Incomplete Perfect Mendelsohn Designs

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Abstract. Let v, k and n be positive integers. An incomplete perfect Mendelsohn design, denoted by k-IMPD(v, n), is a triple $(X, Y, \mathbf{B})$ where S is a v-set (of *points*), Y is an n-subset of X, and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair $(a, b) \in (X \times X) \setminus (Y \times Y)$ appears t-apart in exactly one block of $\mathbf{B}$ and no ordered pair $(a, b) \in Y \times Y$ appears in any block of $\mathbf{B}$ for any t, where $1 \leq t \leq k-1$. In this paper some basic necessary conditions for the existence of a k-IMPD(v, n) are easily obtained, namely, $(v-n)(v-(k-1)n-1) \equiv 0 \pmod{k}$ and $v \geq (k-1)n+1$. It is shown that these basic necessary conditions are also sufficient for the case $k = 3$, with the one exception of $v = 6$ and $n = 1$. Some problems relating to embeddings of perfect Mendelsohn designs and associated quasigroups are mentioned.

1. Introduction

A set of k distinct elements $\{a_1, a_1, \cdots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the pair a_i, a_{i+t} are said to be *t-apart* in a cyclic k-tuple $(a_1, a_2, \cdots, a_k)$ where $i + t$ is taken modulo k.

Let v and k be positive integers. A $(v, k, 1)$-Mendelsohn design (briefly $(v, k, 1)$-MD) is a pair $(X, \mathbf{B})$ where X is a v set (of *points*) and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly one block of $\mathbf{B}$. If for all $t = 1, 2, \cdots, k-1$, every ordered pair of points of X are t-apart in exactly one block of $\mathbf{B}$, then the $(v, k, 1)$-MD is called *perfect* and is denoted by $(v, k, 1)$-PMD.

We wish to remark that the concept of a perfect cyclic design was introduced by N. S. Mendelsohn [15], and this concept was further developed and studied in subsequent papers by various authors (see, for example, [2]-[7], [10], [11], [16]). We have adapted the terminology and notation of Hsu and Keedwell [10], where the designs have been called Mendelsohn designs. In graph theroretic terms, a $(v, k, 1)$-PMD is equivalent to the decomposition of the complete directed graph

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K_v^* on v vertices into k-circuits such that for any r, $1 \leq r \leq k-1$, and for any two distinct vertices x and y, there is exactly one circuit along which the (directed) distance from x to y is r. It is easy to see that the number of blocks in a $(v, k, 1)$-PMD is $v(v-1)/k$ and hence an abvious necessary condition for its existence is $v(v-1) \equiv 0 \pmod{k}$. This condition is not always sufficient. For example, it is known [13] that no $(6, 3, 1)$-PMD exists. Note that a $(v, 3, 1)$-MD is necessarily perfect by definition, and this design is now more commonly called a *Mendelsohn triple system* (briefly MTS), due to Mathon and Rosa [12].

Let v, k and n be positive integers. An *incomplete perfect Mendelsohn design*, denoted by k-IPMD(v, n), is a triple $(X, Y, \mathbf{B})$ where X is a v-set (of *points*), Y is an n-subset of X, and $\mathbf{B}$ is a collection of cyclically ordered k-subsets of X (called *blocks*) such that every ordered pair $(a, b) \in (X \times X) \setminus (Y \times Y)$ appears t-apart in exactly one block of $\mathbf{B}$ and no ordered pair $(a, b) \in Y \times Y$ appears in any block of $\mathbf{B}$ for any t, where $1 \leq t \leq k-1$. For all practical purposes, the k-IPMD(v, n) can be viewed as a $(v, k, 1)$-PMD with a *hole* of size n based on the set Y. In this paper some basic necessary conditions for the existence of a k-IPMD(v, n) are easily obtained, namely, $(v-n)(v-(k-1)n-1) \equiv 0 \pmod{k}$ and $v \equiv (k-1)n+1$. It is shown that these basic necessary conditions are also sufficient for the case $k = 3$, with the one exception of $v = 6$ and $n = 1$. We also mention some problems relating to embeddings of Mendelsohn designs and their associated quasigroups.

2. Preliminaries

In what follows, we obtain some basic necessary conditions for the existence of a k-IPMD(v, n).

