Fusion schemes and partially balanced incomplete block designs

Joe Hemmeter
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716

and

Andrew Woldar[1]
Department of Mathematical Sciences
Villanova University
Villanova, Pennsylvaina 19085

Abstract. The notion of fusion in association schemes is developed and then applied to group schemes. It is shown that the association schemes derived in [1] and [4] are special cases of A-fused schemes of group schemes $\mathbf{X}(G)$, where G is abelian and A is a group of automorphisms of G. It is then shown that these latter schemes give rise to PBIB designs under constructions identical to those found in [1] and [4].

1. Introduction

Let X be a finite nonempty set and $R_0, R_1, \cdots, R_d$ nonempty subsets of $X \times X$ which satisfy

(i) $R_0 = \{(x,x) : x \in X\}$
(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$ if $i \neq j$
(iii) For all i, ${}^tR_i = R_j$ for some j, where ${}^tR_i = \{(x,y) : (y,x) \in R_i\}$
(iv) For all i, j, k, the cardinality p_{ij}^k of the set $\{z \in X : (x,z) \in R_i, (z,y) \in R_j\}$ is constant whenever $(x,y) \in R_k$.

Such a configuration $\mathbf{X} = (X, \{R_i\})$ is called an *association scheme of class* d. The subsets R_i are termed *relations* and the p_{ij}^k *intersection numbers* for the scheme. As p_{ij}^k depends only on the relation R_k, and not on the particular choice of (x,y), we say p_{ij}^k is *representative-independent*. An association scheme is *commutative* if its intersection numbers satisfy $p_{ij}^k = p_{ji}^k$ and *symmetric* if its relations satisfy ${}^tR_i = R_i$. A symmetric association scheme is necessarily commutative.

In this paper we develop techniques for generating certain association schemes, called fusion schemes, from a fixed scheme. The notion of fusion scheme is quite general, but we shall be mainly interested in the fusion schemes of group schemes (section 3), in part as they are a source for constructing partially balanced incomplete block (PBIB) designs. Constructions of such designs from association schemes appear in [1], and are later extended in [4] to a larger class of schemes

---

[1] Research completed while A. Woldar was a visitor at the University of Delaware

(see section 4 and the closing remark of section 3). In section 4, we show the class of schemes to which these constructions apply is quite substantial.

For general terminology on association schemes and designs, the reader is referred to Bannai and Ito [2] and Raghavarao [5], respectively.

2. Fusion schemes and automorphisms

Let $\mathcal{X} = (X, \{R_i\})$ and $\mathcal{W} = (W, \{S_j\})$ be association schemes. For $\theta : X \to W$ a bijection, define $R_i^\theta = \{(x^\theta, y^\theta) : (x, y) \in R_i\}$, $i \geq 0$. We call θ a *fusion mapping from* $\mathcal{X}$ *onto* $\mathcal{W}$ if, for each i, $R_i^\theta \subseteq S_j$ for some j. We call $\mathcal{W}$ a *fusion scheme* of $\mathcal{X}$ if there is a fusion mapping from $\mathcal{X}$ onto $\mathcal{W}$. If, in the above, we replace $R_i^\theta \subseteq S_j$ by $R_i^\theta = S_j$, then θ is called a *scheme isomorphism*. A *scheme automorphism* of $\mathcal{X}$ is defined to be a scheme isomorphism of $\mathcal{X}$ onto itself. Observe that a scheme automorphism induces a permutation of the relations of the scheme. We call the automorphism trivial if $R_i^\theta = R_i$ for all i. Clearly, the set of all automorphisms of $\mathcal{X}$ forms a group under composition, which we denote by $Aut\mathcal{X}$.

The reader will observe that, in essence, a fusion scheme $\mathcal{W}$ of $\mathcal{X}$ is obtained by "fusing" or combining relations from $\mathcal{X}$ in some admissible fashion. Clearly, one does not obtain a fusion scheme of $\mathcal{X}$ by arbitrarily fusing relations. It is necessary, for example, that R_0 be fused with no other relation, and that ${}^tR_i$ be fused with ${}^tR_j$ whenever R_i and R_j are fused. Still, adhering to these rules in no way guarantees that the resulting object will be an association scheme. What is needed is a more precise recipe for fusing relations.

