REGULAR TRIANGULATIONS OF NON-COMPACT SURFACES

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Abstract. A triangulation of a surface is δ-regular if each vertex is contained in exactly δ edges. For each $\delta \geq 7$, δ-regular triangulations of arbitrary non-compact surfaces of finite genus are constructed. It is also shown that for $\delta \leq 6$ there is a δ-regular triangulation of a non-compact surface Σ if and only if $\delta = 6$ and Σ is homeomorphic to one of the following surfaces: the Euclidean plane, the two-way-infinite cylinder, or the open Möbius band.

1. Introduction.

Recently, some authors tried to construct infinite triangulations of (non-compact) surfaces, which have the property that each vertex belongs to exactly δ triangles. Such triangulations are called *δ-regular*, since their graphs are δ-regular. For example, Lavrenchenko [2] constructs δ-regular triangulations of the torus with finitely or countably many points removed, for any $\delta \geq 7$. He also shows that such triangulations do not exist for $\delta \leq 6$.

In this paper we construct for each $\delta \geq 7$, δ-regular triangulations of *all* non-compact surfaces of finite genus. We also prove that δ-regular triangulations with $\delta \leq 6$ do not exist, with three obvious exceptions which admit 6-regular triangulations. Let us mention that for each $g \geq 0$ there are infinitely many pairwise non-homeomorphic noncompact surfaces of genus g.

A *surface* is assumed to have no boundary unless explicitly stated otherwise. It is known [1,4] (see also [3]) that every non-compact surface Σ of finite genus is homeomorphic to $\Sigma_0 \setminus A$ where Σ_0 is a compact surface, orientable or non-orientable, and A is a non-empty subset of Σ_0 which is homeomorphic to an arbitrary closed subset of the Cantor set. Equivalently, A is a totally disconnected compact metric space. Two such surfaces $\Sigma_0 \setminus A$ and $\Sigma_0' \setminus A'$ are homeomorphic if and only if (Σ_0, A) and (Σ_0', A') are homeomorphic pairs. The *genus* of Σ is equal to the genus of the corresponding compactification Σ_0. The corresponding set A is called the set of *ends* (also

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the set of *ideal points*) of Σ. Note that the set of ends may be uncountable, and that also its topology is important.

Closed subsets of the Cantor set can be represented by subtrees of the *binary tree*. Recall that the binary tree is a tree T_2 with a root $r \in V(T_2)$ such that each vertex $v \in V(T_2)$ has exactly two neighbours which are further from the root than v. A subtree of T_2 is a rooted tree T with the same root r and $V(T) \subseteq V(T_2)$, $E(T) \subseteq E(T_2)$. Vertices in T at distance i from the root are said to be at *level* i. The set of all 1-way-infnite paths in T, starting at the root, is called the set of *ends* of T. Each end P of T is determined by the infinite sequence of the Left-Right turns of P in the binary tree T_2. If we write 0 for Left and 2 for Right we get an infinite word consisting of $0's$ and $2's$. Let $\psi(P) \in [0,1]$ be the real number between 0 and 1 with the fraction part, if written in the *ternary system*, equal to the obtained sequence. It can be shown that the set $\{\psi(P);\ P$ an end of $T_2\}$ is the Cantor set, and that each closed subset of the Cantor set can be represented in this way as the set of ends of some subtree T of T_2.

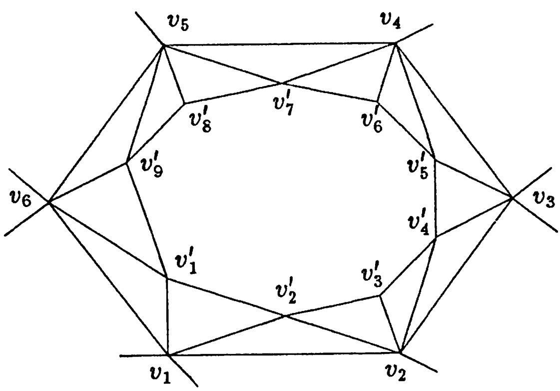

Figure 1. A 6-rim addition

In the construction of δ-regular triangulations the following construction, called *adding* the δ-*rim*, will be used. Let S be a triangulated surface with boundary. Choose a boundary component of S, and let C be the cycle of the triangulation lying on this boundary. Assume that each of the vertices $v_1, v_2, \ldots, v_k$ on C has degree $deg(v_i) \leq \delta - 1$. Denote by $\delta_i := \delta - deg(v_i)$ the number of edges missing in order for v_i to be of degree δ. Add δ_i new edges at v_i $(i = 1, 2, ..., k)$ and add also triangles containing these edges so that each vertex v_i is contained in δ triangles. Of course, one edge at v_i and one edge at v_{i+1} $(i = 1, 2, ..., k)$ have a common endpoint, but the endpoints of the new edges are different elsewhere. See Figure 1 for an example of the addition of the 6-rim. In this Figure the "outside" is ment as a triangulated surface. Whenever we will be adding a δ-rim, no two vertices u, v on C with $deg(u) = deg(v) = \delta - 1$ will be adjacent. Consequently, the degrees of all new vertices on the rim will be at most 5.