Suppose there exists a k-IPMD(v, n) $(X, Y, \mathbf{B})$, where $Y = \{y_1, y_2, \ldots, y_n\}$ $X = \{y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_m\}$ and $m = v n$. By the definition of a k-IPMD(v, n), no block in $\mathbf{B}$ can contain two or more points from Y. Now let us consider those ordered pairs of points which appear 1-apart in blocks of $\mathbf{B}$. For any given $y \in Y$ and any $x \in X \setminus Y$, there exists one block of the form $(x, y, \ldots)$ and therefore, there are exactly m blocks in $\mathbf{B}$ containing y. Consequently, there are mn blocks in $\mathbf{B}$ intersecting Y and these blocks contain altogether $mn(k-2)$ ordered pairs with two points both in $X \setminus Y$. However, there are $m(m-1)$ ordered pairs in $X \setminus Y$ and the remainlng $m(m-1) - mn(k-2)$ ordered pairs must appear in some blocks, each of which contains k ordered pairs. Thus the total number of blocks in $\mathbf{B}$ is $(m(m-1) - mn(k-2))/k + mn$. It follows that $m(m-1) - mn(k-2)$ must be divisible by k, and we readily obtain the following necessary conditions:

Theorem 2.1. *A necessary condition for the exisence of a k-IPMD(v, n) is*

$$(v-n)(v-(k-1)n \equiv 0 \pmod{k}, \text{ and } v \geq (k-1)n+1.$$

For the special case $k = 3$, which is fully investigated in this paper, we observe the following:

Corollary 2.2. *A necessary condition for the existence of a 3-PMD(v, n) is*

$$(v-n)(v-2n-1) \equiv 0 \pmod 3, \text{ and } v \geq 2n+1.$$

Before proceeding, we wish to point out that our study of IPMDs is very much related to the problem of embeddings of PMDs. An $(n, k, 1)$-PMD$(X, \mathbf{B})$ is said to be *embedded* in a $(v, k, 1)$-PMD$(X^*, \mathbf{B}^*)$ provided that $X \subseteq X^*$ and $\mathbf{B} \subseteq \mathbf{B}^*$. It is a trivial matter to see that by unplugging a subdesign of order n from a $(v, k, 1)$-PMD, we readily obtain a k-IPMD(v, n). On the other hand, if we have a k-IPMD(v, n) $(X, Y, \mathbf{B})$ and there exists an $(n, k, 1)$-PMD$(Y, \mathbf{B}')$, then we can easily fill in the hole of the IPMD to get a $(v, k, 1)$-PMD$(X, \mathbf{B} \cup \mathbf{B}')$. Evidently, the problem of constructing IPMDs is more general than that of embeddings of PMDs. It is worth mentioning that Hoffman and Lindner [9] have completely solved the embedding problem for Mendelsohn triple systems. Thus, in particular, we have the following result from [9, Theorem 2.7]:

Lemma 2.3. *Let v and n be positive integers and $v \geq 2n+1$. If $v, n \equiv 0$ or 1 $\pmod 3$, then there exists a 3-IPMD(v, n) except for the case of $(v, n) = (6, 1)$.*

Lemma 2.3 in conjunction with Corollary 2.2 essentially reduces our investigation of the problem of existence of a 3-IPMD(v, n) to the case where $v - n \equiv 2 \pmod 3$, $v \geq 2n+1$, for which we shall use the notion of a resolvable Mendelsohn design.

Definition 2.4. *If the blocks of a $(v, k, 1)$-MD for which $v \equiv 0 \pmod k$ can be partitioned into $v-1$ sets each containing v/k blocks which are pairwise disjoint (as sets), we say that the $(v, k, 1)$-MD is resolvable and each set of v/k pairwise disjoint blocks will be called a parallel class of the resolution.*

We shall make use of the following result due to Bermond, Germa and Sotteau [8].

Lemma 2.5. *If $v \equiv 0 \pmod 3$ and $v \not\equiv 6$, then there exists a resolvable $(v, 3, 1)$-PMD.*

Instead of listing all the blocks of a design, it suffices to give the group G acting on a set of base blocks. We shall adapt the following notation:

$$\text{dev } \mathbf{B} = \{B + g | B \in \mathbf{B}] \text{ and } g \in G\},$$

where $\mathbf{B}$ is the collection of base blocks of the design.