Given an association scheme $\mathcal{X} = (X, \{R_i\})$ of class d and a group A of automorphisms of $\mathcal{X}$, it is useful to consider the equivalence relation on $\{0, 1, \cdots, d\}$ defined by

$$i \sim j \text{ if and only if } R_i^\theta = R_j \text{ for some } \theta \in A.$$

We denote by $[i]$ the equivalence class containing i, and we define

$$R_{[i]} = \bigcup_{j \in [i]} R_j.$$

The following result suggests a relationship between scheme automorphisms and fusion.

Theorem 2.1. *Let* $\mathcal{X} = (X, \{R_i\})$ *and* A *be as above. Then* $\mathcal{W} = (X, \{R_{[i]}\})$ *is a fusion scheme of* $\mathcal{X}$ *whose intersection numbers* $\mathcal{P}_{[i][j]}^{[k]}$ *satisfy*

$$\mathcal{P}_{[i][j]}^{[k]} = \sum_{\substack{r \in [i] \\ s \in [j]}} p_{rs}^k$$

where p^k_{rs} are the intersection numbers of X. *Moreover, if* X *is commutative (resp., symmetric), then* $\mathcal{W}$ *is commutative (resp., symmetric).*

Proof: Clearly $\{R_{[i]}\}$ partitions the set $X \times X$, and $R_{[0]}$ is the diagonal relation as $R_0^\theta = R_0$ for all $\theta \in A$. It is trivial to verify that ${}^tR_{[i]} = R_{[j]}$, where j is determined by ${}^tR_i = R_j$. In particular if X is symmetric then ${}^tR_i = R_i$ for all i, whence $\mathcal{W}$ is symmetric as well. We complete the proof by showing the $\mathcal{P}^{[k]}_{[i][j]}$ are representative-independent and satisfy the formula given in the theorem statement.

Let $(x_1, y_1), (x_2, y_2) \in R_{[k]}$. We first consider the case where (x_1, y_1), (x_2, y_2) both lie in R_k. Here we have

$$\begin{aligned}
&|\{z \in X : (x_1, z) \in R_{[i]}, (z, y_1) \in R_{[j]}\}| \\
&= \sum_{\substack{r\in[i]\\ s\in[j]}} |\{z \in X : (x_1, z) \in R_r, (z, y_1) \in R_s\}| \\
&= \sum_{\substack{r\in[i]\\ s\in[j]}} p^k_{rs} \\
&= \sum_{\substack{r\in[i]\\ s\in[j]}} |\{z \in X : (x_2, z) \in R_r, (z, y_2) \in R_s\}| \\
&= |\{z \in X : (x_2, z) \in R_{[i]}, (z, y_2) \in R_{[j]}\}|
\end{aligned}$$

since the p^k_{rs} are representative-independent. Assume now (the general case) that $(x_1, y_1) \in R_k$ and $(x_2, y_2) \in R_t$ for some $t \in [k]$. Then $(x_2^\theta, y_2^\theta) \in R_k$ for some $\theta \in A$. Using our result from above, we have

$$\begin{aligned}
&|\{z \in X : (x_1, z) \in R_{[i]}, (z, y_1) \in R_{[j]}\}| \\
&= |\{z \in X : (x_2^\theta, z) \in R_{[i]}, (z, y_2^\theta) \in R_{[j]}\}| \\
&= |\{z \in X : (x_2^\theta, z^\theta) \in R_{[i]}, (z^\theta, y_2^\theta) \in R_{[j]}\}| \\
&= |\{z \in X : (x_2, z) \in R_{[i]}, (z, y_2) \in R_{[j]}\}|
\end{aligned}$$

where the last equality follows from the fact that $R_{[i]}$ and $R_{[j]}$ are invariant under the action of A. Thus the $\mathcal{P}^{[k]}_{[i][j]}$ are representative-independent and satisfy

$$\mathcal{P}^{[k]}_{[i][j]} = \sum_{\substack{r\in[i]\\ s\in[j]}} p^k_{rs}.$$

From this we easily conclude that $\mathcal{W}$ is commutative whenever X is, and the proof is complete.

Remark: We refer to the fused scheme $\mathcal{W}$ of Theorem 2.1 as the A-fused scheme of X, and we write $\mathcal{W} = \mathrm{X}_A$.

