CYCLE-BOOK RAMSEY NUMBERS

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Abstract. Let $B_n = K(1,1,n)$ denote the n-book. In this paper we (i) calculate $r(C_5, B_n)$ for all n, (ii) prove that if m is an odd integer ≥ 7 and $n \geq 4m - 13$ then $r(C_m, B_n) = 2n + 3$, and (iii) prove that if $m \geq 2n + 2$ then $r(C_m, B_n) = 2m - 1$.

1. Introduction.

The complete tripartite graph $B_n = K(1,1,n)$ is called the *n-book*. Interest in Ramsey numbers involving books grew out of the discovery of a link between book Ramsey numbers and the theory of *strongly regular graphs* [6]. In [7] the following formula is established for book-path Ramsey numbers:

$$r(B_m, P_n) = \max\left\{(k+2)(n-1)+1, 2\left[\frac{m-1}{k+1}\right]+m\right\},$$
$$k = \left[\frac{n-1}{m-1}\right].$$

Book-star Ramsey numbers were studied in [7], and additional results concerning these Ramsey numbers are contained in a paper of Chartrand et al. [2]. There are several interesting unsolved problems concerning book-star Ramsey numbers and so there are certain to be additional papers on this subject. Indeed, Ramsey number problems involving books have provided serveral fruitful studies. The whimsical title given to reference [4] is an expression of this fact. Cycle-book Ramsey numbers are no exception to this rule. In this paper we (i) calculate $r(C_5, B_n)$ for all n, (ii) prove that if m is an odd integer exceeding 5 and if $n \geq 4m - 13$ then $r(C_m, B_n) = 2m - 1$. Some results concerning $r(C_4, B_n)$ were given in [4], but we know practically nothing about $r(C_m, B_n)$ when m is even and greater than 4. Also, the problem of computing $r(C_m, B_n)$ when m is odd and m and n are nearly equal provides an unanswered test of strength.

2. Terminology and Notation.

For the most part, our use of graph theoretic terminology and notation will conform with that used in [1]. However, there are certain special conventions which we shall follow in treating the problem at hand, and these are now described.

Let $V = \{v_1, v_2, \ldots, v_p\}$ denote a set of vertices. Then $[V]^2$ denotes the set of all un-ordered pairs of these vertices. By a *two-coloring* we mean a partition $[V]^2 = (R, B)$. Equivalently, we ascribe to each edge of the complete graph of order p a color, either red or blue. This two-coloring defines two edge-induced graphs of order p and we use $\langle R \rangle$ and $\langle B \rangle$ as symbols for these graphs. For each nonempty set of vertices $X \subseteq V$, there are vertex-induced subgraphs of $\langle R \rangle$ and $\langle B \rangle$, and these are denoted $\langle X \rangle_R$ and $\langle X \rangle_B$ respectively. Let F and G be graphs without isolated vertices. The *Ramsey number* $r(F, G)$ is the smallest value of $|V|$ such that in every possible two-coloring (R, B) of $[V]^2$, either $\langle R \rangle$ contains (a subgraph isomorphic to) F or $\langle B \rangle$ contains G. We are here concerned with the Ramsey number $r(C_m, B_n)$. In this regard, it is well to remind the reader that, in accordance with [1], the scheme $C : x_1, x_2, \ldots, x_m, x_1$ is used to denote a cycle of order m. (A *path* of order m is denoted $P : x_1, x_2, \ldots, x_m$.) Also, it is convenient to introduce a special symbol, namely $\Lambda(uv)$, to represent for a given $uv \in [V]^2$ the set of all vertices w which are commonly adjacent to u and v in $\langle B \rangle$. Thus, the occurrence of B_n as a subgraph of $\langle B \rangle$ means that $|\Lambda(uv)| \geq n$ for some $uv \in B$.

