$ on the number of 4-contractible edges in every 4-connected maximal planar graph with at least eight vertices.

Throughout the discussion here, G will stand for a 4-connected maximal planar graph with at least 8 vertices. The graph obtained by contracting an edge e in G is denoted by $G \cdot e$, and if $G \cdot e$ is 4-connected, then we say e is *4-contractible* (note that an edge is 4-contractible if and only if it does not lie on a separating 4-cycle). Let $\gamma(G)$ denote the number of 4-contractible edges in G. For any vertex v in G, let $I(v)$ denote the set of 4-contractible edges incident with v, and let $W(v)$ denote the set of 4-contractible edges in the wheel surrounding v (clearly $I(v) \subseteq W(v)$). Let $X_i = \{v \mid |I(v)| = i\}, i \geq 0$. We say that G is of *insufficient contractibility* if, for *every* 4-contractible edge e in G, it holds that $\gamma(G \cdot e) \geq \gamma(G)$. Clearly, it suffices to prove the result for insuffciently contractible graphs G and for graphs G with exactly 8 vertices.

Lemma 1. *For every degree-4 vertex v in G, it holds that $|I(v)| \in \{2, 4\}$, and when $|I(v)| = 2$, the 4-contractible edges incident with v touch two non-adjacent vertices of the separating 4-cycle surrounding v.*

Proof: Let (x_1, x_2, x_3, x_4) be the 4-cycle connected to v. Suppose there is at least one non-contractible edge incident with v, say, (x_1, v). If there were the separating 4-cycle (x_1, v, x_2, z), then (x_1, x_2, z) would be a separating triangle, and so the separating 4-cycle passing though (x_1, v) must pass through (v, x_3) as well. Similarly, if either (x_2, v) or (x_4, v) is 4-contractible, then both of them are 4-contractible. Assume for the sake of contradiction that there are separating 4-cycles (x_1, v, x_3, z) as well as (x_4, v, x_2, z'); Then it must be that $z' = z$, which further implies that G has only 6 vertices. ∎

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Lemma 2. *If $\gamma(G \cdot e) \geq \gamma(G)$ for a 4-contractible edge e in G, then e is incident with a degree-4 vertex.*

Proof: If $\gamma(G \cdot e) \geq \gamma(G)$ for a 4-contractible edge e in G, then there is an edge which is contractible in $G \cdot e$ but was not contractible in G. That edge must then lie on a separating 4-cycle in G whose interior (or exterior) contains no vertices when e is contracted. This can only happen if e is incident with the 4-cycle as well as with a single degree-4 vertex in its interior. ∎

Lemma 3. *If G is insufficiently contractible, then there does not exist a path $P = (v, w, y)$ in G where each vertex on P is of degree 4.*

Proof: Suppose there exists such a path P, and its vertices are connected to x_1 and x_3 as in Figure 1 (x_1 and x_3 cannot be neighbours in the 4-cycle surrounding w, as this would imply that G has only 6 vertices). Let u and z be the other vertices connected to v and to y respectively. Clearly, u cannot be z, (v, w) and (w, y) are 4-contractible, and $\gamma(G \cdot (v, w)) = \gamma(G \cdot (w, y)) = \gamma(G) - 1$, a contradiction. ∎

Lemma 4. *Let G be insufficiently contractible. Let x_1 be a vertex that is connected by a non-contractible edge to a degree-4 vertex v in G. Then there is a triangle (x_1, v, v') where v, v' are of degree 4, and the path $P = (s', v, v', s'')$ in the wheel surrounding x_1 is made of three 4-contractible edges.*

Proof: Let (x_1, u, x_3, w) be the 4-cycle surrounding v, and (x_1, v, x_3, z) be a separating 4-cycle passing through $(x_1, v)(z \notin \{u, w\})$. By Lemma 1, edges (u, v) and (v, w) are 4-contractible as shown in Figure 2. Assume for the sake of contradiction that neither u nor w is of degree 4; Then there is at least one vertex in the strict interior of (x_1, w, x_3, z) and one in the strict exterior of (x_1, u, x_3, z); Consider contracting the edge (w, v) in G into a vertex w'; Since G is insufficiently contractible, then as in the proof of Lemma 2, at least one of the four edges (x_1, u), (u, x_3), (x_3, w'), (w', x_1) must be 4-contractible in $G \cdot (v, w)$, which they are not. The rest follows from Lemma 1. ∎

Theorem 1. *Let G be insufficiently contractible. Then for every vertex x_1 in G, it holds that*

(a) $|I(x_1)| \geq 2$, *or*
(b) $|I(x_1)| = 1$ *and* $|W(x_1) - I(x_1)| \geq 3$, *or*
(c) $|I(x_1)| = 0$ *and* $|W(x_1)| \geq 6$.

