On the Construction of Cordial Graphs

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Abstract. Ho and Shee [5] showed that for a graph G of order n (≥ 4) and size m to be cordial, it is necessary that m must be less than $n(n-1)/2 - \lceil n/2 \rceil + 2$. In this paper, we prove that there exists a cordial graph of order n and size m, where $n-1 \leq m \leq n(n-1)/2 - \lceil n/2 \rceil + 1$.

1. Introduction.

Let $G = (V, E)$ be a simple graph. A binary labeling of G is a mapping $f: V \to \{0, 1\}$. For each $v \in V(G)$, $f(v)$ is called the label of v under f, and for an edge $(u, v) \in E(G)$, the induced edge label is defined by $|f(u) - f(v)|$. We denote by $v_f(0)$ (resp. $e_f(0)$) and $v_f(1)$ (resp. $e_f(1)$) the number of vertices (resp. edges) with labels 0 and 1 under a binary labeling f of G respectively. We say f is *cordial* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

A graph G is cordial if it admits a cordial labeling. Cordial graphs were first introduced by Cahit [2] as a weaker version of both graceful graphs [7] and harmonious graphs [3].

Cordialness of several families of graphs was investigated by various researchers; see [**1, 2, 4**]. Recently, characterization of cordial graphs was carried out by Ho and Shee [5], and Kirchher [6]. In particular, Ho and Shee gave a necessary condition for a cordial graph of order n and size m, which we restate as the following Theorem.

Theorem 1. *If a graph G of order $n \geq 4$ and size m is cordial, then $m < n(n-1)/2 - \lceil n/2 \rceil + 2$.*

We see from Theorem 1 that the size m of a cordial graph on n vertices is at most $n(n-1)/2 - \lceil n/2 \rceil + 1$. In this paper, we show that it is possible to construct a cordial graph of order n and size m, where $n-1 \leq m \leq n(n-1)/2 - \lceil n/2 \rceil + 1$. We thus have the following Theorem.

Theorem 2. *For a given integer $n \geq 4$, there exists a cordial graph G of order n and size m, where $n-1 \leq m \leq n(n-1)/2 - \lceil n/2 \rceil + 1$.*

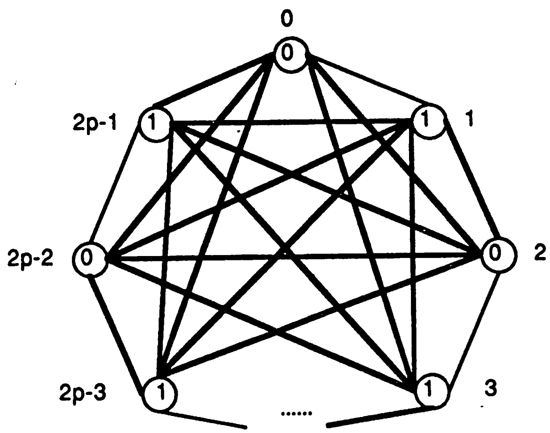
Research supported by NSERC grant OGP0036870.

2. Graphs of even order.

To prove Theorem 2, we first of all consider graphs of even order, that is, n even. The case of odd order is fairly similar, and is treated in the next section.

Consider a complete graph on $2p$ vertices, K_{2p}. To obtain a cordial labeling for K_{2p}, we must label p of the vertices 0, and the remaining p vertices $1(v_f(0) = v_f(1) = p)$. It is easy to verify that the number of edges labeled 0 is $[p(p-1)/2]2 = p(p-1)$, and the number of edges labeled 1 is p^2. That is, $e_f(0) = p(p-1)$ and $e_f(1) = p^2$, and K_{2p} is not cordial for $p \geq 2$. We note that this result was shown in [2].

A one-factor g of K_{2p} is a collection of p edges of K_{2p} that span the vertex set of K_{2p}. Denote by $K_{2p} - g$ the graph of K_{2p} with a one-factor deleted, and denote the $2p$ vertices by $0, 1, \ldots, 2p-1$. We label the vertices using an alternate sequence of 0 and 1, as shown in the following figure.