3. Canonical Colorings.

Let $|V| = 2(p-1)$ and consider the two-coloring (R, B) of $[V]^2$ in which $\langle R \rangle \simeq 2K_{p-1}$. In this two-coloring, $\langle R \rangle$ contains no connected graph of order p and $\langle B \rangle$ contains no odd cycle. This is an example of a *canonical coloring*. By letting $p = m$, we find that

$$r(C_m, B_n) \geq 2m - 1.$$

By letting $p = n + 2$ and reversing the roles of R and B, we find that

$$r(C_m, B_n) \geq 2n + 3$$

whenever m is *odd*. In what follows, we shall establish certain cases in which the above statements hold with equality.

4. Odd Cycles.

Our first theorem gives $r(C_m, B_n)$ completely for $m = 3$ and $m = 5$.

Theorem 1.

$$r(C_3, B_n) = \begin{cases} 6 & \text{if } n = 1, \\ 2n+3 & \text{if } n > 1. \end{cases}$$

$$r(C_5, B_n) = \begin{cases} 9 & \text{if } n = 1, 2, \\ 10 & \text{if } n = 3, \\ 2n+3 & \text{if } n > 3. \end{cases}$$

Proof: The calculation of $r(C_3, B_n)$ is given in [6]. The statement of the result is included here for the sake of completeness.

With one exception, the colorings needed to establish $r(C_5, B_n)$ are the canonical ones discussed in the last section. The exception occurs in the case of $n = 3$. The fact that $r(C_5, B_3) \geq 10$ comes from the two-coloring in which $\langle R \rangle$ contains no C_5 and $\langle B \rangle$ contains no B_3.

It is very easy to prove that $r(C_5, B_1) = r(C_5, B_2) = 9$ and these two Ramsey numbers are already recorded in [2]. We shall describe a general scheme which may be used to prove that $r(C_5, B_3) \leq 10$ and $r(C_5, B_n) \leq 2n + 3$ for $n > 3$. It will be clear to the reader that straightforward arguments based on

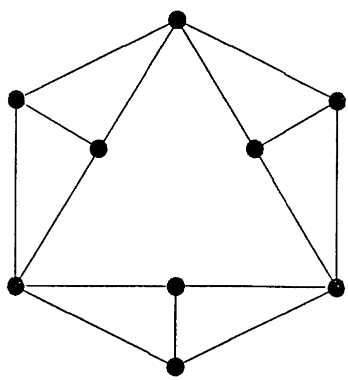

Figure 1.
Critical Coloring for $r(C_5, B_3)$

this scheme will take care of the special cases $n = 3, 4$ and 5. Although straightforward, the complete arguments for these special cases involve many details. For this reason, these proofs are left to the reader. In what follows, we shall rely on the validity of the statements $r(C_5, B_4) = 11$, $r(C_5, B_5) = 13$ and confine our attention to the case of $r(C_5, B_n)$ where $n \geq 6$.

With $|V| = 2n + 3$, let us assume the existence of a two-coloring (R, B) of $[V]^2$ in which $\langle R \rangle$ contains no C_5 and $\langle B \rangle$ contains no B_n. We shall demonstrate that this assumption leads ultimately to a contradiction. A useful element of our argument is a simple forbidden subgraph result which is established by induction. We claim that in the two-coloring whose existence we have assumed there is no K_4

in $\langle R \rangle$. To see this, suppose to the contrary that $\langle X \rangle_R \simeq K_4$ and let $W = V - X$. The induction hypothesis together with the fact that $|W| = 2(n-2)+3$ imply that $\langle W \rangle_B$ contains a B_{n-2}. Since $\langle R \rangle$ contains no C_5, each vertex of W is adjacent in $\langle R \rangle$ to at most one vertex of X. It follows that any two vertices $w_i, w_j \in W$ are commonly adjacent to at least two vertices of X in $\langle B \rangle$. But this gives rise to a B_n in $\langle B \rangle$ and so must be rejected. Thus $\langle R \rangle$ contains no K_4. Since $r(C_5, B_4) = 11$ and $r(C_5, B_5) = 13$, the induction argument is complete.

