A Bijective Proof for the Parity of Stirling Numbers

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Abstract. We give a bijective proof for the identity $S(n,k) \equiv \binom{n-j-1}{n-k} \pmod 2$ where $j = \lfloor \frac{k}{2} \rfloor$ is the largest integer $\leq \frac{k}{2}$.

In [1], page 46, problem 17b, Richard Stanley asks for a combinatiorial proof of the identity

$$S(n,k) \equiv \binom{n-j-1}{n-k} \pmod 2$$

Here $j = \lfloor \frac{k}{2} \rfloor$ is the largest integer $\leq \frac{k}{2}$. It is the purpose of this note to provide such a proof. Recently, Sagan [2] has found a different bijection for the q-analogue of $S(n,k)$.

For n a positive integer, let [n] denote the set $\{1, \ldots, n\}$. Let $P_{n,k}$ denote the set of partitions of [n] into k parts, so that the cardinality of $P_{n,k}$ is $|P_{n,k}| = S(n,k)$. We are going to define involutions $f_{n,k} : P_{n,k} \to P_{n,k}$ by induction on n and k. Clearly, $S(n,k)$ will have the same parity as the number of fixed points of the involutions $f_{n,k}$.

Let $f_{n,n}$ and$f_{n,1}$ be the identity mapping for all n. Now suppose we have an element in $P_{n,k}$, say $\pi = \{B_1, \ldots, B_k\}$. Suppose the number n is in block B_r. We define $s = \max([n] - B_r)$. Let the block that s is in be B_i. By defintion, $i \neq r$. Clearly the numbers $s+1, s+2, \ldots, n-1, n$ are all in B_r, since s is the biggest number not in B_r. Our idea is to switch s with the set of numbers $s+1$ to n to get a new partition, that is, let

$$f_{n,k}(\pi) = \{B'_1, \ldots, B'_k\}$$

where

$$B'_l = \begin{cases} B_l & \text{if } l \neq i, r \\ (B_i - \{s\}) \cup ([n] - [s]) & \text{if } l = i \\ (B_r - ([n] - [s]) \cup \{s\} & \text{if } l = r \end{cases}$$

However this won't work when s is the only element of B_i and B_r is exactly the numbers from $s+1$ to n, since we will simply interchange the two blocks without altering them. In this case, we will forget about the numbers from s to n and work on the smaller set $[s-1]$. Therefore, let

$$f_{n,k}(\pi) = f_{s-1,k-2}(\pi - \{B_i, B_r\}) \cup \{B_i, B_r\}$$

Then $f_{n,k}$ is clearly an involution.

As an example, suppose that $\pi = \{\{6,5\},\{4\},\{3,1\},\{2\}\}$. Then

$$f(\pi) = \{\{6,5\},\{4\},\{3\},\{2,1\}\}$$

Now we will count the number of partitions fixed under $f_{n,k}$. Suppose $f_{n,k}(\pi) = \pi$. If k is even, π must look like

$$\{[n]-[s_1],\{s_1\},[s_1-1]-[s_2],\{s_2\},[s_2-1]-[s_3]\ldots,[s_{j-1}-1]-[s_j],\{s_j\}\}$$

Where $n > s_1 > s_2 > \cdots > s_j = 1$ and $s_i - 1 > s_{i+1}$. We have that $s_j = 1$ and so $s_{j-1} \geq 3$. Hence the number of such π is equal to the number of ways of choosing $j-1$ non-consecutive dots out of $n-3$ dots in a row.

If k is odd, then π must look like

$$\pi = \{[n]-[s_1],\{s_1\},[s_1-1]-[s_2],\{s_2\},[s_2-1]-[s_3],$$
$$\ldots,[s_{j-1}-1]-[s_j],\{s_j\},[s_j-1]\}$$

where $n > s_1 > s_2 > \cdots > s_j > 1$ and $s_i - 1 > s_{i+1}$. We have that $s_j > 1$ and hence the number of such π is the number of ways of choosing j non-consecutive dots out of $n-2$ dots in a row.

Let $g(m,t)$ be the number of ways of choosing t non-consecutive dots from m dots in a row. It is well-known and easy to show that

$$g(m,t) = \binom{m-t+1}{t}$$

Hence, if k is even, $S(n,k) \equiv \binom{n-3-(j-1)+1}{j-1} (\bmod 2)$, and this equals $\binom{n-j-1}{n-k}$. If k is odd, $S(n,k) \equiv \binom{n-2-j+1}{j} (\bmod 2)$, and this is equal to $\binom{n-j-1}{n-k}$.

References

1. Richard Stanley, *Enumerative Combinatorics*, Wadsworth and Brooks/Cole Advance Books and Software, Monterey, Califorinia, 1986.
2. Bruce Sagan, personal communication (1988).