Domination Number of Products of Graphs

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Abstract. Let $G = (X, E)$ be any graph. Then $D \subset X$ is called a dominating set of G if for every vertex $x \in X - D$, x is adjacent to at least one vertex of D. The domination number, $\gamma(G)$, is $\min\{|D| | D$ is a dominating set of $G\}$. In 1965 Vizing gave the following conjecture: For any two graphs G and H

$$\gamma(G \times H) \geq \gamma(G) \cdot \gamma(H).$$

In this paper, it is proved that $\gamma(G \times H) \geq \gamma(G) \cdot \gamma(H)$ if H is either one of the following graphs: (a) $H = G^-$ *i.e.*, complementary graph of G, (b) $H = C_m$, *i.e.*, a cycle of length m or (c) $\gamma(H) \leq 2$.

1. Introduction.

All graphs considered in this paper are assumed finite, simple, undirected and without loops.

If G is a graph, then $V(G)$ will denote the vertex set of G and $E(G)$ will denote the edge set of G. A subset D of $V(G)$ is called a dominating set of G if for each x in $V(G) - D$ there is y in D such that xy is in $E(G)$. The dommination number, $\gamma(G) = \min\{|D| | D$ is a dominating set of $G\}$, where $|D|$ denotes the number of elements of D. Finally, for graphs G and H, the product, $G \times H$, is the graph with vertex set

$$V(G \times H) = V(G) \times V(H) = \{(x, y) | x \in V(G), y \in V(H)\}$$

and edge set

$$E(G \times H) = \{(x_1, y_1)(x_2, y_2) | \text{ either } x_1 x_2 \in E(G) \text{ and } y_1 = y_2 \text{ or } x_1 = x_2 \text{ and } y_1 y_2 \in E(H)\}$$

In [3], Vizing conjectured: For graphs G and H, $\gamma(G \times H) \geq \gamma(G) \cdot \gamma(H)$.

In [1] and [2] Jacobson and Kinch have shown that the Vizing's conjecture is true for the product of the paths, and if one of the graphs is a tree.

In this paper, we show that the conjecture holds if G is any graph and H is either G^-, C_m (cycle of length m) or $\gamma(H) \leq 2$. Our results improve some of the results of [1] and [2].

Before, proving the main results we give some auxilary results and state a few observations.

Theorem 1. *If $\gamma(G) = k$ then there is a partition of $V(G)$ into k classes, say, $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ such that each V_i is a dominating set for G^-.*

Proof: Let $D = \{v_1, v_2, \ldots, v_k\}$ be a dominating set of G. For $A \subset V(G)$ define

$$m(A) = |\{\{y_1, y_2\} | y_1, y_2 \in A, y_1 \neq y_2 \text{ and } y_1 y_2 \notin E(G)\}|$$

Now, consider a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ satisfying:

(a) $v_i \in V_i$;
(b) If $y \in V_i, y \neq v_i$ then $yv_i \in E(G)$;
(c) $\sum_{i=1}^{k} m(V_i)$ is minimum.

We claim that each V_i is a dominating set of G^-. For, if V_i is not a dominating set of G^- there is a $y \notin V_i$ such that $yx \notin E(G^-)$ for each x in V_i. That is $yx \in E(G)$ for each x in V_i. Suppose $y \in V_j$ for $i \neq j$. But, then there is z in V_j such that $zy \notin E(G)$. If this is not true then $D' = (D - \{v_i, v_j\}) \cup \{y\}$ would be a dominating set of G of cardinality less than k, a contradiction. Hence, there is z in V_j such that $zy \notin E(G)$.

Now, define a new partition $V_1' \cup V_2' \cup \cdots \cup V_k'$ where

$$V_t' = V_t \ t \neq i, j \text{ and } V_i' = V_i \cup \{y\} \text{ and } V_j' = V_j \backslash \{y\}.$$

Evidently, this partition satisfies (a) and (b) and $\sum_{i=1}^{k} m(V_i') < \sum_{i=1}^{k} m(V_i)$. This contradicts (c). This completes the proof of the theorem.