Here is the general scheme mentioned earlier. With $uv \in B$ set $X = V - \{u, v\}$ and form the partition $X = (RR, RB, BR, BB)$ by placing $x \in X$ in the appropriate part according to whether the pair (xu, xv) is an element of $R \times R$, $R \times B$, $B \times R$, or $B \times B$. Let $W = X - BB$. By assumption, $|BB| \leq n-1$ and so $W \geq n+2$. We claim that $|RR| \leq 2$. Suppose, to the contrary, that $|RR| \geq 3$ and note that in this case we must find $w_i w_j \in B$ for every pair of distinct vertices $w_i \in RR$, $w_j \in W$ in order to avoid a C_5 in $\langle R \rangle$. It follows that $w_i \in RR$ and $|\Lambda(w_i w_j)| \geq n$ for any two vertices $w_i, w_j \in RR$. This is in contradiction to our assumption that $\langle B \rangle$ contains no B_n and so our claim that $|RR| \leq 2$ is justified. Now select $uv \in B$ so that the corresponding $|RR|$ is *maximal*. There are three cases to consider.

Case 1, $RR = \phi$. In view of the maximality condition, it follows that $xw \in B$ for every $x \in BB$ and $w \in W$. Since $\langle B \rangle$ contains no B_n, it follows that BB must span a complete graph in $\langle R \rangle$. Again in view of the maximality condition, RB and BR span complete graphs in $\langle R \rangle$. Since $2n+3 \geq 15$, the coloring just described cannot fail to have a C_5 in $\langle R \rangle$ and so we have a contradiction.

Case 2, $RR = \{w_1\}$. In view of the maximality of $|RR|$, each vertex $w_i \in BB$ is adjacent in $\langle R \rangle$ to at most one vertex of RB or BR. Moreover, the maximality condition implies that if $w_i w_1 \in R$ then w_i must be adjacent in $\langle B \rangle$ to all the vertices of RB and BR. A quick count shows that $|\Lambda(w_i w_j)| \geq n$ for arbitrary $w_i, w_j \in BB$. Hence, BB must span a complete graph in $\langle R \rangle$. Since there can be no K_4 in $\langle R \rangle$ and since $n \geq 6$, it follows that $|BB| \leq n-3$ and so $|RB| + |RB| \geq n+3$. Since $\langle R \rangle$ contains no C_5, $w_i w_j \in B$ for every $w_i \in RB$ and $w_j \in BR$. Now observe that no vertex can have degree ≥ 2 in $\langle RB \rangle_R$ or $\langle BR \rangle_R$. The reason is that a vertex of degree ≥ 2 in one of these induced subgraphs leads to either a K_4 in $\langle R \rangle$ or else a violation of the maximality condition. In view of the last two observations, we find that $|\Lambda(w_i w_j)| \geq n$ whenever $w_i, w_j \in RB$ or $w_i, w_j \in BR$. Consequently, we must assume that RB and BR span complete graphs in $\langle R \rangle$ and so find a contradiction as in case 1.

Case 3, $RR = \{w_1, w_2\}$. In this case, the fact that $\langle R \rangle$ contains no C_5 implies that $w_1 w, w_2 w \in B$ for every $w \in RB \cup BR$. Consequently, $w_1 w_2 \in R$ since $\langle B \rangle$ contains no B_n. If $w_i \in BB$ is adjacent in $\langle R \rangle$ to either w_1 or w_2, then w_i is adjacent in $\langle B \rangle$ to every vertex $w \in RB \cup BR$. Otherwise, we find a C_5 in $\langle R \rangle$. We now claim that there cannot be two distinct vertices $w_i, w_j \in BB$ such that the edges $w_i w_1$ and $w_i w_2$ are both in R. Suppose that two such vertices did exist. Using the observation just made, a count shows that $|\Lambda(w_iw_j)| \geq n$. Now if $w_iw_j \in R$, we find a C_5 in $\langle R\rangle$, namely $C: u, w_1, w_i, w_2, u$. Otherwise, of course, there is a B_n in $\langle B\rangle$. Hence, our claim is justified. In view of this fact, we may without loss of generality assume that w_1 is adjacent in $\langle R\rangle$ to at most one vertex of BB. Consequently, w_1 has degree ≤ 4 in $\langle R\rangle$. Now observe that no vertex in BB can be adjacent in $\langle R\rangle$ to two vertices of RB or BR. The reason is that, were this to occur, we would find a vertex w_3 which plays the same role as does w_1 and so has degree ≤ 4 in $\langle R\rangle$. Moreover, $w_1w_3 \in B$ and, since there is a vertex (u or v) to which w_1 and w_3 are commonly adjacent in $\langle R\rangle$, $|\Lambda(w_1w_3)| \geq (2n+1) - 7 \geq n$. Now, as in the proof of case 2, we find that BB must span a complete graph in $\langle R\rangle$ in order to avoid a B_n in $\langle B\rangle$. Also, as in case 2, there can be no vertex of degree ≥ 2 in $\langle RB\rangle_R$ or $\langle BR\rangle_R$. The reason now is that such a vertex in one of these induced subgraphs gives rise to either a K_4 in $\langle R\rangle$ or else the existence of a vertex w_4 which acts just as w_3 did to produce a B_n in $\langle B\rangle$. The proof is now completed as in case 2. ∎