It is clear by construction that the number k' of vertices on the rim, after adding a δ-rim, is equal to

$$k' = \sum_{i=1}^{k} (\delta_i - 1) = \Delta - k \tag{1.1}$$

where $\Delta = \sum_{i=1}^{k} \delta_i$. The new defficiencies δ'_j ($j = 1, 2, ..., k'$) depend on the degrees of v_i. If $\delta_i \geq 2$ for each i, then exactly k among the new k' vertices are of degree 4, and the remaining $k' - k$ of degree 3. In general the new sum Δ' is equal to

$$\Delta' = \sum_{j=1}^{k'} \delta'_j = (\delta - 3)\Delta - (\delta - 2)k \tag{1.2}$$

which follows by an easy counting argument using the fact that the total number of edges from v'_j to v_i is equal to Δ. Later on we will need the following simple consequence of (1.1) and (1.2). If $\Delta \geq 2k + 1$ and $\delta \geq 6$ then $k' > k$ and $\Delta' \geq 2k' + 1$.

2. The construction.

The main result of this Section is:

Theorem 1. *Let Σ be a non-compact surface of finite genus, and let $\delta \geq 7$. Then there exist δ-regular triangulations of Σ.*

The rest of this Section is the proof of Theorem 1. Fix Σ and $\delta \geq 7$. Let Σ_0 be the compactification of Σ, and let A be the set of ends of Σ, so $\Sigma = \Sigma_0 \backslash A$. Next, represent A by a subtree T of the binary tree T_2 as explained in the introduction. We shall construct triangulations of bordered surfaces, $R_0, R_1, R_2, ...$ with the following properties:

(i) $R_0 \subset R_1 \subset R_2 \subset ...$ and, moreover, $\partial R_n \subseteq \text{int } R_{n+1}$, $n \geq 0$.

(ii) R_n ($n \geq 0$) has the same genus and the same orientability type as Σ_0.

(iii) The boundary components of R_n are in a bijective correspondence with the vertices at level n in T, i.e. vertices at distance n from the root. Moreover, this correspondence admits the following requirement. If $v \in V(T)$ is at level n and $u \in V(T)$ is its son at level $n + 1$ then the corresponding boundaries of R_n and R_{n+1} lie in the same connected component of $R_{n+1} \setminus \text{int } R_n$.

(iv) Each vertex in the interior of R_n has degree δ, and each boundary vertex of R_n has degree at most 5.

Properties (i) and (iv) imply that $R := \bigcup_{n=0}^{\infty} R_n$ is a δ-regular triangulation of a non-compact surface without boundary, while (ii) and (iii) imply that this surface is homeomorphic to $\Sigma_0 \setminus A = \Sigma$.

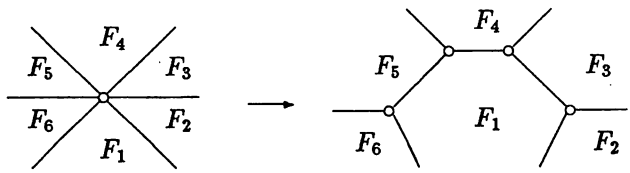

Figure 2. Producing vertices of degree 3

It remains to show how to construct the triangulations R_n. The initial one, R_0, is obtained as follows. If Σ_0 is the 2-sphere, then let R_0 be the triangulation consisting of one inner vertex of degree δ and δ boundary vertices, each of degree three. Otherwise, choose an arbitrary cellular dissection of Σ_0 having only one 2-cell. Such dissections are easy to find if Σ_0 is not the 2-sphere. It may be assumed that every 0-cell of this dissection is incident with 3 edges (see Figure 2). Now, replace each vertex of such a dissection by a triangle, and each edge by a triangulated strip connecting the corresponding triangles as shown on Figure 3. Note that this can be done in such a way that all vertices of the obtained triangulation R_0 have degree at most 5 and that each vertex lies on the boundary of the obtained simplicial 2-complex. Then R_0 has the required properties (i)–(iv). In addition, it satisfies

(v) Each boundary component of R_n contains two non-incident edges e_1, e_2 such that their end-vertices have degree at most four.

