REGULI IN TRANSLATION PLANES DEFINED BY FLOCKS OF HYPERBOLIC QUADRICS

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1. Introduction

In Gevaert and Johnson [2] a correspondence was given between flocks of quadratic cones in $PG(3,q)$ and translation planes of order q^2 and kernel $\supseteq K \cong GF(q)$ that admit an affine elation group E such that some component orbit union the axis of E is a regulus in $PG(3,K)$. Furthermore, Gavaert, Johnson and Thas [3] show that in the associated translation plane π there are q reguli that share a component and if π is not Desarguesian then any regulus in π is one of these q reguli. A corollary to this result is that the collineation group of π must permute the basic set of reguli.

In Johnson [4], a correspondence was obtained between flocks of hyperbolic quadratics in $PG(3,q)$ and translation planes of order q^2 and kernel $\supseteq K \cong GF(q)$ that admit an affine homology H such that some component orbit union the axis and coaxis (recall the coaxis is the component whose extension contains the center for the homology group) of H is a regulus in $PG(3,K)$. Such translation planes, in turn, are equivalent to spreads that are covered by $q+1$ reguli that mutually share two lines.

In this article, we consider if a translation plane of order q^2 and kernel $\supseteq K \cong GF(q)$ whose spread is covered by $q+1$ reguli that mutually share two lines (components) could admit reguli not among the basic set. Since the regular nearfield planes are André planes and correspond to flocks of hyperbolic quadrics, it follows that the regular nearfield planes admit a set of $q-1$ disjoint reguli (disjoint on lines) as well as a set of $q+1$ reguli that mutually share two components. Our main result is that the Desarguesian and regular nearfield planes are exactly the translation planes with "extra" reguli that correspond to flocks of hyperbolic quadrics.

2. Extra Reguli

We first recall some results which are required in this section.

(2.1) Theorem. (see Gevaert, Johnson, and Thas[3] for (i) and Johnson[4] for (ii)).

Let R_1, R_2, be any two distinct reguli in $PG(3, q)$. If

(i) *R_1 and R_2 share a line, or*

(ii) *R_1 and R_2 share two lines then there exists a unique Desarguesian spread containing R_1 and R_2.*

(2.2) Theorem. (See, e.g. Lüneburg [6] for (i)).

Let R be a regulus in $PG(3, q)$ and Σ any Desarguesian spread (affine translation plane) containing R. Let $\mathcal{G}$ denote the collineation group of R which fixes the opposite regulus of R linewise. (i) Then $\mathcal{G}$ is also a collineation group of Σ.

(ii) Any group H of affine central collineations with axis (coaxis) in R and leaving R invariant is a collineation group of Σ.

Proof: (ii) Consider R as a translation net. H clearly leaves any Baer subplane of R (incident with the axis of H) invariant. These Baer subplanes are the lines of the opposite regulus when considered projectively.

(2.3) Theorem. (Johnson [4], and Thas[5])

There is a correspondence between flocks of hyperbolic quadratics in $PG(3, q)$ and translation planes of order q^2 and kernel $\supseteq K \cong GF(q)$ whose spreads are covered by a set S of $q+1$ reguli mutually sharing two components $\mathcal{L}, \mathcal{M}$. Furthermore, the translation plane admits an affine homology group H with axis $\mathcal{L}$ and coaxis $\mathcal{M}$ which fixes each regulus $\mathcal{R}$ of S and acts regularly on the components of $\mathcal{R} - \{\mathcal{L}, \mathcal{M}\}$.

(2.4) Theorem. (Zeischang [7]).

Let $\mathcal{F} = \{C_i; i = 1, \ldots, q+1\}$ be a flock of hyperbolic quadratic in $PG(3, q)$, q odd, and let $\pi_i \supseteq C_i$ denote the planes containing the conics for $i = 1, 2, \ldots, q+1$. Let $T \subseteq \{\pi_i; i = 1, \ldots, q+1\}$ be a subset of planes that intersect in a line of $PG(3, q)$. If $|T| \geq \frac{q+1}{2}$ then the flock $\mathcal{F}$ is either linear (all planes intersect in a line) or the flock of Thas[5].

By (2.3) (see Johnson[4]), (2.4) has the

(2.5) Corollary.