Corollary 2. (Jaeger and Payan [4]). *If G is any graph, then $\gamma(G) \cdot \gamma(G^-) \leq |V(G)|$.*

Theorem 3. *For graphs G and H and D any dominating set of $G \times H$ either $(\{x\} \times V(H)) \cap D \neq \phi$ for each $x \in V(G)$ or $(V(G) \times \{y\}) \cap D \neq \phi$ for each $y \in V(H)$.*

The proof follows from the fact that if $(x, y) \notin D$ then either there is a member of D in $\{x\} \times V(H)$ or in $V(G) \times \{y\}$ otherwise D would not be a dominating of $G \times H$.

Corollary 4. *For any two graphs G and H, $\gamma(G \times H) \geq \min(|V(G)|, |V(H)|)$.*

As a consequence of Corollaries 2 and 4 we have the following:

Theorem 5. *If G is any graph, then $|V(G)| = \gamma(G \times G^-) \geq \gamma(G) \cdot \gamma(G^-)$.*

The proof follows by theorem 3 and the observation that for any graph G, $D = \{(x, x) | x \in V(G)\}$ is a dominating set of $G \times G^-$.

Corollary 6. *If G and H are graphs such that $\gamma(G) = \gamma(H) = 1$, then $\gamma(G \times H) = \min(|V(G)|, |V(H)|)$.*

The proof follows by Corollary 3 and the fact that if $\{x_0\}$ is a dominating set of G then $D = \{(x_0, y) | y \in V(H)\}$ dominates $G \times H$.

Theorem 7. *If G is any graph and H is a graph such that $\gamma(H) = 2$, then $\gamma(G \times H) \geq 2\gamma(G)$.*

Proof: Assume $\gamma(G) = k$. For any $y \in V(H)$, let $G_y = V(G) \times \{y\}$ and for $x \in V(G)$, let $H_x = \{x\} \times V(H)$. Let A be any dominating set for $G \times H$. We show that $|A| \geq 2k$.

By Theorem 1, there is a partition $V(H) = V_1 \cup V_2$ such that V_i $(i = 1,2)$ is a dominating set of H^-. Now, define

$$B_0 = \{x \in V(G) | H_x \cap A = \phi\},$$
$$B_1 = \{x \in V(G) || H_x \cap A| = 1\},$$
$$B_2 = \{x \in V(G) || H_x \cap A| \geq 2\}.$$

Clearly, $B_0 \cup B_1 \cup B_2 = V(G)$ is a partition. Let $B_1' = \{x \in B_1 | H_x \cap A = \{(x, y)\}$ with $y \in V_1\}$ and $B_1'' = \{x \in B_1 | H_x \cap A = \{(x, y)\}$ with $y \in V_2\}$.

We now show that $B_2 \cup B_1'$ and $B_2 \cup B_1''$ are dominating sets of G.

Let $x_0 \in V(G) - (B_2 \cup B_1')$. Assume first $x_0 \in B_0$. Choose any y in V_1. Observe that H_{x_0} contains no vertex from A. Therefore, A contains a vertex (x', y) for some x' such that $x_0 x' \in E(G)$ and $x' \in B_2 \cup B_1'$. But if $x_0 \notin B_0$ then $A \cap H_{x_0} = \{(x_0, y)\}$ where $y \in V_2$. Since V_1 is a dominating set of for H^- there is a vertex $y_1 \in V_1$ such that $yy_1 \in E(H^-)$ *i.e.*, $yy_1 \notin E(H)$. Now (x_0, y_1) has to be dominated by some vertex (x', y_1). Clearly, $x' x_0 \in E(G)$ and $x' \in B_2 \cup B_1'$. This completes the proof that $B_2 \cup B_1'$ is a dominating set of G. Similarly, one can prove that $B_2 \cup B_1''$ is a dominating set of G.

Now, we have the following:

$$|B_2 \cup B_1'| \geq k \text{ and } |B_2 \cup B_1''| \geq k,$$

that is, $|B_2| + |B_1'| \geq k$ and $|B_2| + |B_1''| \geq k$. Therefore, $|A| \geq |B_1| + 2|B_2| = |B_1'| + |B_2'| + 2|B_2| \geq 2k$. Hence,

$$\gamma(G \times H) \geq 2k = 2\gamma(G).$$

In the next theorem, it is proved that the Vizing's conjecture is true if one of the graphs is a cycle.