This property will be shown to hold also for other R_n, since it is needed for our iterative construction.

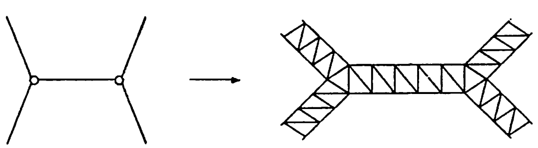

Figure 3. Constructing R_0

Having constructed R_n, define R_{n+1} as follows. For each vertex v of T at level n which has two sons, subdivide the boundary component in R_n corresponding to v into two parts as shown on Figure 4. In order not to get vertices of degree more than 6 on the boundary, one can use for the initial edges of the subdividing strip the two edges e_1, e_2 whose existence is guaranteed by (v). Add now, to each of the boundary components (including those which were not subdivided) a δ-rim, and call the obtained 2-complex R_{n+1}. Although there might have been vertices of degree 6 on the boundary before the δ-rims were added, no two such vertices are adjacent. Therefore all boundary vertices of R_{n+1} have degree at most 5.

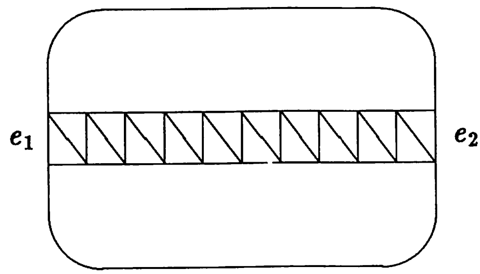

Figure 4. Subdividing

Using (iv), (v), and the assumption $\delta \geq 7$ one can show by applying (1.1) and (1.2) (see also the remark after (1.2)) that the number of vertices on each of the boundary components gets larger and larger, and therefore no boundary component can disappear after adding a rim. Now it is clear that R_{n+1} has the required properties (i)–(v), and our proof is complete.

3. Non-existence of δ-regular triangulations.

Theorem 2. *Let Σ be a non-compact surface. Then Σ admits no δ-regular triangulations for $\delta \leq 5$, and Σ admits a 6-regular triangulation if and only if either*
(a) Σ is homeomorphic to the 2-sphere with 1 or 2 points removed, or
(b) Σ is homeomorphic to the projective plane with one point removed.

The rest of this section is devoted to the proof of Theorem 2. Let us first construct 6-regular triangulations of Σ in cases (a) and (b). The two cases for the 2-sphere are well-known 6-regular tessellations of the plane, or the infinite cylinder. The projective plane is also simple. Start with a 4-regular triangulation of the Möbius band (with all vertices on the boundary; see Figure 5 where the edge xy on the left is identified with the edge on the right), and then add 6-rims one after another ad infinitum. Any 6-rim addition preserves the number of vertices on the boundary, and all of them are of degree four, hence the construction is correct.

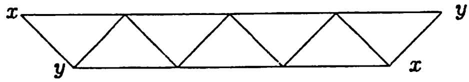

Figure 5. Möbius band

In proving the non-existence part of Theorem 2, the following lemma will be needed. Let us recall that a subcomplex K_1 of a 2-dimensional triangulation is said to be *pure* if each simplex in K_1 is contained in some 2-simplex of K_1. For a vertex v in K_1, let $\kappa(v; K_1)$ denote the difference between the degree of v in K_1 and the number of 2-simplices of K_1 which contain v. The vertex v is a *boundary vertex* of K_1 if $\kappa(v; K_1) > 0$. We shall use the following notation. Let $B(K_1)$ be the set of boundary vertices of K_1, $D(K_1)$ the sum of the degrees in K_1 of the vertices in $B(K_1)$, and let $k(K_1) := \sum_{v \in B(K_1)} \kappa(v; K_1)$.

Lemma 1. *Let K be a δ-regular triangulation of a non-compact surface. Let K_0 be a pure subcomplex of K with finite boundary $B(K_0)$. Then there is a pure subcomplex K_1 of K such that $K_0 \subseteq K_1$, $K_1 \setminus K_0$ is finite, and it satisfies*

$$D(K_1) \leq |B(K_1)|\delta - 2\,k(K_1)\ . \tag{3.1}$$

Proof. Assume that there is a counterexample, and let k be the smallest possible number for which there is a subcomplex K_0 of K not having a required extension, and with $k(K_0) = k$. Obviously, $k > 0$. If a connected component of $K \backslash K_0$ is finite we may add it to K_0 obtaining a smaller counterexample since any finite extension of the obtained complex is also a finite extension of K_0. If there is a 2-simplex T in $K \backslash K_0$ which has two of its edges in K_0, let K_0' be the complex obtained by adding T to K_0. Since $k(K_0') < k$, and any extension of K_0' is also an extension of K_0, we contradict the minimality of k. So we may assume that none of the above cases holds for K_0.