Let π be a translation plane of odd q^2 and kernel $\supseteq K \cong GF(q)$ and have its spread covered by a set $\{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ of $q+1$ reguli that mutually share two lines (components). If a subset $\mathcal{T}$ of $\{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ of at least $\frac{q+1}{2}$ reguli may be embedded in a Desarguesian spread, then π is either Desarguesian of the regular nearfield plane.

We may now give our main result:

(2.6) Theorem.

Let π be a translation of order q^2 and kernel $\supseteq K \cong GF(q)$ whose spread is covered by a set $\{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ of $q+1$ reguli that mutually shere two components $\mathcal{L}, M$. If there is a regulus $\mathcal{R} \subseteq \pi$ and $\mathcal{R} \notin \{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ then π is either Desarguesian or the regular nearfield plane.

Proof:

Thas[5] has shown that any hyperbolic flock of even order corresponds to a Desarguesian plane. Hence, we may assume that q is odd.

If $\mathcal{R} \neq \mathcal{R}_i$ for $i = 1, \ldots, q+1$ then $\mathcal{R}$ can share ≤ 2 components with any $\mathcal{R}_j$. Since $\mathcal{R} \subseteq \bigcup_{i=1}^{q+1} \mathcal{R}_i$, it is not possible that $\mathcal{L}$ and $\mathcal{M}$ can both be components of $\mathcal{R}$. Hence there are at least q components of $\mathcal{R} - \{\mathcal{L}, \mathcal{M}\} \cap \mathcal{R}$ which lie in the $\cup \mathcal{R}_i$ and no three can be in any one $\mathcal{R}_j$. Hence, $\mathcal{R}$ shares components $\neq \mathcal{L}$ or $\mathcal{M}$ with at least $\frac{q-1}{2} + 1 = \frac{q+1}{2}$ reguli (recall q is odd). Let $S_1, \ldots, S_{q+1/2}$ denote reguli in $\{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ that share at least one component different from $\mathcal{L}, \mathcal{M}$ with $\mathcal{R}$.

By (2.1), there exists a unique Desarguesian affine plane Σ containing $\mathcal{R}$ and S_1. Furtermore, Σ admits as a collineation group any collineation group of S_1 which fixes the opposite regulus of S_1 linewise. By (2.3), π admits a homology group H with axis $\mathcal{L}$ and coaxis $\mathcal{M}$ that fixes each $\mathcal{R}_i$ and acts regularly on the components of $\mathcal{R}_i - \{\mathcal{L}, \mathcal{M}\}$ for $i = 1, \ldots, q+1$. By (2.2)(ii), H is also a collineation group of Σ.

Let $\mathcal{L}_i$ be a common component of $\mathcal{R}$ and $S_i \neq \mathcal{L}$ or $\mathcal{M}$ for $i = 1, 2, \ldots, \frac{q+1}{2}$. Then $\mathcal{L}_i H = S_i - \{\mathcal{L}, \mathcal{M}\}$ so that $\mathcal{L}_i H \subseteq (\mathcal{R} \cup S_1) H \subseteq \Sigma H = \Sigma$. That is, $S_i \subseteq \Sigma$ for $i = 1, 2, \ldots, \frac{q+1}{2}$. By (2.5), π is either Desarguesian or the regular nearfield plane which proves (2.6).

(2.7) Corollary.

Let π be a translation plane of q^2 and kernel $\supseteq K \cong GF(q)$ whose spread is covered by a set $\{\mathcal{R}_1, \ldots, \mathcal{R}_{q+1}\}$ of $q+1$ reguli that mutually share two components $\mathcal{L}, \mathcal{M}$. If π is not Desarguesian or a regular nearfield plane, then

(1) *The full collineation group $\mathcal{G}$ permutes the reguli $\mathcal{R}_i, i = 1, \ldots, q+1$.*

(2) *Either there is a normal subgroup H of $\mathcal{G}$ of order $q-1$ consisting of homologies with axis $\mathcal{L}$ and coaxis $\mathcal{M}$ or there is a collineation which interchanges $\mathcal{L}$ for $\mathcal{M}$.*

Proof: $\{\mathcal{L}, \mathcal{M}\}$ and $\{\mathcal{R}_i | i = 1, \ldots, q+1\}$ is invariant by (2.6). if $\mathcal{L}$ and $\mathcal{M}$ are invariant then H is normal since $H^g = gHg^{-1}$ is a homology group with axis $\mathcal{L}$ and coaxis $\mathcal{M}$ such that the orbits of H^g are reguli. So the orbits of H^g are the orbits of H, which implies $H^g = H$.

