Some Small Packing Numbers

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Abstract

The packing number D(2,k, v) is defined to be the maximum cardinality of a family of k-subsets, chosen from a set of v elements, such that no pair of elements appears in more than one k-subset. We examine D(2,k, v) for $v < k(k-1)$ and determine such numbers for the case $k = 5, v < 20$.

1 Introduction

A (2,k) packing of a v-set V is a design **D** consisting of b blocks, each with k elements chosen from the set V, such that no pair of elements appears in more than one block of **D**. The maximum number of blocks in a (2,k) packing is generally denoted by $D(2, k, v)$. It is well known that $D(2, k, v) \leq \left[\frac{v}{k}\left[\frac{v-1}{k-1}\right]\right] = B(2, k, v)$, where $[x]$ denotes the greatest integer $\leq x$. The problem of determining D(2,k, v) for $k \leq 4$ has been solved. Apart from the well known result that D(2,5,v) = B(2,5,v) for $v \equiv 0$, 1, 4, or 5 (mod 20), which was mentioned in [1], very little seems to be known for the case $k = 5$. In a recent communication, R.C. Mullin suggested that he would be interested in knowing the values of D(2,5,v) for small values of v. In this paper we restrict our attention to designs in which $v < k(k-1)$ and we determine these numbers for the case $k = 5$.

Let **D** be a design consisting of b blocks, each of size k, on a set of v elements, such that no pair of elements appears in more than one of the blocks. We define the *frequency* of an element to be the number of times which that element appears in **D**. Let r_i denote the number of elements of frequency i in a design. Clearly $i \leq \left[\frac{v-1}{k-1}\right]$.

If we denote the frequency numbers for a design **D** by the $n-tuple$ $(r_1, r_2, \ldots r_n)$ then these numbers satisfy the system of equations

$$\left.\begin{array}{rcl} \sum r_i & = & v \\ \sum i r_i & = & bk \end{array}\right\} \qquad (1)$$

where the summation is over all non-negative integers less than or equal to $\left[\frac{v-1}{k-1}\right]$. By examining restrictions on occurrences of elements of frequency $k-1$ for designs with $v < k(k-1)$ we establish two theorems which together with the system of equations given above are helpful in determining packing numbers for small values of v.

2 Designs with $v < k(k-1)$

In this section we consider designs with block size k for which $v < k(k-1)$. By examining the possible occurrences of elements of frequency $k-1$, we show that such a design can have at most $k-2$ elements of frequency $k-1$, no pair of which appear together. We also prove a theorem which will give the value of D(2,k,v) for $v \equiv -1 \bmod k(k-1)$ if there exists a balanced incomplete block design with parameters $(k(k-1)+1, k, 1)$.

Theorem 2.1 *If a design* **D** *with block size* k *has* $v < k(k-1)$ *then* **D** *contains at most* $k-2$ *elements of frequency* $k-1$ *and no pair of such elements can appear in the same block of* **D**. *If any element in* **D** *has frequency* $k-1$ *then* **D** *has at most* $(k-1)(k-2)$ *blocks.*

Proof Let **D** be a design on v elements, $v < k(k-1)$, and let x be an element of frequency $k-1$. Including the element x, there are $(k-1)^2+1$ distinct elements in the blocks containing x. Any other block in **D** can contain at most $k-1$ of these elements and hence an additional element is required to make that block into a k-subset. There are at most $k-3$ such elements available to fill these additional blocks. Hence there are at most $k-2$ elements of frequency $k-1$ and **D** can contain at most $(k-1)(k-2)$ blocks.

Suppose y is one of the elements which appears with x. Among the $(k-1)^2$ elements appearing with x, only one element from each block not containing the pair $\{x, y\}$, or at most $k-2$ elements, can appear in another block along with y. Hence an additional element which has not appeared with x is needed to complete any block containing y. If y is to have frequency $k-1$ then an additional $k-2$ elements for a total of $k(k-1)$ elements are required. Since $v < k(k-1)$, this is impossible and hence no pair of elements of frequency $k-1$ appear together in **D**. □

Corollary If $v \equiv j \bmod (k-1)$, $v < k(k-1)$, then any design **D** on v elements with an element of frequency $k-1$ has at most j elements of frequency $k-1$ and at most $j\left[\frac{v-1}{k-1}\right]$ blocks.