Continuation of the process begun in Theorem 1, *i.e.* complete calculation of $r(C_m, B_n)$ for $m = 7, 9, 11, \ldots$, does not appear to be tractable. Instead, we limit ourselves to a proof that $r(C_m, B_n) = 2n+3$ for all $n \geq 4m - 13$. Even this demonstration relies on some very special devices. Therefore, it is well for us to describe the genesis of the proof before getting into any of its details. As is our custom, with $|V| = 2n+3$ we assume the existence of a two-coloring (R, B) of $[V]^2$ in which $\langle R\rangle$ contains no C_m and $\langle B\rangle$ contains no B_n. Our aim is to show that this assumption leads to ultimately to a contradiction. From Theorem 1 we know that $\langle R\rangle$ contains C_3 (and C_5) so it is certainly not bipartite. There are then pairs of vertices which are connected in $\langle R\rangle$ by paths of both even and odd lengths. The key idea of the proof is to start with an edge $uv \in B$ which is an especially good candidate for satisfying $|\Lambda(uv)| \geq n$ as a result of constraints which arise because u and v are connected in $\langle R\rangle$ by paths of the appropriate lengths, both even and odd. To give this idea precision, we make the following definitions. A path $P: x_1, x_2, \ldots, x_k$ in $\langle R\rangle$ will be called a *switching path* if, for some i, $1 \leq i \leq k-2$, there is a two-chord $x_ix_{i+2} \in R$. An edge $uv \in B$ will be called a *key* if there exists a switching path $P: x_1, x_2, \ldots, x_{m-2} = v$ connecting u and v in $\langle R\rangle$. In what follows, we first prove that a key edge exists and then exploit its properties.

Theorem 2. *Let m be an odd integer ≥ 7 and suppose that $n \geq 4m - 13$. Then* $r(C_m, B_n) = 2n+3$.

Proof: The fact that $r(C_m, B_n) \geq 2n+3$ was brought out in § 3 and, in the discussion just preceding, the stage has been set for the remainder of the proof. So we now take aim at our target, the desired contradiction.

We first claim the $\langle R\rangle$ contains a switching path of order m. Let $P: x_1, x_2, \ldots, x_t$ be a switching path in $\langle R\rangle$ which is of maximal order. Since $\langle R\rangle$ must contain a triangle, $t \geq 3$. We wish to show that $t \geq m$. Suppose, to the contrary, that $3 \leq t < m$. Let $X = \{x_1, x_2, \ldots, x_t\}$ and $W = V - X$. In view of the path maximality, for every $w \in W$ we have $x_1 w, x_t w \in B$. Since $|W| > n$ and $\langle B \rangle$ contains no B_n, $x_1 x_t \in R$. Consequently, $xw \in B$ for every $x \in X$ and $w \in W$; otherwise, the maximality of the path $P : x_1, x_2, \ldots, x_t$ is violated. Finally, X must span a complete graph in $\langle R \rangle$ in order to avoid a B_n in $\langle B \rangle$. Repetition of this argument yields a partition $V = (X_1, X_2, \ldots, X_\ell, X_{\ell+1} = W)$ such that