Let C_m denote a cycle of length $m \geq 3$, *i.e.*, $V(C_m) = \{v_1, v_2, \ldots, v_m\}$ and $v_i v_{i+1} \in E(C_m)$ for $i = 1, 2, \ldots, m$ (subscripts are interpreted modulo m). Note that $\gamma(C_m) = \lceil \frac{m}{3} \rceil$.

Theorem 8. *For any graph* H,

$$\gamma(C_m \times H) \geq \gamma(C_m) \cdot \gamma(H).$$

This theorem is proved by induction on m.

Proof: For $m = 3, \gamma(C_m) = 1$ and the result follows trivially. For $4 \leq m \leq 6$, $\gamma(C_m) = 2$ and by Theorem 7 the formula holds. Suppose $m \geq 7$ and H is a graph with $\gamma(H) = k$. Suppose $A \subset V(C_m) \times V(H)$ is a dominating set for $C_m \times H$. For $i = 1, 2, \ldots, m$ define $A_i = \{y \in V(H) | (v_i, y) \in A\}$. Put $D = A_m \cup A_{m-1} \cup (A_{m-2} \cap A_1)$. D is a dominating set of H. For if $y \in V(H)$ and y is not adjacent to any vertex in $A_m \cup A_{m-1}$, then the vertex (v_m, y) can only be dominated by (v_1, y) and (v_{m-1}, y) can only be dominated by (v_{m-2}, y). Therefore, $y \in A_1 \cap A_{m-2}$. Hence D is a dominating set of H. Now, it follows that $|D| \geq k$, that is,

$$|A_m| + |A_{m-1}| - |A_m \cap A_{m-1}| + |A_1 \cap A_{m-2}| \geq k. \tag{1}$$

Now, consider a new graph K obtained from $C_m \times H$ by deleting the vertices $(v_m \times h)$ and (v_{m-1}, h) for each $h \in V(H)$ and identifying vertices (v_1, h) and (v_{m-2}, h) for each $h \in V(H)$. It is evident that $K \cong C_{m-3} \times H$.

Choose $A' \subseteq V(K)$ as follows:

$$\begin{aligned} A' =& \{(v_i, y) | 2 \leq i \leq m-3 \text{ and } (v_i, y) \in A\} \\ &\cup \{(v_1, y) | (v_1, y) \in A \text{ or } (v_{m-2}, y) \in A\} \\ &\cup \{(v_1, y) | (v_m, y) \in A \text{ and } (v_{m-1}, y) \in A\}. \end{aligned}$$

We claim that A' is a dominating set of K. Let $(v_i, y) \in K - A'$. If $i \neq 1$, it is easy to see that (v_i, y) is adjacent to some vertex of A'. Suppose $i = 1$. Recall that (v_1, y) corresponds to two vertices of $C_m \times H$, namely (v_1, y) and (v_{m-2}, y). Suppose (v_1, y) is not dominated in K by a vertex from $A' \cap (\{v_1\} \times V(H))$. Therefore, (v_1, y) is not dominated in $C_m \times H$ by a vertex in $A_1 = A \cap (\{v_1\} \times V(H))$ in which case at least one of (v_2, y) or (v_m, y) must be in A. Similarly, (v_{m-2}, y) is not dominated in K by a vertex from $A \cap (\{v_{m-2}\} \times V(H))$ in which case at least one of (v_{m-3}, y) or (v_{m-1}, y) must be in A. Both, (v_m, y) and (v_{m-1}, y) cannot belong to A for otherwise $(v_1, y) \in A'$ which is a contradiction. Therefore, $(v_2, y) \in A$ or $(v_{m-3}, y) \in A$, hence either, $(v_2, y) \in A'$ or $(v_{m-3}, y) \in A'$. But, (v_1, y) is adjacent in K to each of (v_2, y) and (v_{m-3}, y). Hence A' dominates K. Now, by the induction hypothesis, $|A'| \geq \lceil \frac{m-3}{3} \rceil k$. That is,

$$\sum_{i=2}^{m-3} |A_i| + |A_1| + |A_{m-2}| - |A_1 \cap A_{m-2}| + |A_m \cap A_{m-1}| \geq \left\lceil \frac{m-3}{3} \right\rceil k. \tag{2}$$

Combining (1) and (2) we get

$$|A| \geq \left\lceil \frac{m}{3} \right\rceil k.$$

This completes the proof of the theorem.

References

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