Theorem 2.2 *If $v = k(k-1) - 1$ then $D(2,k,v) \leq (k-1)(k-2)$, and if there exists a BIBD$(k(k-1)+1,k,1)$ then $D(2,k,v) = (k-1)(k-2)$.*

Proof Let D be a design on v elements with $v = k(k-1) - 1$. It follows from Theorem 2.1 that if **D** has an element of frequency $k-1$ then **D** has at most $(k-1)(k-2)$ blocks. We now show that no larger design exists. Suppose **D** is a design which has at least $b = (k-1)(k-2)+1 = k^2-3k+3$ blocks of size k. Then **D** can have no element of frequency $k-1$. There are $k[k^2-3k+3] = k^3-3k^2+3k$ spaces in these blocks, but if each element of D has a frequency of at most $k-2$ we are able to fill only $(k-2)(k^2-k-1) = k^3-3k^2+k+2$ of these spaces. Hence $D(2,k,v) \leq (k-1)(k-2)$.

If there exists a BIBD$(k(k-1)+1,k,1)$ then we can construct a design with $(k-1)(k-2)$ blocks by choosing any pair of elements in the BIBD and deleting all blocks containing these elements. We are left with a design on $k(k-1)-1$ elements with $(k-1)(k-2)$ blocks in which no pair of elements is repeated. □

Conjecture $D(2,k,v) < B(2,k,v)$ for $v < k(k-1)$.

3 Packing designs with $k = 5$ and $v \leq 19$.

In this section we restrict our attention to designs with $k = 5$ in which $v \leq 19$. For simplicity we refer to D(2,5,v) as D(v) and to B(2,5,v) as B(v). B(v) gives the trivial bound of one for $v \leq 8$. It is also easy to see that D(v) = 2 for $9 \leq v \leq 11$. For $v = 12$, it is easy to show by construction that the result is three. The solution is unique up to isomorphism. For $v = 13$, we can construct two different designs, depending on whether $r_3 = 0$ or $r_3 = 1$. In each case the number of blocks is three. For $v = 14$, we have $r_3 = 0$, and we can show that D(14) =4, uniquely.

At $v = 15$, if $r_3 > 0$, we can get at most five blocks. Using elements of frequency two, however, we get the design with six blocks shown below on the left. For $v = 16$, if $r_3 = 0$ then $b \leq 6$. It is also easy to show that $r_3 \leq 1$, and that if $r_3 = 1$ then $b \leq 6$. A design with $r_3 = 1$ having six blocks is given below on the right. A solution with $r_3 = 0$ can easily be constructed from the design on 15 elements.

1	2	3	4	5
1	6	7	8	9
2	6	10	11	13
3	7	10	12	14
4	8	11	14	15
5	9	12	13	15

1	2	3	4	5
1	6	7	8	9
1	10	11	12	13
2	6	10	14	15
3	7	11	14	16
4	8	12	15	16

For $v = 17$, if we wish to get a design in which $b > 4$ then by the Corollary to Theorem 2.1, we must have $r_4 = 0$. If we also have $r_3 = 0$

then $b \leq 6$. By allowing an element of frequency three, since no design with $v = 17$ and $b > 7$ exists, the design on the left shows that D(17) = 7.

1	2	3	4	5
1	6	7	8	9
1	10	11	12	13
2	6	10	14	15
3	7	11	14	16
4	8	12	15	17
5	9	13	16	17

1	2	3	4	5
1	6	7	8	9
1	10	11	12	13
2	6	10	14	16
2	7	11	15	17
3	8	12	14	17
3	9	13	15	16
4	6	12	15	18
5	7	13	16	18

If $v = 18$ and $r_4 \neq 0$, then by the Corollary to Theorem 2.1, we have $b \leq 8$. Such a design in which $r_4 = 2$ can easily be constructed. If $r_4 = 0$ system (1) gives a bound of $b \leq 10$. Since no design with $v = 18$ and $b = 10$ exists the design on the right above shows that D(18) = 9. D(19) = 12 follows directly from Theorem 2.2

4 Conclusion.

We summarize these results for $k = 5$ in a table showing the values of B(v) and D(v) for $12 \leq v \leq 21$.

v	12	13	14	15	16	17	18	19	20	21
B(2,5,v)	4	7	8	9	9	13	14	15	16	21
D(2,5,v)	3	3	4	6	6	7	9	12	16	21

References

[1] J. L. Allston and R. G. Stanton, *A note on Pair Coverings with Maximal Block Length Five,* Utilitas Mathematica, 28(1985), 211-217.