Let K_1 be obtained from K_0 by adding to K_0 all triangles of K which have a vertex in common with K_0. Let E_0 be the set of boundary edges of K_0, and let E_1 be the set of edges lying on the boundary of K_1. Each vertex $v \in B(K_0)$ is contained in exactly $2\kappa(v; K_0)$ edges from E_0. Consequently,

$$|E_0| = \frac{1}{2} \sum_{v \in B(K_0)} 2\,\kappa(v; K_0) = k \tag{3.2}$$

and similarly we see that $|E_1| = k(K_1)$. Each edge in E_1 lies in a unique 2-simplex which has its third vertex on $B(K_0)$. Each vertex $v \in B(K_0)$ determines in this way at most $\delta - d(v) - \kappa(v; K_0)$ edges of E_1, where $d(v)$ is the degree of v in K_0. (Here we used the assumption that no 2-simplex of $K \setminus K_0$ has two of its edges in K_0.) Using this and $|B(K_0)|\delta - D(K_0) < 2k$ (since K_0 is a trivial extension of itself) we conclude that

$$\begin{aligned} k(K_1) = |E_1| &\leq \sum_{v \in B(K_0)} (\delta - d(v) - \kappa(v; K_0)) \\ &= |B(K_0)|\,\delta - D(K_0) - k < k \,. \end{aligned}$$

By the minimality of k there are extensions of K_1 satisfying the inequality (3.1) which contradicts the non-existence of such extensions for K_0.

The rest of the proof of Theorem 2 goes as follows. Assume we have a δ-regular triangulation K of $\Sigma = \Sigma_0 \setminus A$, where $\delta \leq 6$. If Σ has a finite genus then let K_0 be a finite subtriangulation of K which has the same genus as Σ and $r' < \infty$ compact boundary components. Clearly there is no question about the existence of K_0 since Σ is of finite genus. On the other hand, if the genus of Σ is infinite, let K_0 be a finite subtriangulation of K with large finite genus (≥ 2). We may assume that each component of $K \setminus K_0$ is infinite. Let K_1 be a finite extension of K_0 satisfying (3.1). Since all components of $K \setminus K_0$ are infinite, the number of boundary components did not decrease. Denote their number by r. Also denote by Σ_0' the closed surface of the same genus and orientability type as K_1.

Denote by n the number of vertices of K_1, and let $B = B(K_1)$, $D = D(K_1)$, and $k = k(K_1)$. Standard counting arguments show that

$$\#\text{edges in } K_1 = \frac{1}{2}(N\delta + D) \tag{3.3}$$

and

$$\#\text{2-simplices in } K_1 = \frac{1}{3}(N\delta + D - k) \tag{3.4}$$

where $N = n - |B|$ is the number of inner vertices of K_1. In proving (3.4) we also used (3.2). Now we may apply the Euler's formula to get:

$$\chi(\Sigma_0') = n - \frac{1}{2}(N\delta + D) + \frac{1}{3}(N\delta + D - k) + r$$

where $\chi(\Sigma_0')$ is the Euler characteristic of Σ_0'. After a short calculation this reduces to

$$6(\chi(\Sigma_0') - r) = (6 - \delta)N + 6|B| - D - 2k \,. \tag{3.5}$$

Since $6|B| - D \geq 2k$, (3.5) implies

$$(\chi(\Sigma_0') - r) \geq \frac{6-\delta}{6} N \,. \tag{3.6}$$

Since $r \geq 1$ and N is arbitrarily large, it follows that $\delta = 6$ and either $\chi(\Sigma_0') = 2$ with $r = 1$, or 2, or $\chi(\Sigma_0') = 1$ with $r = 1$. In both cases $\Sigma_0 = \Sigma_0'$ by our choice of Σ_0'. If K has more than r ends we might have taken K_1 with more boundary components. Hence we may conclude that all of the admissible possibilities are our exceptional cases (a) and (b), and we are done.

References

1. B. Kerekjarto, *Vorlesungen über Topologie*, Springer Verlag, Berlin, 1923.
2. S. Lavrenchenko, *Infinite regular toroidal triangulations*, preprint,1989.
3. B. Mohar, *Embeddings of infinite graphs*, J. Combin. Theory, Ser. B 44 (1988) 29–43.
4. I. Richards, *On the classification of non-compact surfaces*, Trans.Amer. Math. Soc. 106 (1963) 259–269.