(i) $3 \leq |X_1| < m$,
(ii) $|X_1| \geq |X_2| \geq \cdots \geq |X_\ell|$,
(iii) $x_i x_j \in B$ for every $x_i \in X_i$, $x_j \in X_j$ with $i \neq j$,
(iv) $X_1, X_2, \ldots, X_\ell$ span complete graphs in $\langle R \rangle$, and
(v) $\langle W \rangle_R$ contains no triangle.

There are two cases for us to consider.

Case 1, $W = \phi$. Since $2n + 3 > 3(m - 1)$ it follows that $\ell \geq 4$. Select $u \in X_{\ell-1}$ and $v \in X_\ell$. Then $uv \in B$ and $|\Lambda(uv)| \geq 2n + 3 - 2\left\lceil\frac{2n+3}{\ell}\right\rceil > n$. Therefore, we reject this possibility.

Case 2, $W \neq \phi$. Of necessity, $\langle W \rangle_B$ is not an empty graph. In fact, if $|W| = s \geq 6$ then $\langle W \rangle_B$ contains B_j where $j \geq \left[\frac{s-3}{2}\right]$. In this case, $\langle B \rangle$ contains B_k where $k \geq (2n + 3) - s + \left[\frac{s-3}{2}\right] \geq n$. In the case where $\langle W \rangle_B$ contains an edge but no triangle, $s \leq 5$ and $\langle B \rangle$ contains B_k where $k \geq 2n - 2 > n$. Reaching this contradiction, we have thus proved the existence of a switching path of order m.

We now claim the existence of a key edge. To verify this claim, we start with a switching path $P : x_1, x_2, \ldots, x_m$ in $\langle R \rangle$ where the two-chord $x_k x_{k+2} \in R$. By symmetry, we may assume that $k \leq \frac{m-1}{2}$. There are three cases to consider.

Case 1, $k \geq 3$. If $x_1 x_{m-2} \in B$ then it is a key edge. Similar observations hold for the edges $x_2 x_{m-1}$ and $x_3 x_m$. One of these alternatives must hold, for, otherwise, $C : x_1, x_2, x_{m-1}, x_m, x_3, x_4, \ldots, x_{m-2}, x_1$ is a C_m in $\langle R \rangle$.

Case 2, $k = 2$. If $x_1 x_{m-2} \in B$ then it is a key edge. On the other hand, if $x_1 x_{m-2} \in R$ and $m > 7$, then the switching path $P : x_m, x_{m-1}, x_{m-2}, x_1, x_2, \ldots, x_{m-3}$ falls under case 1. If $m = 7$, consider the edge $x_2 x_6$. If $x_2 x_6 \in B$ then it is a key edge. Otherwise, the path $P : x_7, x_6, x_2, x_3, x_4, x_5, x_1$ falls under case 1.

Case 3, $k = 1$. Again, if $x_1 x_{m-2} \in B$ then it is a key edge. Otherwise, the path $P : x_m, x_{m-1}, x_{m-2}, x_1, x_2, \ldots, x_{m-3}$ falls under either case 1 or case 2.

Now we are ready to exploit the properties of the key edge whose existence has just been proved. Let $uv \in B$ be a key edge where u and v are connected in $\langle R \rangle$ by the switching path $P : u = x_1, x_2, \ldots, x_{m-2} = v$. Set $X = \{x_1, x_2, \ldots, x_{m-2}\}$ and $W = V - X$. Form the partition $W = (RR, RB, BR, BB)$ by placing $w \in W$ in the appropriate set according to whether the pair (uw, vw) is an element of $R \times R$, $R \times B$, $B \times R$, or $B \times B$. Further, let us define the sets $R_u = \{w | w \in W,$

$uw \in R\}$ and $R_v = \{w | w \in W, vw \in R\}$. In view of the assumption that $\langle R \rangle$ contains no C_m, the following basic facts emerge:

(i) $w_i w_j \in B$ for every pair of distinct vertices $w_i \in R_u, w_j \in R_v$,
(ii) for every $w \in BB$ either $\{w_i | w_i \in R_u, ww_i \in R\}$ or $\{w_j | w_j \in R_v, ww_j \in R\}$ is empty.