If we delete p edges $(0,1), (2,3), \ldots$ and $(2p-2, 2p-1)$ from $K_{2p} - g$, we see that $e_f(1) = p^2 - p = e_f(0)$. Consequently, $K_{2p} - g$ is cordial. Adding any one of the p edges (say e) in g back to $K_{2p} - g$, we have a graph $K_{2p} - g + e$ that is still cordial, as $v_f(0) = v_f(1)$ and $e_f(1) = e_f(0) + 1$. We note that the size of $K_{2p} - g + e$ is $2p(2p-1)/2 - p + 1$, which is the largest permissible value for a given p (see Theorem 1).

We have just constructed cordial graphs of order $2p$, with sizes $2p(2p-1)/2-p$ and $2p(2p-1)/2 - p + 1$. We now give an algorithm for constructing a cordial graph of order $2p$ and size $m = 2p(2p-1)/2 - p - 1, 2p(2p-1)/2 - p - 2, \ldots, 2p, 2p-1$. The idea is to start out with the cordial labeling of $K_{2p} - g$, deleting one edge at a time, making sure that the intermediate graphs remain cordial in the process. Note also that in deleting the edges, we want the graph to remain connected. We achieve this by constructing a path P of $2p-1$ edges before the deletion takes place, and selecting only edges not on P for deletion. There are two cases:

(1) p is odd ($2p = 4s+2$): we select $4s+1$ edges such that the vertex labels form the sequence 0011, 0011, ..., 0011, 01 (note that there are $2s$ edges labeled 0, and $2s+1$ edges labeled 1). There are many ways to do it. An example is to traverse the vertices in the following order: 2 4 1 3, 6 8 5 7, ..., $4s-4$ $4s$ $4s-3$ $4s-1$, 0 $4s+1$.

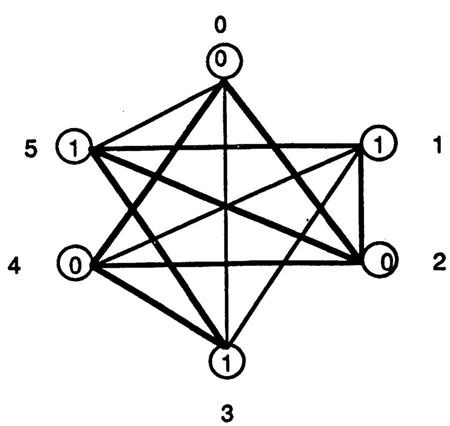
(2) p is even ($2p = 4s$): the vertex sequence is 0011, 0011, ..., 0011, 0011 (there are $2s$ edges labeled 0, and $2s-1$ edges 1). An example is to traverse the vertices in the following order: 0 2 1 3, 4 6 5 7, ..., $4s-4$ $4s-2$ $4s-3$ $4s-1$.

The following is an algorithm for constructing cordial graphs of order $2p$ and size $m = 2p(2p-1)/2-p-1, 2p(2p-1)/2-p-2, \ldots, 2p, 2p-1$, where p is odd.

(1) Start out with the cordial labeling of $K_{2p}-g$ as described above.
(2) Select a path P of $2p-1$ edges.
(3) Select and delete an edge labeled 0.
(4) If the number of edges in the graph is $2p-1$, stop.
(5) Select and delete an edge labeled 1.
(6) Go to step (3).

For p even ($2p = 4s$), the algorithm is identical to above except that steps (3) and (5) are interchanged.

