Almost Selfcomplementary Graphs I

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Abstract. The concept of selfcomplementary (s.c.) graphs is extended to almost selfcomplementary graphs. We define an n-vertex graph to be almost selfcomplentary (a.s.c) if it is isomorphic to its complement with respect to $K_n - e$, the complete graph with one edge removed. A.s.c. graphs with n vertices exists if and only if $n \equiv 2$ or $3 \pmod 4$, i.e., precisely when s.c. graphs do not exist. We investigate various properties of a.s.c. graphs.

1. Introduction

We consider only finite, simple and undirected graphs. For a graph G, we denote by $V(G)$, $E(G)$ and $\overline{G}$, the vertex set, the edge set and the complement of G, respectively.

A simple graph G is selfcomplementary (s.c.) if it is isomorphic with its complement $\overline{G}$ (cf., e.g., [2,3,4]). For a s.c. graph with n vertices to exist, the number of edges in the complete graph K_n must be even, and thus an s.c. graph G with n vertices has necessarily $n \equiv 0$ or $1 \pmod 4$. It is natural to attempt to modify the complete graph K_n slightly to remove the "trivial obstacle" for the existence of s.c. graphs with n vertices when $n \equiv 2$ or $3 \pmod 4$, and to obtain s.c. like graphs for these orders.

In this paper we consider one such possible modification. We delete from K_n one edge, which we always denote by e, and consider, for a graph G, its complement $\widetilde{G}$ with respect to $\widetilde{K}_n = K_n - e$. This leads to the following definition.

Definition. A simple graph G with n vertices is *almost selfcomplementary* (a.s.c.) if it is isomorphic with its complement $\widetilde{G}$ with respect to the graph $\widetilde{K}_n = K_n - e$, the complete graph from which one edge e has been deleted. The edge e is called the *missing edge*.

We get immediately

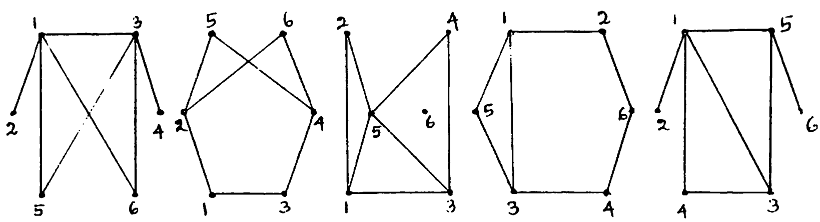
Theorem 1.1. *An almost selfcomplementary graph with n vertices exists if and only if $n \equiv 2$ or $3 \pmod 4$.*

Proof: Necessity is obvious. For sufficiency, first suppose $n \equiv 2 \pmod 4$, i.e., n is even. Take a s.c. graph G' with $n - 1$ vertices. By properties of s.c. graph with odd number of vertices, such a G' always exists and there is a permutation of $V(G')$ taking G' to $\overline{G'}$ which fixes exactly one vertex, say v, of G'. Now taking a vertex, say x, not in $V(G')$ and joining x to all those vertices of G' already joined to v, the resulting graph with vertex set $V(G') \cup \{x\}$ is an a.s.c. graph with n vertices where (x, v) is the missing edge. Next, for $n \equiv 3 \pmod 4$ take a s.c. graph G_1 with $n-2$ vertices which again has a vertex, say u, fixed by a permutation of $V(G_1)$ taking G_1 to $\overline{G}_1$. Then take two vertices, say x and y, not in $V(G_1)$. Join both x and y to the vertices of G_1 already joined to u and also join u to one of x and y. The graph obtained on the vertex set $V(G_1) \cup \{x, y\}$ is an a.s.c. graph with n vertices, where (x, y) is the missing edge.

It may be remarked that not all a.s.c. graphs with given number of vertices can be constructed by the method discussed in the above proof.