If (i) does not hold, then $\langle R \rangle$ contains the m-cycle $C : w_i, x_1, x_2, \ldots, x_{m-2}, w_j, w_i$. To see the validity of (ii), first recall the existence of the two-chord $x_k x_{k+2}$ in the switching path. If (ii) does not hold, then $\langle R \rangle$ contains the m-cycle $C : w, w_i, x_1, x_2, \ldots, x_k, x_{k+2}, \ldots, x_{m-2}, w_j, w$.

We now begin to draw conclusions concerning the size and nature of the sets RR, RB, BR, and BB. Of course $|BB| \leq n-1$ since $\langle B \rangle$ contains no B_n. We also claim that $|RR| \leq 2$. Suppose, to the contrary, that $|RR| \geq 3$ and note that fact (ii) then implies that there are two vertices $w_i, w_j \in RR$ such that $\Lambda(w_i w_j)$ contains at least one-third of the vertices of BB. This in combination with the consequences of fact (i) and the observation that $|BB| \leq n-1$ shows that $w_i w_j \in B$ and $|\Lambda(w_i w_j)| \geq (2n+3) - 2 - (m-2) - \lceil \frac{2(n-1)}{3} \rceil > n$. Since this conclusion must be rejected, our claim that $|RR| \leq 2$ is justified.

Let $Y = BR$. (A similar proof will hold with $Y = RB$.) We assert that either $\langle Y \rangle_B$ is an empty graph or else there exist two vertices $w_i, w_j \in Y$ such that $w_i w_j \in B$ and there are at least $|Y| - (m-1)$ vertices of $X \cup Y$ which are elements of $\Lambda(w_i w_j)$. To prove this assertion, we first note that since $\langle R \rangle$ contains no C_m, $\langle Y \rangle_R$ contains no path of order $m-1$. Let $P : w_1, w_2, \ldots, w_\ell$ be a maximal length path in $\langle Y \rangle_R$. Then $w_1 w, w_\ell w \in B$ for every $w \in Y - P$. If $w_1 w_\ell \in B$ our assertion is already proved. Otherwise, $w_1 w_\ell \in R$ and the maximality of the path yields the fact that $w_i w \in B$ for every $w_i \in P$ and $w \in Y - P$. Continuing in this manner, we find that $\langle Y \rangle_R$ is the union of s disjoint complete graphs. If $s = 1$, $\langle Y \rangle_B$ is an empty graph and our assertion is proved. If $s > 1$, select two vertices $w_i, w_j \in Y$ such that $w_i w_j \in B$. Then w_i is a vertex of a complete graph K_a within $\langle Y \rangle_R$. Likewise, w_j is a vertex of a complete graph K_b within $\langle Y \rangle_R$. Of course, $a, b \leq m-2$ and, without loss of generality, we may assume that $a \leq b$. Since $\langle R \rangle$ contains no C_m, it follows that $\Lambda(w_i w_j)$ contains the set of vertices $\{x_1, x_2, \ldots, x_{a-1}\}$ from the switching path. Consequently, the number of vertices of $X \cup Y$ which are members of $\Lambda(w_i w_j)$ is at least $|Y| - (a+b) + (a-1) \geq |Y| - (m-1)$.