Definition

Let $\mathcal{F}_1, \mathcal{F}_2$ be flocks of a hyperbolic quadratic $\mathcal{H}$ in $PG(3, q)$. $\mathcal{F}_1$ is said to be isomorphic to $\mathcal{F}_2$ if and only if there is an element $g \in P\Gamma L(4, q)$ which fixes

$\mathcal{H}$ and maps the conics of $\mathcal{F}_1$ onto the conics of $\mathcal{F}_2$. If $\mathcal{F}_1 = \mathcal{F}_2$, g is said to be a collineation of $\mathcal{F}_1$.

2.9 Theorem.

Let $\mathcal{F}$ be a flock of a hyperbolic quadratic $\mathcal{H}$ in $PG(3,q)$ and let $\mathcal{G}_{\mathcal{F}}$ denote the full collineation group of $\mathcal{F}$. Let π denote the associated translation plane and assume π is neither Desarguesian nor a regular nearfield plane. Let $\mathcal{G}_\pi$ denote the full collineation group of π. Let K denote the kernel $\cong GF(q)$ of π and let K^* denote the kernel homology group of order $q-1$. Let H denote the affine homology group of order $q-1$ with axis $\mathcal{L}$ and coaxis $\mathcal{M}$ where the spread for $\pi = \bigcup_{i=1}^{q+1} \mathcal{R}_i$ where $\mathcal{R}_i$ are the reguli sharing $\mathcal{L}$ and $\mathcal{M}$ for $i = 1, \ldots, q+1$. Then $\mathcal{G}_{\mathcal{F}}$ is isomorphic to $\mathcal{G}_\pi / K^*H$.

Proof: By the construction of $\mathcal{F}$ from π, using the Klein quadric (see Johnson [4]), the conics of $\mathcal{G}$ are the $(q+1)^2$ Baer subplanes incident with the zero vector of the the $q+1$ reguli $\mathcal{R}_i, i = 1, 2, \ldots, q+1$. Clearly, the group $\Gamma L(4,q)$ in π induces a corresponding group in the space $PG(5,q)$ which leaves invariant the Klein quadric. The associated group is $\Gamma O_+(6,q)$ so that $\Gamma L(4,q)/\{I,-I\} \cong \Gamma O_+(6,q)$. Let the subgroup of π which fixes each Baer subplane of $\mathcal{R}_i$ be denoted by $\mathcal{B}$. Clearly, $\mathcal{B} \supseteq K^*H$ as each Baer subplane is a K-space since $\mathcal{R}_i$ is a regulus (also see (2.2)). Since $\mathcal{G}_\pi$ permutes the reguli $\mathcal{R}_i$ by (2.7), $\mathcal{B} \trianglelefteq \mathcal{G}_\pi$.

Note that $\mathcal{G}_\pi/\{I,-I\}/\mathcal{B}/\{I,-I\} \cong \mathcal{G}_\pi/\mathcal{B}$, so that $\mathcal{G}_\pi$ induces a collineation group on $\mathcal{F}$ isomorphic to $\mathcal{G}_\pi/\mathcal{B}$. The collineation subgroup of $\mathcal{R}_1$ which fixes each Baer subplane of $\mathcal{R}$ incident with $\mathcal{O}$ *and* fixes a component $\mathcal{L}$ has order $|q(q-1)^2$ by Foulser [1]. Hence, $|\mathcal{B}_{\mathcal{L}}| \,|\, (q-1)^2$ since $\mathcal{M}$ is fixed by $\mathcal{B}_{\mathcal{L}}$. Hence if $\mathcal{B} = \mathcal{B}_{\mathcal{L}}$ then $\mathcal{B} = K^*H$.

Suppose there exists an element $g \in \mathcal{B}$ such that $\mathcal{L} \overset{g}{\longleftrightarrow} \mathcal{M}$. Then $g^2 \in \mathcal{B}_{\mathcal{L}}$ so that $|g^2| \,|\, (q-1)^2$, $|\mathcal{B}| = 2(q-1)^2$. Let $\mathcal{N}$ be a component of $\mathcal{R}_1 - \{\mathcal{L}, \mathcal{M}\}$ so that $\mathcal{B}_{\mathcal{N}} = \langle g \rangle \cdot K^*$.