2. Complementing permutation and the cycle structure

Let G be an a.s.c. graph with n vertices and missing edge e. Then, just like for s.c. graphs, an isomorphism between G and $\widetilde{G}$ is given by a permutation τ : $V(G) \rightarrow V(G)$, called a *complementing permutation* (c.p.) of G. However (unlike in the case of s.c. graphs), in the case of a.s.c. graphs there exists two kinds of c.p.'s, depending on whether or not the missing edge e is fixed by the c.p. If $\tau(e) = e$ then τ is a *strong c.p.*, otherwise τ is a *weak c.p.* As a permutation, τ can be written as a product of disjoint cycles. Note that a given a.s.c. graph G may admit more than one c.p., while, on the other hand, nonisomorphic a.s.c. graphs may have the same c.p. (this is precisely what happens to s.c. graphs as well). Fig. 1 below shows all a.s.c. graphs with 6 vertices together with their (weak or strong) c.p's.



(a) (b) (c) (d) (e)

(a),(b)&(c) are all with strong c.p. (1234)(56) or (1432)(56) and same missing edge (5,6)

(d) is with strong c.p. (1234)(5)(6) or (1432)(5)(6) and missing edge (5,6)

(e) is weak c.p. (123456) or (163452) and missing edge (3,6) or (2,5) respectively.

Fig. 1

The following elementary observations regarding the cycle structure of a complementing permutation τ of an a.s.c. graph G are parallel to those for s.c. graphs, and their proofs are left to the reader.

Lemma 2.1. *τ has no cycle of odd length > 3. Also if τ has a cyclw of length 3 then τ is a weak c.p. and has no other cycle of odd length.*

(A c.p. τ of a s.c. graph has no odd cycle of length >1)

This is due to the fact that there exists an a.s.c. graph G with 3 vertices and a weak c.p. consisting of a single cycle, where $E(G)$ contains exactly one edge. Moreover, this (disconected) a.s.c. graph with three vertices can always be taken as an induced a.s.c. graph of at least one a.s.c. graph with $4k + 3(k \geq 1)$ vertices such that deletion of these three vertices results in a s.c. graph.

Lemma 2.2. *τ fixes at most two vertices of G. If τ fixes two vertices u, v then $e = (u, v)$ is the missing edge, and τ is a strong c.p. of even degree.*

(A c.p. of a s.c. graph fixes at most one vertex.)

Lemma 2.3. *τ has at most one cycle of length $\ell > 1$ such that $\ell \equiv 2 \pmod 4$. If τ has a cycle of length $\ell \equiv 2 \pmod 4 > 2$ then τ fixes at most one vertex of the corresponding a.s.c. graph, and τ is in this case a weak c.p.*

(For a s.c. graph, we have $\ell \equiv 0 \pmod 4$ for every cycle of length > 1 of a c.p.).

Lemma 2.4. *The order of an a.s.c. graph with a weak c.p. may be odd or even whereas that of an a.s.c. graph with a stong c.p. containing two cycles of length 1 is always even.*

Remark. An a.s.c. graph with more than 3 vertices is disconnected if and only if it has exactly two components of which one is pancyclic and the other is an isolated vertex. Also the associated c.p. is a strong c.p. containing a unique cycle of length 2. (A proof is given after Lemma 3.4)

The a.s.c. graphs with two or three vertices are always disconnected and we call these trivial a.s.c. graphs.

Henceforth an a.s.c. graph in our discussion will always mean a connected a.s.c. graph.

Lemma 2.5. *If τ is a weak c.p. then τ^2 is not necessarily an automorphism.*

This is due to the presence of a unique cycle of length $\ell \equiv 2 \pmod 4 > 2$ where the image under τ^2 of the missing edge is not itself. But τ^2 is an automorphism if τ is a strong c.p.

(For a c.p. τ of a s.c. graph, τ^2 is always an automorphism).