For completeness, we need the following lemmas.

Lemma 2.6. *There exists a 3-IPMD*$(8,2)$.

Proof: Let $G = Z_6$. $Y = \{\infty_1, \infty_2\}$ and $X = Z_6 \cup Y$. Let $\mathbf{B}$ be the following base blocks:

$$\mathbf{B} = \{(0,1,3),(\infty_1,0,4),(\infty_2,0,5)\}.$$

Then it is readily checked that $(X, Y, \text{dev}\mathbf{B})$ is a 3-IPMD(8,2).

Lemma 2.7. *For every positive integer n, there exists a 3-IPMD*$(2n+1, n)$.

Proof: Let $G = Z_{n+1}$. $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$ and $X = Z_{n+1} \cup Y$. Let $\mathbf{B}$ be the following base blocks:

$$\mathbf{B} = \{(\infty_i, 0, i) \mid i \in A_{n+1} \setminus \{0\}\}.$$

Then it is readily verified that $(X, Y, \text{dev}\mathbf{B})$ is a 3-IPMD$(2n+1, n)$.

3. Existence of 3-IPMD(v, n)

In this section, we establish that the necessary condition for the existence of a 3-IPMD(v,n) given in Corollary 2.2 is also sufficient, with the exception of $(v,n) = (6,1)$. To complete our investigation, we need

Lemma 3.1. *Suppose v and n are positive integers and $v \geq 2n+1$. If $v-n \equiv 0 \pmod 3$ and $v-n \neq 6$, then there exists a 3-IPMD*(v,n).

Proof: Since $v-n \equiv 0 \pmod 3$ and $v-n \neq 6$, then there exists a resolvable $(v-n, 3, 1)$-PMD $(X, \mathbf{B})$ from Lemma 2.5. In this resolvable PMD, there are $v-n-1$ parallel classes of blocks. Let $v = n = 3t$. For $1 \leq j \leq 3t-1$, let P denote the j-th parallel class consisting of the blocks:

$$P_j = \{(x_{1j}, y_{1j}, z_{1j}), (x_{2j}, y_{2j}, z_{2j}), \ldots, (x_{tj}, y_{tj}, z_{tj})\}.$$

Let $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$. We shall adjoin the "infinite" point ∞_j to P_j and reform the blocks as follows. From each block (x_{ij}, y_{ij}, z_{ij}), $1 \leq i \leq t$, we obtain the collection of three blocks $\{(x_{ij}, y_{ij}, \infty_j), (y_{ij}, z_{ij}, \infty_j), (z_{ij}, x_{ij}, \infty_j)\}$. Denote by P_j^* the collection of $3t$ blocks resulting from the adjoining of ∞_j to P_j. We now define

$$\mathbf{B}^* = \left(\bigcup_{1 \leq j \leq n} P_j^*\right) \cup \left(\bigcup_{n < j \leq v-n-1} P_j\right).$$

Then it is fairly straightforward to verify that $(X \cup Y, Y, \mathbf{B}^*)$ is a 3-IPMD(v,n). Note that any ordered pair (x, ∞_j) for $x \in X$ appears in exactly one block of P_j^* because of the resolvability of $(X, \mathbf{B})$ and the same applies to the ordered pair (∞_j, x). On the other hand, P_j^* has the same pairs of type $(x,y) \in X \times X$ as originally found in P_j. This completes the proof of the lemma.

We are now in a position to prove

Theorem 3.2. *The necessary condition for the existence of a 3-IPMD*(v, n), *namely*

$$(v-n)(v-2n-1) \equiv 0 \pmod 3 \text{ and } v \geq 2n+1,$$

is also sufficient, except for the case $(v, n) = (6, 1)$.

Proof: As already mentioned, the case $(v, n) = (6, 1)$ is impossible. If $v, n \equiv 0$ or $1 \pmod 3$ and $v \geq 2n+1$, the result follows from Lemma 2.3. If $v \equiv n \equiv 2 \pmod 3$, $v \geq 2n+1$, $v - n \neq 6$, then the result follows from Lemma 3.1. If $v \equiv n \equiv 2 \pmod 3$, $v \geq 2n+1$ and $v-n = 6$, then $(v, n) \in \{(8, 2), (11, 5)\}$ and the result follows from Lemmas 2.6 and 2.7. It is readily checked that all the possible cases have been covered and the proof of the theorem is complete.

