A PROPERTY OF T-WISE BALANCED DESIGNS

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1. Introduction

Let $\mathcal{X}$ be a finite v-set of objects called treatments and $\mathcal{B}$ a family of non-null subsets (called blocks) of $\mathcal{X}$. For each subset S of $\mathcal{X}$, let $\lambda(S)$ denote the number of blocks containing S. Then the system $(\mathcal{X}, \mathcal{B})$ is called a t-wise balanced design d, if for any t-subset S of $\mathcal{X}$, $\lambda(S)$ is a constant, λ_t, say. Moreover, if for any x-subset S of $\mathcal{X}$, for given $x \leq t$, $\lambda(S) = \lambda_x$, then the design d is said to be regular t-wise balanced. Note that a 2-wise balanced design with the equal block size is a balanced incomplete block design.

Several combinatorial properties for t-wise balanced designs can be found in Mullin (1974) and Kramer (1983) with definitions of other terms used here. Here a new property on the inner structure of t-wise balanced designs is provided.

2. Statement

It is known (see Kramer (1983)) that a t-wise balanced design is not necessarily $(t-1)$-wise balanced. Kramer also gave an example of a $13-(24, \{14, 15\}, 11)$ design which is t-wise balanced for $t = 1, 2, 3, 4, 5, 13$, but not-balanced for $6 \leq t \leq 12$.

Below is a property on inner structure of t-wise balanced designs.

Proposition 1. *If $d = d_1 \cup d_2$ is a t-wise and $(t-1)$-wise balanced design with subdesigns d_i having block size k_i, $(i = 1, 2;\ k_1 \neq k_2)$, then each d_i is a $(t-1)$-design.*

Proof: Without loss of generality, let $k_1 > k_2$. For the usual treatment-block incidence matrices $((n_{ij}^{(2)}))$ and $((n_{ij}))$ of designs d_2 and d, respectively, it follows that for any distinct $i_1, \ldots, i_{t-1}$

$$\sum_{j \in d_2} n_{i_1 j}^{(2)} \cdots n_{i_{t-1} j}^{(2)} = \frac{1}{k_1 - k_2} \sum_{j \in d} (k_1 - n_{1j} - \cdots - n_{vj})\, n_{i_1 j} \cdots n_{i_{t-1} j}$$

$$= \frac{1}{k_1 - k_2} \{(k_1 - t + 1)\lambda_{t-1} - (v - t + 1)\lambda_t\}$$

$$(= \lambda_{t-1}^{(2)}, \text{ say})$$

where $j \in d$ denotes the running over blocks belonging to the design d, and v is the number of treatments in d, which implies that d_2 is a $(t-1)-(v, k_2, \lambda_{t-1}^{(2)})$ design, and hence d_1 is a $(t-1)-(v, k_1, \lambda_{t-1} - \lambda_{t-1}^{(2)})$ design.

When $t = 3$ in Proposition 1, we have the following.

Corollary 1. *In a regular 3-wise balanced design with two distinct block sizes, each of subdesigns with all blocks of the equal block size is a 2-design.*

The technique used in the proof of Proposition 1 can not be applied to designs with more than two distinct block sizes for the similar purpose. But we can say a little. Based on Proposition 1, in a t-wise and $(t-1)$-wise balanced design $d = d_1 \cup d_2 \cup d_3$ with three distinct block sizes, if one of d_i for some i is a t-design, it follows that other two subdesigns are $(t-1)$-designs. Even so this is not powerful.

Though a complement of a t-wise balanced design is not necessarily t-wise balanced, as also pointed out in Mullin (1974), a combination of a regular 2-wise balanced design and its complement yields a regular 3-wise balanced design. This observation may reveal a kind of a converse to Corollary 1 when the original design has more than two distinct block sizes.

Recently, the fact described in Corollary 1 is used to characterize statistical block designs with supplemented balance (see Kageyama and Majumdar (1989)).

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