A New Assignment Model and its Algorithm [1,2]

Song Zeng-min
(Department of Mathematics
Southeast University
Nanjing, 210018, People's Republic of China)

Zhang Ke-min
(Department of Mathematics, Nanjing University,
Nanjing, 210008, People's Republic of China)

Abstract. In [1], [2], there are many assignment models. This paper gives a new assignment model and an algorithm for solving this problem.

In a certain company, n workers $A_1, \ldots, A_n$ are available for m jobs $B_1, \ldots, B_m$, the effectiveness of the workers in these various jobs may be distinct; we wish to take an assignment that maximises the total effectiveness of the workers. The problem of finding such an assignment is known as the *optimal assignment problem*. This problem is equivalent to that of finding a maximum-weight matching in a weighted complete bipartite graph; we shall refer to such a matching as an *optimal matching*. We suppose that $m = n$.

It is clear that if the effectiveness of every worker in these various jobs is greater than 0, then the optimal matching must be a perfect matching. Otherwise the optimal matching is not necessarily a perfect matching; that is, it is possible that there exists an assignment that maximises the total effectiveness without needing n workers. In this case, one is interested in an assignment that maximises the total effectiveness and minimizes the number of workers. This problem is equivalent to finding a maximum-weight matching M such that $|M|$ is as small as possible. We shall refer to such a matching as a *minimum cardinality optimal matching*. In the following, we prove several Theorems, and then present a good algorithm for finding a minimum cardinality optimal matching in a weighted complete bipartite graph.

Let $G = (X, Y, W)$ be a weighted complete bipartite graph. As in [1,p.86], a *feasible vertex labelling* is a real-valued function l on the vertex set $X \cup Y$ such that for all $x \in X$ and $y \in Y$

$$l(x) + l(y) \geq w(xy). \tag{1}$$

[1] The project supported by National Natural Science Foundation of China.

[2] This paper was accepted to appear in the Proceedings of the Second International Conference on Combinatorial Mathematics and Computing (Ars Combinatorial, vol. 26 A/B), but was omitted due to an error of the proceedings editor.

The real number $l(v)$ is called the label of the vertex v. No matter what the edge weights are, there always exists a feasible vertex labelling.

Let $E_\ell = \{xy \mid l(x) + l(y) = w(xy)\}$, where l is feasible vertex labelling. The spanning subgraph of G with edge set E_ℓ is denoted by G_ℓ.

Theorem 1. *Let l be a feasible vertex labelling of G. If G_ℓ contains a perfect matching M, then*

(a) M is an optimal matching of G;

(b) Any optimal matching M^* of G is contained in G_ℓ.

Proof: According to Theorem 5.5 in [1] we only need to prove (b). Let M^* be any optimal matching and M any perfect matching of G_ℓ. It follows from (a) that

$$w(M^*) = w(M) \tag{2}$$

If M^* is not contained in G_ℓ, let $M^* = E_1 \cup E_2$, where $E_1 \subseteq E \backslash E_\ell$, $E_2 \subseteq E_\ell$ and E_1 is nonempty. It follows from [1] that

$$w(M^*) = \sum_{e \in M^*} w(e) = \sum_{e \in E_1} w(e) + \sum_{e \in E_2} w(e) < \sum_{v \in V} l(v) \tag{3}$$

Since M is a perfect matching of G,

$$w(M) = \sum_{e \in M} w(e) = \sum_{v \in V} l(v). \tag{4}$$

By (3) and (4), we have $w(M^*) < w(M)$. This contradicts (2), hence M^* is contained in G_ℓ. #

Let M be an optimal matching of G and $P = x_1y_1 \ldots x_ky_k$ an M-alternating path; if $x_1y_1, x_ky_k \in M$ and

$$\sum_{e \in E(P) \backslash M} w(e) = \sum_{e \in E(P) \cap M} w(e),$$

then such a path P is known as an *M-adjusting path.* Clearly, $M' = M \Delta E(P)$ is also an optimal matching of G and $|M'| = |M| - 1$, where $A \Delta B$ denotes the symmetric difference of A and B.

Theorem 2. *Let M be an optimal matching of G. Then M is a minimum cardinality optimal matching of G if and only if G contains no M-adjusting path.*

proof: Let M be an optimal matching in G, and suppose that G contains an M-adjusting path P. By the definition, we may obtain another optimal matching M' and $|M'| = |M| - 1$. Thus M is not a minimum cardinality optimal matching in G.

Conversely, suppose that M is not a minimum cardinality optimal matching in G. Let M' be a minimum cardinality optimal matching; then

$$|M'| < |M|. \tag{5}$$

Let $H = G[M \Delta M']$. Each vertex of H has degree either one or two in H. Thus each component of H is either an even cycle or a path with edges alternately in M' and M. For each component H_1 of H,

$$\sum_{e \in E(H_1) \cap M} w(e) = \sum_{e \in E(H_1) \cap M'} w(e) \tag{6}$$

since both M and M' are the optimal matching. By (5), the number of edges of M, which H contains, is greater than that of M', and there exists some path component P of H which starts and ends edges of M. By (6) and the definition of M-adjusting path, the path is an M-adjusting path in G. #

Theorem 1 provides the basis of a good algorithm for finding a minimum cardinality optimal matching in a bipartite weighted graph. If a feasible vertex labelling l is found such that G_ℓ contains a perfect matching M, then by theorem 1, this matching is optimal and any optimal matching in G is contained in G_ℓ. By theorem 2, we start with an arbitrary perfect matching M of G_ℓ. If G_ℓ contains no M-adjusting path, then M is. If not, we choose an M-adjusting path P in G_ℓ. $\hat{M} = M \Delta E(P)$ is another optimal matching which satisfies $|\hat{M}| < |M|$. So it remains to determine whether G contains an M-adjusting path, and to find one if one exists.