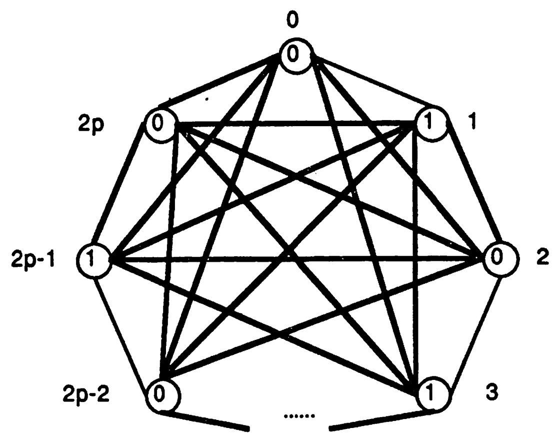
We illustrate the algorithm using K_6-g. We choose the path P that traverses vertices 2, 4, 1, 3, 0 and 5. We may for example delete the seven edges in the following order, to obtain cordial graphs of sizes 11, ..., 7, 6 and 5: (0, 2), (1, 2), (0, 4), (2, 5), (1, 5), (3, 4), and (3, 5).



3. Graphs of odd order.

For graphs of odd order, we consider the complete graph K_{2p+1}. The argument and algorithm are fairly similar to those of the even order. We will note the differences and give the algorithm.

Without loss of generality, we label $p+1$ vertices 0 and p vertices 1, as indicated in the following figure.



The number of edges labeled $0 = p(p+1)/2 + p(p-1)/2 = p^2$, and the number of edges labeled $1 = p(p+1) = p^2 + p$. As in the previous section, we delete a near one-factor g of p edges, which span all but one vertex. Here, we let $g = \{(0,1),(2,3),\dots,(2p-2,2p-1)\}$. Again, we need to select a path P of $2p$ edges. There are two cases.

(1) p is odd ($2p = 4s+3$): the vertex sequence is 0, 0011, 0011, ... , 0011, 01 (note that there are $2s+1$ edges labeled 0, and $2s+1$ edges labeled 1). An example is the following vertex sequence: $0, 2\ 4\ 3\ 1,\ 6\ 8\ 7\ 5, \dots, 4s-2\ \ 4s\ \ 4s-1\ \ 4s-3,\ 4s+2\ \ 4s+1$.

(2) p is even ($2p = 4s+1$): the vertex sequence is 0011, 001, ... , 0011, 0 (there are $2s$ edges labeled 0, and $2s$ edges 1). An example is the following vertex sequence: $0\ 2\ 1\ 3,\ 4\ 6\ 5\ 7, \dots, 4s-4\ \ 4s-2\ \ 4s-3\ \ 4s-1,\ 4s$.

The algorithm is very similar to the case of even order, except that in both cases, we can pick an edge labeled either 0 or 1 at the outset of the algorithm.

(1) Start out with the cordial labeling of $K_{2p+1} - g$ as described above.

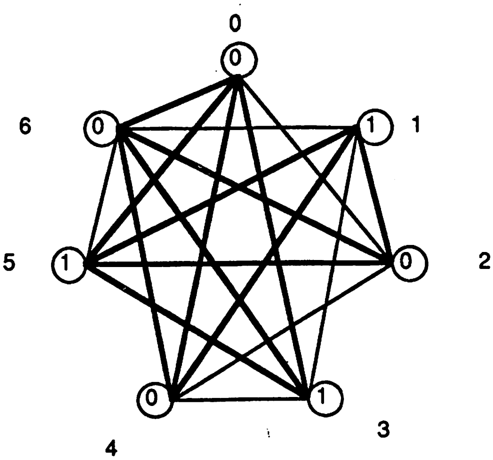
(2) Select a path P of $2p$ edges.

(3) Select and delete an edge labeled 0.

(4) Select and delete an edge labeled 1.

(5) If the number of edges in the graph is $2p$, stop. Otherwise, go to step (3).

We illustrate the algorithm using $K_7 - g$. The path P is 0 2 4 3 1 6 5. As an example, we may delete the twelve edges in the following order to obtain cordial graphs of sizes 17, 16, ..., 8, 7 and 6: (0,3), (0,4), (0,5), (0,6), (1,2), (1,5), (1,4), (2,6), (2,5), (3,5), (3,6) and (4,6).



4. Conclusion.

We prove that there exists a cordial graph of order n and size m, where $n-1 \leq m \leq n(n-1)/2 - \lceil n/2 \rceil + 1$.

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