3. Fusion in group schemes

In the previous section we saw how scheme automorphisms could be used to generate examples of fusion schemes. Nonetheless, we have yet to establish any satisfactory source for these automorphisms. In the case of group schemes this is remedied below, where we establish a link between automorphisms of $\mathrm{X}(G)$ and those of G.

For any finite group G one obtains an association scheme $\mathrm{X}(G)$, called the group scheme of G, as follows. The underlying set is G and the relations $\{R_i\}$ are defined by

$$R_i = \{(x, y) : yx^{-1} \in C_i\}$$

where $C_0, C_1, \cdots, C_d$ are the conjugate classes of G taken in some fixed order. (We do insist that C_0 be the singleton class consisting of the identity element of the group.) It is well known that $\mathrm{X}(G)$ is commutative, and it is symmetric if and only if all elements of G are real (*i. e.*, conjugate to their respective inverses).

Theorem 3.1. *Any automorphism of a group induces an automorphism of its group scheme. Moreover, inner automorphisms of the group induce trivial scheme automorphisms.*

Proof: Let $\mathrm{X}(G)$ be a group scheme and ϕ an automorphism of G. Clearly ϕ is a bijection on G, so we need only show that, for each i, $R_i^\phi = R_j$ for some j. But this follows immediately as group automorphisms permute the conjugate classes of the underlying group. If ϕ is inner, then ϕ fixes each conjugate class setwise, so induces a trivial automorphism of $\mathrm{X}(G)$.

As the following general example illustrates, it is possible to obtain an automorphism of $\mathrm{X}(G)$ from a bijection ϕ on G which is not itself a group automorphism. Indeed let g and h be distinct elements of G and let λ_g and $\rho_{h^{-1}}$ be the maps $(x)^{\lambda_g} = gx$ and $(x)^{\rho_{h^{-1}}} = xh^{-1}$ $(x \in G)$ respectively. Clearly the composition of these maps gives a bijection ϕ on G which is not an automorphism, as ϕ moves the identity element. Still, ϕ induces an automorphism (in fact a trivial automorphism) of $\mathrm{X}(G)$, as can be seen by the following simple argument:

$$(x^\phi, y^\phi) \in R_i \Leftrightarrow (gxh^{-1}, gyh^{-1}) \in R_i \Leftrightarrow gyh^{-1}(gxh^{-1})^{-1} \in C_i$$
$$\Leftrightarrow gyx^{-1}g^{-1} \in C_i \Leftrightarrow yx^{-1} \in C_i \Leftrightarrow (x, y) \in R_i.$$

If G is a finite group and A is a group of automorphisms of G, we can form a new group $G{:}A$, called the external semidirect product of G by A [6]. There is an interesting relationship between the group scheme $\mathrm{X}(G{:}A)$ and the A-fused scheme $\mathrm{X}(G)_A$ of $\mathrm{X}(G)$. Before this can be made precise, we need one more definition.

Let $\mathcal{Y} = (Y, \{S_j\})$ be an association scheme of class d. We define a *subscheme of* $\mathcal{Y}$ to be any association scheme $\mathrm{X} = (X, \{T_i\})$ of class $e \leq d$ which satisfies

(i) $X \subseteq Y$ and
(ii) the S_j can be rearranged so that $T_i = S_i \cap (X \times X), 0 \leq i \leq e$.

Theorem 3.2. *Let G and A be as above. Then the A-fused scheme $\mathrm{X}(G)_A$ of $\mathrm{X}(G)$ is a subscheme of $\mathrm{X}(G{:}A)$.*

Proof: Let $\mathrm{X}(G) = (G, \{R_i\})$ and $\mathrm{X}(G{:}A) = (G{:}A, \{S_j\})$ be the group schemes for G and $G{:}A$, respectively. Clearly, as G is normal in $G{:}A$, every $G{:}A$-conjugate class which meets G is a union of G-classes. Order the nonidentity G-conjugate classes $C_1, \cdots, C_e, \cdots, C_t$ and the nonidentity $G{:}A$-conjugate classes $K_1, \cdots, K_e, \cdots, K_d$ so that $K_1, \cdots, K_e$ is the totality of $G{:}A$-classes which meet G and $C_i \subseteq K_i$ for $i = 1, \cdots, e$. We claim, under this ordering, that $T_0, \cdots, T_e$ are precisely the relations for $\mathrm{X}(G)_A$, where $T_i = S_i \cap (G \times G)$. The result will then follow as $(G, \{T_i\})$ is clearly a subscheme of $\mathrm{X}(G{:}A)$.