Proof: The claim trivially holds if there are at least two 4-contractible edges incident with x_1 (as in the case of a vertex of degree 4 by Lemma 1). So assume that x_1 is of degree at least 5, and $x_1 \in X_0 \cup X_1$. Then there exists at least one separating 4-cycle $\Psi(x_1)$ (Ψ for short) passing through x_1. Of all separating 4-cycles having their interior contained in the interior of Ψ, pick a cycle $C = (x_1, x_2, x_3, x_4)$

such that no other separating 4-cycle passing through x_1 has its interior contained in the interior of C.

Case (i): The number of vertices inside C is *more* than 1:
Let (x_1, x_2, y) be the face incident with the edge (x_1, x_2) inside C as in Figure 3. By the minimal interior assumption, there cannot be a separating 4-cycle that passes through (x_1, y) and does *not* contain a vertex from the exterior of C. If it were the case that there exists a path (x_1, y, x_3), then the same assumption on C would be violated. Therefore, if there were a separating 4-cycle C' that passes through (x_1, y) and contains a vertex strictly from the exterior of C, it must pass through either x_2 or x_4 and thus must induce a separating triangle. The edge (x_1, y) is therefore 4-contractible.

Case (ii): There is exactly one vertex v inside C:
If (x_1, v) is not 4-contractible, then by Lemma 4, there is a triangle (x_1, v, v') where v, v' are both of degree 4 and neither is in the strict exterior of C, and there is a path $P = (s', v, v', s'')$ in the wheel surrounding x_1 made of the three 4-contractible edges.

To complete the proof of the theorem, reembed the graph G such that the outside of Ψ becomes its inside, and argue as before on the inside of Ψ. We then find that

(a) there are at least two 4-contractible edges incident with x_1, or
(b) there is one 4-contractible edge incident with x_1, and at least three more 4-contractible edges in the wheel surrounding x_1, or
(c) no 4-contractible edges are incident with x_1, but there are at least six 4-contractible edges in the wheel around x_1 (note that the two paths, each with 3 edges, gotten from arguing on the inside and the outside of Ψ have to be edge disjoint by Lemma 3).

∎

Observe that if follows from Theorem 1 that, if v is a vertex in an insufficiently contractible graph G such that $|I(v)| = 0$, then degree $(v) \geq 6$. McCuaig, in his thesis [11], has proven a dual version of Theorem 1 for *all* cycles in *any* cyclically 4-connected cubic graph without the assumption of insufficient contractibility.

Corollary 1. *If an insufficiently contractible graph G has no vertices of degree 4, then it has at least $|V(G)|$ 4-contractible edges.*

Proof: In the proof of Theorem 1, case (ii) applies only when v is of degree 4. Since G has no degree-4 vertices, it holds that $|I(v)| \geq 2$ for every vertex v. ∎

Theorem 2. *Let G be insufficiently contractible. Then G admits at least $\lceil 3n/4 \rceil$ 4-contractible edges.*

Proof: Let $p = \{v|\ |I(v)| \geq 2\}$. Counting the 4-contractible edges by their incidence, we have that

$$2 \cdot \gamma(G) \geq \sum_{i \geq 2} i \cdot |X_i| \geq |X_1| + 2p.$$

Since a 4-contractible edge (u, v) which is in two triangles (u, v, x) and (u, v, y) can appear in at most 4 wheels (viz., those of u, v, x and y), Theorem 1 implies that

$$4\gamma(G) \geq 6|X_0| + 4|X_1| + \sum_{i \geq 2} i \cdot |X_i| \geq 6|X_0| + 4|X_1| + 2p.$$

Then,

$$\gamma(G) \geq \min \max\{\lceil (|X_1| + 2p)/2 \rceil, \lceil (6(n - |X_1| - p) + 4|X_1| + 2p)/4 \rceil\},$$

where the minimum is taken over all possible values of $|X_1| + 2p$. This is $\lceil 3n/4 \rceil$ when $|X_1| + 2p = 3n/2$. ∎

Corollary 2. *Every n-vertex 4-connected maximal planar graph with at least eight vertices admits at least $\lceil 3n/4 \rceil$ 4-contractible edges.*

Proof: There is exactly one 4-connected maximal planar graph with seven vertices, and it has precisely five 4-contractible edges (there are no 4-connected maximal planar graphs with six or less vertices with 4-contractible edges using the definition of connectivity from [2]). Therefore a graph G with eight vertices which is *not* insufficiently contractible has at least six 4-contractible edges, from the definition of insufficient contractiblity; For an insufficiently contractible graph G with eight vertices, the lower bound follows from Theorem 2. Induction using Theorem 2 then completes the proof for graphs with more than eight vertices. ∎

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