4. Concluding Remarks

1. It is fairly well-known [13] that a Mendelsohn triple system can be associated with a variety of quasigroups satisfying the identities $x^2 = x$ (idempotent) and $x(yx) = y$ (semisymmetric). Quasigroups satisfying the identity $x(yx) = y$ are called *semisymmetric* and are known to exist for all orders. Moreover, semisym-symmetric quasigroups are not necessarily indempotent and, in fact, may have no idempotents at all (see, for example, [1]). Evidently, Theorem 3.2 provides an effective tool for the embedding of semisymmetric quasigroups of all orders satisfying the necessary conditions. Equivalently, the theorem provides a construction of incomplete idempotent semisymmetric Latin squares with a hole, that is, missing a subsquare.

2. While the problem of existence of k-IPMD(v, n) is completely settled for the case $k = 3$, there is much work left to be done for the other values of $k \geq 4$, which are currently under investigation. In particular, the nonexistence of a (4,4,1)-PMD has already made the case $k = 4$ much more challenging than that for $k = 3$. We also wish to remark that there is an almost complete solution to the problem of existence of a $(v, 4, 1)$-PMD (see [7]). This problem was originally studied by Mendelsohn [14], who associated with the designs a variety of quasigroups satisfying the identities $x^2 = x$ and $(x(yx))y = x$. Since the identity $(x(yx))y = x$ is conjugate equivalent to the identity $(yx)(xy) = x$, called Stein's third law, it is evident that an investigation of 4-IPMD(v, n) will provide for embeddings of an interesting variety of quasigroups.

References

1. F. E. Bennett, *Extended cyclic triple systems*, Discrete Math. **24** (1978), 139–146.
2. F. E. Bennett, *Direct constructions for perfect 3-cyclic designs*, Annals of Discrete Math. **15** (1982), 63–68.
3. F. E. Bennett, *On r-fold perfect Mendelsohn designs*, Ars Combinatoria **23** (1987), 57–68.
4. F. E. Bennett, Du Beiliang and L. Zhu, *On the existence of $(v, 7, 1)$-perfect Mendelsohn designs*, Discrete Math. (to appear).
5. F. E. Bennett, E. Mendelsohn and N. S. Mendelsohn, *Resolvable perfect cyclic designs*, J. Combinatorial Theory (A) **29** (1980), 142–150.
6. F. E. Bennett, K. T. Phelps, C. A. Rodger and L. Zhu, *Constructions of perfect Mendelsohn designs*, Discrete Math. (to appear).
7. F. E. Bennett, Zhang Xuebin and L. Zhu, *Perfect Mendelsohn designs with block size four*, Ars Combinatoria **29** (1990), 65–72.
8. J. C. Bermond, A. Germa and D. Sotteau, *Resolvable decomposition of K_n^**, J. Combinatorial Theory (A) **26** (1979), 179–185.
9. D. G. Hoffman and C. C. Lindner, *Embeddings of Mendelsohn triple systems*, Ars Combinatoria **11** (1981), 265–269.
10. D. F. Hsu and A. D. Keedwell, *Generalized complete mappings, neofields, sequenceable groups and block designs. II*, Pacific J. Math. **117** (1985), 291–312.
11. A. D. Keedwell, *Circuit designs and Latin squares.*
12. R. Mathon and A. Rosa, *A census of Mendelsohn triple systems of order nine*, Ars Combinatoria **4** (1977), 309–315.
13. N. S. Mendelsohn, *A natrual generalization of Steiner triple systems*, Computers in Number Thoery (edited by A.O.L. Atkin and B. J. Birch, Academic Press, New York, 1971), 323–338.
14. N. S. Mendelsohn, *Combinatorial designs as models of universal algebras*, in "Recent Progress in Combinatorics", Academic Press, New York and London, 1969, pp. 123–132.
15. N. S. Mendelsohn, *Perfect cyclic designs*, Discrete Math. **20** (1977), 63–68.
16. Zhang Xuebin, *On the existence of $(v, 4, 1)$-PMD*, Ars Combinatoria **29** (1990), 3–12.