THEOREM 3. *If G_ℓ contains an M-adjusting path, then there exists an edge with weight 0 belongs to G_ℓ.*

Proof: Let $P = x_1 y_1 \ldots x_k y_k$ be an M-adjusting path of G_ℓ; then

$$\sum_{e \in M \cap \hat{E}(P)} w(e) = \sum_{e \in E(P) \setminus M} w(e) \tag{7}$$

According to the definition of G_ℓ,

$$l(x_1) + l(y_k) = l(x_1) + l(y_k) + \sum_{i=1}^{k-1} (l(x_{1+i}) + l(y_i)) - \sum_{i=1}^{k-1} (l(x_{i+1}) + l(y_i))$$

$$= \sum_{i=1}^{k} w(x_i y_i) - \sum_{i=1}^{k-1} w(y_i x_{i+1}),$$

from (7) we have $l(x_1) + l(y_k) = 0$. Also from (1) and $w \geq 0$, it follows that

$$0 \leq w(x_1 y_k) \leq l(x_1) + l(y_k) = 0.$$

Thus, $x_1 y_k \in E_\ell$ and $w(x_1 y_k) = 0$. #

Let $P = x_1 y_1 \ldots x_k y_k$ be an M-alternating path, and $x_1 y_k, x_k y_k \in M$, then P is called an *(x_1, y_k)-path.*

Theorem 4. *Let x_1, y_k be M-saturated, $x_1 y_k \in E_\ell$ and $w(x_1 y_k) = 0$, then*

(a) G_ℓ contains an M-adjusting path if and only if $G_\ell - \{x_1 y_1, x_k y_k\}$ contains an M-augmenting path from y_1 to x_k;

(b) If G_ℓ contains no M-adjusting (x_1, y_k)-path then $x_1 y_k \notin E(P)$ for any M-adjusting path P of G_ℓ.

Proof: (a) Necessity is clear. Now we prove the sufficiency as follows.

Let $P_1 = y_1 x_2 y_2 \ldots x_k$ be an M-augmenting path in $G_\ell - \{x_1 y_1, x_k y_k\}$. Thus $P = P_1 \cup \{x_1 y_1, x_k y_k\}$ is a (x_1, y_k)-path in G_ℓ. Since $x_1 y_k \in E$ and $w(x_1 y_k) = 0$ we have that $l(x_1) + l(y_k) = 0$. With a proof similar to that of Theorem 3, it is easily obtained that P is an M-adjusting path.

(b) If G_ℓ contains an M-adjusting path P starting at $x(\neq x_1)$ and ending at $y(\neq y_k)$, and $x_1 y_k \in E(P)$, by the proof of Theorem 3, $w(xy) = 0$. Note that $x_1 y_k \notin M$, and we have an M-adjusting (x_1, y_k)-path from (y_k, y)-path in P to x to (x, x_1)-path in P. This is a contradiction.

By Theorem 3 and 4, we may get an algorithm for finding an M-adjusting path. By this algorithm, either G_ℓ contains no M-adjusting path, or an M-adjusting path in G_ℓ is obtained.

Algorithm 1

Step 0: Let $G_0 = G_\ell[V(M)]$, that is, G_0, is a subgraph of G_ℓ induced by the ends of edges in M.

Step 1: If G_0 contains no edge with weight 0. by Theorem 3, G_0 contains no M-adjusting path and hence, by Theorem 2, M is minimum, in this case, stop. Otherwise, let $x_1 y_k \in E(G_0)$ and $w(x_1 y_k) = 0$, go to step 2.

Step 2: Let $e_1 = x_1 y_1 \in M$ and $e_k = x_k y_k \in M$, by the Hungarian method [1,p. 82], we find an M-augmenting path from y_1 to x_k in $G_0 - \{e_1, e_k\}$. If $G_0 - \{e_1, e_k\}$ contains no such path, then replace G_0 by $G_0 - \{x_1 y_k\}$ and go to step 1. Otherwise let P be an M-augmenting path, then $P_1 = P \cup \{e_1, e_k\}$ is an M-adjusting path.
Stop.

We see that Algorithm 1 is a good algorithm.

By Algorithm 1, it is not difficult to give an algorithm for finding a minimum cardinality optimal matching.

Algorithm 2

Step 1: By the Kuhn-Munkres algorithm [1,p. 87] we find a feasible vertex labelling such that G_ℓ contains perfect matching. Let M be any optimal matching in G_ℓ, go to step 2.

Step 2: By Algorithm 1, we find an M-adjusting path P_1 in G_ℓ. If it exists, then replace M by $M \Delta E(P_1)$ and go to step 1. Otherwise, M is a minimum cardinality optimal matching, stop.

It is easy to see that Algorithm 2 is a good algorithm too.

References

[1] J.A. Bondy and U.S.R. Murty:, *Graph Theory with Applications*, Amer. Elsevier (1976), New York.

[2] L. Lovasz and M.D. Plummer, *Matching Theory*, North-Holland, Elsevier (1986).