Without loss of generality, we may assume that $|RB| \leq |BR|$. Since $|RR| \leq 2$, $|BB| \leq n-1$, and $n \geq 4m-13$, it follows that $|BR| \geq m-1$. Consequently, $\langle BR \rangle_B$ is not an empty graph. We now claim that either RR or RB is an empty set. Making the contrary supposition, we select $w_i \in RR$ and $w_j \in RB$ such that w_j is of degree at most $m-2$ in $\langle RB \rangle_R$. The latter condition is fulfilled by choosing w_j to be an end-vertex of a maximal length path in $\langle RB \rangle_R$. Also, we select $w_k, w_\ell \in BR$ in accordance with the result proved in the last paragraph so that $w_k w_\ell \in B$

and $\Lambda(w_k w_\ell)$ has at least $|BR| - (m-1)$ members excluding vertices from RR, RB, and BB. Let $S = \Lambda(w_i w_j) \cap BB$ and $T = \Lambda(w_k w_\ell) \cap BB$ and note that fact (ii) implies that $|S| + |T| \geq |BB|$. In view of the observations made thus far, we have

$$|\Lambda(w_i w_j)| \geq |RR| + |RB| + |BR| + |S| - m.$$

and

$$|\Lambda(w_k w_\ell)| \geq |RR| + |RB| + |BR| + |T| - (m-1)$$

Adding these two equations and invoking the assumption that $\langle B \rangle$ contains no B_n, we find that $n \leq 4m - 14$, contrary to the hypothesis of theorem. Thus, either RR or RB is empty.

By a similar argument, we now prove that $\langle RB \rangle_B$ is necessarily an empty graph and so $|RB| \leq m - 2$. Suppose that $\langle RB \rangle_B$ is not an empty graph and choose $w_i, w_j \in RB$ in accordance with the result proved in the paragraph before last. Choose $w_k, w_\ell \in BR$ as in the last paragraph and define S and T as before. Then $|S| + |T| \geq |BB|$,

$$|\Lambda(w_i w_j)| \geq |RR| + |RB| + |BR| + |S| - (m-1)$$

and

$$|\Lambda(w_k w_\ell)| \geq |RR| + |RB| + |BR| + |T| - (m-1).$$

These inequalities lead to a contradiction as in the last paragraph.

By combining the last two results, we find that $|RR| + |RB| \leq m - 2$. Let $S = \{v\} \cup BR \cup BB$. Then S is contained in the neighborhood of u in $\langle B \rangle$ and, by the result just obtained, $|S| \geq (2n+3) - (m-3) - (m-2) = 2n - 2m + 8$. We now claim that $\langle S \rangle_R$ contains no triangles. Observe that since $\langle B \rangle$ contains no B_n every vertex of S is of degree at least $|S| - n \geq n - 2m + 8 > m$ in $\langle S \rangle_R$. If $\langle S \rangle_R$ contained a triangle, we could first find a switching path of order m and then another key edge as in the argument at the outset of this proof. Applying the argument up to now to this new key edge, we could find a vertex $w \in S$ which is adjacent to u in $\langle B \rangle$ and whose degree in $\langle B \rangle$ is at least $2n - 2m + 8$. Since, by assumption, $|\Lambda(uw)| \leq n - 1$, we must have $2(2n - 2m + 8) - (2n + 1) \leq n - 1$ and so $n \leq 4m - 14$. As this is contrary to the hypothesis of the theorem, our claim that $\langle S \rangle_R$ contains no triangles is now justified. In particular, this means that BR spans a complete graph in $\langle B \rangle$.

In what follows, we shall use the following theorem of

Jackson [5]. *With* $k \geq 2$, *let* G *be a bipartite graph with bipartation* $V(G) = (A, B)$ *where* $|A| = a \geq 2$ *and* $|B| = b \geq k$ *and suppose that each vertex of* A

has degree at least k *where* $b \leq 2k-2$. *Then* G *contains a cycle of length* 2ℓ *for all* ℓ *satisfying* $2 \leq \ell \leq \min(a,k)$.