Embed $\mathcal{R}_1 \cup \mathcal{R}_2$ in a Desarguesian plane Σ so that by (2.2), $\mathcal{B}$ is also a collineation group of Σ. $\mathcal{B}_{\mathcal{N}}$ fixes all 1-dimensional K-spaces on $\mathcal{N}$ since $\mathcal{R}_1$ is a regulus. Let P be any affine point on $\mathcal{N} - \{\mathcal{O}\}$. Then $|\mathcal{B}_{\mathcal{N},P}| = 2$ as K^* is regular on the nonzero vectors of the 1-space containing P. Let $\langle \tau \rangle = \mathcal{B}_{\mathcal{N},P}$ so that $\mathcal{B} = \langle \tau \rangle H K^*$. Thus τ interchanges the components $\mathcal{L}$ and $\mathcal{M}$.

Choose $\mathcal{R}_1$ in Σ to be represented in the form $x = 0, y = x \cdot \alpha, \alpha \in K \cong GF(q)$ where $\{\mathcal{L}, \mathcal{M}\} = \{(x=0), (y=0)\}$ so that coordinatizing Σ by $F \cong GF(q^2), F \supseteq K, \tau$ has the form (in Σ) $(x,y) \longrightarrow (y^\sigma b, x^\sigma c)$ where $b, c \in F, \sigma \in AutF$.

Since $\tau^2 = 1$, then $\sigma^2 = 1$ and $c^\sigma b = 1$. Also τ leaves $\{(a\alpha, a\beta) \,|\, a$ fixed in F, for all $\alpha, \beta \in K\}$ invariant as these subspaces represent Baer subplanes of $\mathcal{R}_1$. Then $(a\alpha, a\beta) \overset{\tau}{\longrightarrow} ((a\beta)^\sigma b, (a\alpha)^\sigma c) = (a^\sigma b\beta, a^\sigma c\alpha)$ since $\sigma^2 = 1$. Hence

$a^\sigma b = a\delta$ for some $\delta \in K$. That is, $a^{\sigma-1}b \in K$ of all $a \in F$ which implies that $\sigma = 1, b \in K$ and similarly that $c \in K$.

So τ has the form $(x, y) \longrightarrow (yb, xb^{-1})$ in Σ and fixes $y = xb^{-1}$ pointwise. So τ is an affine homology of both Σ and π since $y = xb^{-1}$ is a component of $\mathcal{R}_1$. However, τ leaves each Baer subplane incident with $\mathcal{O}$ of each regulus in $\{\mathcal{R}_i, i = 1, 2, \ldots, q+1\}$ which covers π. By Lüneburg [6](4.7), $y = xb^{-1}$ must be a component of each Baer subplane of each regulus $\mathcal{R}_1$, which cannot be the case. This proves (2.9).

(2.10) Theorem.

(1) Let π_1, π_2 *be translation planes of* q^2 *and kernel* $GF(q)$ *which are neither Desarguesian nor regular nearfield planes, each of whose spreads are covered* $q+1$ *reguli sharing two components. Let* $\mathcal{F}_{\pi_i}, i = 1, 2$ *denote the associated flocks of a hyperbolic quadric in* $PG(3, q)$. *Then* π_1 *is isomorphic to* π_2 *if and only if* $\mathcal{F}_{\pi_1}$ *is isomorphic to* $\mathcal{F}_{\pi_2}$.

(2) Let $\mathcal{F}_1, \mathcal{F}_2$ *be flocks of a hyperbolic quadric in* $PG(3, q)$ *which are neither linear nor the flock of Thas. Let* $\pi_{\mathcal{F}_i}, i = 1, 2$ *denote the translation planes constructed from* $\mathcal{F}, i = 1, 2$. *Then* $\mathcal{F}_1$ *is isomorphic to* $\mathcal{F}_2$ *if and only if* $\pi_{\mathcal{F}_1}$ *is isomorphic to* $\pi_{\mathcal{F}_1}$.

Proof: Since there is but one fundamental set of reguli for each plane, a flock isomorphism induces (is induced from) a plane isomorphism by (2.8),(2.9).

Added in Proof

L. Bader and G. Lunardon have recently announced that using recent results of Thas they have determined all flocks of hyperbolic quadrics. Besides the linear flock and flock of Thas corresponding to the Desarguesian and regular nearfield planes, there are exactly three other flocks which correspond to the irregular nearfield planes of orders $11^2, 23^2, 59^2$.

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