For $0 \leq i \leq e$, let $(x, y) \in T_i$ whence $x, y \in G$ and $yx^{-1} \in K_i$. As $C_i \subseteq K_i$ there exists $\theta \in A$ such that $\theta^{-1}(yx^{-1})\theta \in C_i$. But in the group $G{:}A$, conjugation of $g \in G$ by $\theta \in A$ is defined by $\theta^{-1}g\theta = g^\theta$. Thus $(yx^{-1})^\theta \in C_i$. As $x, y \in G$, we have $(yx^{-1})^\theta = y^\theta(x^\theta)^{-1}$ whence $(x^\theta, y^\theta) \in R_i$, *i. e.* $(x, y) \in R_{[i]}$. This proves $T_i \subseteq R_{[i]}, 0 \leq i \leq e$. Conversely, let $(x, y) \in R_{[i]}, 0 \leq i \leq e$. Then $(x^\theta, y^\theta) \in R_i$ for some $\theta \in A$, whence $(yx^{-1})^\theta \in C_i \subseteq K_i$. But then $yx^{-1} \in K_i$ as K_i is A-invariant. Thus $(x, y) \in T_i$ and we have $R_{[i]} = T_i, 0 \leq i \leq e$. As $T_0 \cup \cdots \cup T_e = G \times G$, it follows that the A-fused scheme of $\mathrm{X}(G)$ has the form $(G\{R_{[i]}\})$ relative to our initial ordering of conjugate classes. The result follows.

Before we proceed to the construction of PBIB designs, let us discuss the schemes obtained in [1] by Agrawal and Nair and in [4] by Hultquist, Mullen and Niederreiter.

Essentially, the schemes of Agrawal and Nair are derived as follows. Start with the ring $\mathbf{Z}_n$ of complete residue classes modulo n. Letting $t_0 = 1 < t_1 < t_2 < \cdots < t_s$ be all divisors of n other than n, partition $\mathbf{Z}_n$ into sets $A_0, A_1, \cdots, A_{s+1}$, where $A_0 = \{0\}$ and $A_{i+1} = \{a : (a, n) = t_i\}, 0 \leq i \leq s$. (Here (a, n) denotes the greatest common divisor of a and n.) The relations T_i are now defined by

$$T_i = \{(x, y) : x - y \in A_i\}, \quad 0 \leq i \leq s + 1.$$

Clearly, A_1 is the multiplicative group of units in $\mathbf{Z}_n$, and, as shown in [1], the sets $A_0, A_1, \cdots, A_{s+1}$ are precisely the A_1-orbits of $\mathbf{Z}_n$ under the action of multiplication.

The schemes derived by Hultquist, Mullen and Niederreiter are similar. They replace $\mathbf{Z}_n$ by a quotient ring of the form $\mathbf{F}_q[x]/\langle V \rangle$, where $\mathbf{F}_q[x]$ is the polynomial ring in one indeterminate over the field $\mathbf{F}_q$ of q elements, and $\langle V \rangle$ is the ideal generated by a fixed nonconstant polynomial $V \in \mathbf{F}_q[x]$. If one views the ring $\mathbf{Z}_n$ as the quotient $\mathbf{Z}/\langle n \rangle$, the connection becomes even more apparent: $\mathbf{Z}$ is replaced by $\mathbf{F}_q[x]$, and n by V. The t_i are now taken to be the *monic* divisors of V, and the A_i and T_i are unchanged. As before, A_1 is the group of units of $\mathbf{F}_q[x]/\langle V \rangle$ and $A_0, A_1, \cdots, A_{s+1}$ are the orbits of $\mathbf{F}_q[x]/\langle V \rangle$ under the action of A_1.