Let us apply this result to the bipartite graph G where $V(G) = (BR, BB)$ and $E(G) = \{w_i w_k | w_i \in BR, w_k \in BB, w_i w_k \in R\}$. Note that in this graph every vertex of BR has degree at least $n-2m+7$. Since $n \geq 4m-13$, it follows that $|BB| \leq n-1 \leq 2(n-2m+7)-2$, *i.e.* the hypothesis of Jackson's theorem is fulfilled. As a consequence of Jackson's theorem we find that, starting from an arbitrary vertex in BR as an end-vertex, there are paths in the bipartite graph of each length $2,4,\ldots,m-3$. Since $\langle R \rangle$ contains no C_m, it follows that each of the vertices $x_2, x_4, \ldots, x_{m-3}$ in the original switching path is adjacent in $\langle B \rangle$ to each vertex of BR. Let w_i and w_j be any two vertices in BR. In view of the condition $2(n-2m+7)-2 \geq n-1$, there are vertices $w_k, w_\ell \in BB$ which are commonly adjacent in $\langle R \rangle$ to w_i and w_j. Since there can be no triangle in $\langle S \rangle_R$, it follows that $w_k w_\ell \in B$. Using the cycles guaranteed by Jackson's theorem, one concludes that, starting from w_k or w_ℓ as an end-vertex, there are paths in the bipartite graph of each length $3,5,\ldots,m-4$. Consequently, since $\langle R \rangle$ contains no C_m, the vertices $x_3, x_5, \ldots, x_{m-4}$ are adjacent in $\langle B \rangle$ to both w_k and w_ℓ. Finally, we note that each vertex $w \in BB$ with $w \neq w_k, w_\ell$ belongs to either $\Lambda(w_i w_j)$ or $\Lambda(w_k w_\ell)$. Were this not the case, $\langle S \rangle_R$ would contain a triangle.

Now here is the situation. With the exception of the vertices w_i, w_j, w_k, w_ℓ themselves, every vertex of V belongs to at least one of the two sets $\Lambda(w_i w_j)$, $\Lambda(w_k w_\ell)$. Since there are $2n-1$ such vertices, it must be that $\langle B \rangle$ contains a B_n. We thus obtain the long sought contradiction and the proof is complete. ∎

5. Large Cycles.

In the proof of the theorem which follows, we make use of some standard results concerning Hamiltonian graphs and their generalizations. All of these results are to be found in chapter 7 of [1].

Theorem 3. *For all* $n \geq 1$ *and* $m \geq 2n+2$, $r(C_m, B_n) = 2m-1$.

Proof: The fact that $r(C_m, B_n) = 2m-1$ was noted in § 3. Now let us suppose that with $|V| = 2m-1$ there exists a two-coloring (R, B) of $[V]^2$ in which $\langle R \rangle$ contains no C_m and $\langle B \rangle$ contains no B_n. We claim that in $\langle B \rangle$ there is a vertex, x_0, of degree at least $m-1$. Otherwise, every vertex would have degree at least m in $\langle R \rangle$ and this would make $\langle R \rangle$ pancyclic by a result of Bondy [**1**, p. 150]. Let X denote the neighborhood of x_0 in $\langle B \rangle$. Since $\langle B \rangle$ contains no B_n, every vertex in X has degree at least $|X|-n \geq |X|-[\frac{m-2}{2}] > \frac{|X|}{2}$ in $\langle X \rangle_R$. It follows that $\langle X \rangle_R$ is Hamiltonian connected [**1**, p. 146] and pancyclic [**1**, p. 150]. In view of the latter, we must assume that $|X|$ is precisely $m-1$. In view of the former, we must assume that each vertex $w \in V-X$ is adjacent in $\langle R \rangle$ to at most one vertex of X. If w

is adjacent in $\langle R\rangle$ to x_1 and x_{m-1} and $P: x_1, x_2, \ldots, x_{m-1}$ is the Hamiltonian path connecting x_1 and x_{m-1} in $\langle X\rangle_R$, then $C: w, x_1, x_2, \ldots, x_{m-1}, w$ is a C_m in $\langle R\rangle$. Let $w_i, w_j \in V - X$ be two vertices which are adjacent in $\langle B\rangle$. Since $|V - X| = m$ two such vertices must exist. As a consequence of the observation that neither w_i nor w_j can be adjacent to two distinct vertices of X in $\langle R\rangle$, we see that $|\Lambda(w_i w_j)| \geq |X| - 2 = m - 3 \geq n$. But this gives a B_n in $\langle B\rangle$ and so a contradiction. ∎

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