Each of the aforementioned derivations of schemes is a special case of a single derivation applied to an arbitrary finite ring R with unity. Define the equivalence relation

$$x \sim y \text{ if and only if } x = yu \text{ for some unit } u \in R.$$

Denote the equivalence classes by A_i $(i \geq 0)$ with ordering chosen so that $A_0 = \{0\}$ and $1 \in A_1$. Clearly $A = A_1$ is the multiplicative group of units of R. Letting G denote the additive group of R, it is immediate that A acts as automorphisms on G via right multiplication: $x \to xu, x \in G, u \in A$. (Indeed, this is just the distributive law in R.) Moreover, the A_i are precisely the A-orbits of G under this action. With T_i as above, we conclude from Theorems 2.1 and 3.1 that $\mathrm{X}(G, \{T_i\})$ is an association scheme, viz. the A-fused scheme of $\mathrm{X}(G)$. When $R = \mathbf{Z}_n$, we obtain the schemes in [1]; when $R = \mathbf{F}_q[x]/\langle V \rangle$, we get those in [4]. We observe that the notation of greatest common divisor is incidental to the derivation of these schemes.

In the next section, we show the design constructions in [1] and [4] can be extended to the family of schemes $\{\mathrm{X}(G)_A\}$ where G is *arbitrary abelian* and A is *any* group of automorphisms of G. (One extra condition on A is required in one of these constructions. See Theorem 4.2.) In particular, fusion schemes arising from arbitrary rings with unity are members of this family. Interestingly, the schemes of [1] and [4] occur at opposite ends of the spectrum. In [1], G is *cyclic* and A is the *full* automorphism group of G; in [4], G is *elementary abelian* and A is a very *small* subgroup of $AutG$. Our generalization fills a sizeable middle ground.

4. Construction of PBIB designs

Let $\mathrm{X} = (X, \{S_j\})$ be an association scheme of class $d \geq 2$ with $v = |X|$. Suppose the members of X can be arranged in b subsets of X, called *blocks*, so that

(i) Each block contains k distinct elements of X,
(ii) Each element of X is contained in r blocks,
(iii) Given any $(x, y) \in S_j$, $0 \leq j \leq d$, the number λ_j of blocks in which x and y occur together is independent of (x, y).

Such a configuration is called a *PBIB design based on* X. The numbers v, r, b, k and λ_j $(0 \leq j \leq d)$ are called *parameters* of the design.

Remark: If $d = 1$, the above configuration is called a *balanced incomplete block (BIB) design based on* X.

We are interested in the case where X is the A-fused scheme of a group scheme $X(G)$, G abelian. For our first construction, we follow the one given in 3.2 of [1]. Here we let B be a fixed conjugate class K_i of $G{:}A$ which meets (so lies entirely within) G. Thus $B = \{g^\theta : \theta \in A\}$ where $\{g\}$ is the ith conjugate class of G, and $(x, y) \in R_{[i]}$ if and only if $yx^{-1} \in B$. One now defines the blocks by $x + B = \{x + y : y \in B\}, x \in G$. The following result gives sufficient conditions for such a system of blocks to yield a PBIB design.

Theorem 4.1. *Let G be an abelian group and A any group of automorphisms of G which does not fuse all nonidentity elements of G. Then the system of blocks defined above yields a PBIB design based on $X(G)_A$ with parameters $v = b = |G|, r = k = |B|$, and $\lambda_j = \mathcal{P}^{[j]}_{[i'][i]}$ where $R_{[i']} = {}^tR_{[i]}$.*

Proof: By definition each block contains $k = |B|$ elements of G. Also, for any $t \in G$, t occurs in precisely $r = |B|$ blocks, viz. $t - y + B$, $y \in B$. Now fix $(a, b) \in R_{[j]}$. Then $a, b \in x + B \Leftrightarrow a - x, b - x \in B \Leftrightarrow (a, x) \in R_{[i']}$, $(x, b) \in R_{[i]}$. Thus $\lambda_j = \mathcal{P}^{[j]}_{[i'][i]}$, and λ_j depends only on j. It now follows that the system $\{x + B : x \in G\}$ yields a design based on $X(G)_A$, which is a PBIB design if and only if $X(G)_A$ is of class at least two, *i. e.*if and only if A fails to fuse all nonidentity elements of G. As we clearly have $v = b = |G|$, the proof is complete.

The second method of constructing PBIB designs is based on a procedure of Das and Kulshreshtha [3]. Here we assume G is abelian and A contains the automorphism $\tau : x \to -x$. Let B_0 be any nonempty subset of G, and let $\{\theta_1, \theta_2, \cdots, \theta_t\}$ be a complete set of representatives for the cosets of $\langle\tau\rangle$ in A. Letting $B_i = B_0^{\theta_i}$, our set of blocks is now defined to be $\mathcal{B} = \{x + B_i : x \in G, 1 \le i \le t\}$. (Although it is possible that $x + B_j$ and $y + B_k$ are identical as sets, we regard them as distinct blocks unless $x = y$ and $j = k$.) We now assert

Theorem 4.2. *With notation as above, the set of blocks $\mathcal{B}$ yields a PBIB design with parameters $v = |G|, b = vt, k = |B_0|, r = kt$, and $\lambda_j = t\beta_j / \mathcal{P}^{[o]}_{[j][j]}$ where $\beta_j = |R_{[j]} \cap (B_0 \times B_0)|$.*

Proof: Clearly $v = |G|$, $k = |B_0|$ and $b = vt$. Also $r = kt$, as each $g \in G$ is contained in only those blocks of the form $g - y + B_i$ $(y \in B_i, 1 \le i \le t)$. For (a, b) in the jth relation $R_{[j]}$ of $X(G)_A$, let $\mathcal{B}_i(a, b) = \{x + B_i : a, b \in x + B_i\}$ and $\mathcal{D}_i(a, b) = \{(\alpha, \beta) \in B_i \times B_i : \beta - \alpha = b - a\}$. Because of the difference set construction of $\mathcal{B}$, $|\mathcal{B}_i(a, b)| = |\mathcal{D}_i(a, b)|$. Fix i, j and $(a, b), (c, d) \in R_{[j]}$. Then there exists $\theta \in A$ such that $(b - a)^\theta = d - c$. Let $\theta_k\langle\tau\rangle$ be the coset containing $\theta_i\theta$.

Case one: $\theta_i\theta = \theta_k$. As $B_i^\theta = B_k$, we have $(\alpha, \beta) \in \mathcal{D}_i(a, b) \Leftrightarrow (\alpha^\theta, \beta^\theta) \in \mathcal{D}_k(c, d)$, whence $|\mathcal{B}_i(a, b)| = |\mathcal{B}_k(c, d)|$.

Case two: $\theta_i\theta = \theta_k\tau$. Here $(\alpha, \beta) \in \mathcal{D}_i(a, b) \Leftrightarrow (\beta^\theta, \alpha^\theta) \in \mathcal{D}_k(c, d)$, so again $|\mathcal{B}_i(a, b)| = |\mathcal{B}_k(c, d)|$.

We now have

$$|\mathcal{B}(a, b)| = \sum_i |\mathcal{B}_i(a, b)| = \sum_k |\mathcal{B}_k(c, d)| = |\mathcal{B}(c, d)|,$$

which proves λ_j depends only on j. To compute λ_j, we count the set $\{(a, b, B) : (a, b) \in R_{[j]}, B \in \mathcal{B}, a, b \in B\}$ in two ways. First we note that $|(B \times B) \cap R_{[j]}| = \beta_j$ for all $B \in \mathcal{B}$, as $R_{[j]}$ is invariant under the action of A and that of translation by elements of G. Also $\mathrm{X}(G)_A$ is symmetric (since $\tau \in A$), so that $\mathcal{P}^{[0]}_{[j][j]} = |\{b : (a, b) \in R_{[j]}\}|$ for all $a \in G$. We thereby obtain

$$v\mathcal{P}^{[0]}_{[j][j]}\lambda_j = vt\beta_j$$

and the theorem follows.

References

1. H. C. Agrawal and C. R. Nair, *Reduced residue classes cyclic PBIB designs*, Austral. J. Statist. **26**, no. 3 (1984), 298–309.
2. E. Bannai and T. Ito, "Algebraic Combinatorics I: Association Schemes", Mathematics Lecture Note Series, Benjamin/Cummings, Menlo Park, Ca., 1984.
3. M. N. Das and A. C. Kulshreshtha, *On derivation of initial blocks of BIB designs with more than one initial block*, Austral. J. Statist. **10** (1968), 75–82.
4. R. A. Hultquist, G. L. Mullen and H. Niederreiter, *Association schemes and derived PBIB designs of prime power order*, Ars Combinatoria **25** (1988), 65–82.
5. D. Raghavarao, "Constructions and Combinatorial Problems in Design of Experiments", John Wiley and Sons, New York, N.Y., 1971.
6. J. J. Rotman, "The Theory of Groups", Allyn and Bacon, Boston, Ma., 1973.