3. Construction Method

A simple extension of the construction algorithm for s.c. graphs by Gibbs [2] provides a method of constructing all a.s.c. graphs with a given (strong/weak) c.p. First suppose τ is a weak c.p. with no odd cycle of length > 1, whose elements are the numbers $1, 2, \ldots, n$. Order the cycles (of length > 1) of τ in nondecreasing order of their lengths with the unique cycle of length 1 (if $n = 4k+3$) at the end. If $\tau_1 = (1\ 2 \ldots 4k_1)$ is the first cycle in this ordering then denote by S the set of all numbers $2, 3, \ldots, 2k_1 + 1$; the first $4k_1$ numbers from each subsequent cycle and n (if $n = 4k+3$). If $\tau_1 = (1\ 2 \ldots 4k_1 + 2)$, $k_1 \leq k$, is the first cycle then S consists of numbers $2, 3, \ldots, 2k_1 + 1, 2k_1 + 2$; the first $4k_1 + 2$ numbers from each subsequent cycle and n (if $n = 4k+3$). Now to construct an a.s.c. graph whose vertices are labeled $1, 2, \ldots, n$, decide arbitrarily the unordered pair $(1, j)$ for every $j \in S$ to be an edge or a nonedge in G. Then the same will be true for $(\tau^{2i}(1), \tau^{2i}(j))$ with

$$\begin{aligned} i &= 1, 2, \ldots, 2k_{i_j} \text{ if } j \text{ is in a cycle of length } 4k_{i_j} \\ &= 1, 2, \ldots, 2k_{i_j} + 1 \text{ if } j \text{ is in the cycle of length } 4k_{i_j} + 2,\ k_{i_j} \neq k_1 \\ &= 1, 2, \ldots, 2k_1 + 1 \text{ if } j \neq 2k_1 + 2 \text{ is in the cycle of length } 4k_1 + 2 \\ &= 1, 2, \ldots, k_1 - 1 \text{ if } j = 2k_1 + 2 \text{ is in the cycle of length } 4k_1 + 2 \end{aligned}$$

and i varies from 1 to $2k_1$ or $2k_1 + 1$ according as the first cycle is of length $4k_1$ or $4k_1 + 2$ and $j = n (= 4k+3)$.

This gives all edges of $G(\tau_1)$ and the edges joining the vertices in $G(\tau_1)$ with those in $G(\tau \backslash \tau_1)$. Then delete $G(\tau_1)$ and repeat the process for the c.p. $\tau \backslash \tau_1$ and continue till all cycles of τ are exhausted. Since n is finite, the process will terminate after finitely many steps. Note that if the ℓth cycle is $(x_1 x_2 \ldots x_{4k_\ell + 2})$ then $(x_{2k_\ell+1}, x_{4k_\ell+2})$ is the missing edge provided $(x_1, x_{2k_\ell+2}) \in E(G)$ otherwise $(x_1, x_{2k_\ell+2})$ is the missing edge.

Suppose τ is a weak c.p. whose elements are the numbers $1, 2, 3, \ldots, n (= 4k+3)$ and τ contains a cycle of length 3, say $\tau' = (n-2, n-1, n)$. By lemma 2.1, τ in this case does not have any other cycle of odd length. First construct a s.c. graph G' with $n-3$ vertices and a c.p. $\tau \backslash \tau'$ by the same procedure as above. Then join the vertices labeled $n-2$ and $n-1$ by an edge, and also join each vertex of τ' to every vertex in a complementary half of the vertices in G'. This results in an a.s.c. graph, say G, with n vertices, a weak c.p. τ and missing edge $(n-2, n)$.

Next Suppose τ is a strong c.p. with numbers $1, 2, \ldots, n$. Then, besides the cycles of length divisible by 4, τ contains either

(i) two cycles of length 1 (if $n = 4k+2$) or
(ii) one cycle of length 2 (if $n = 4k+2$) or
(iii) one cycle of length 2 and one of length 1 (if $n = 4k+3$).

In this case the construction of an a.s.c. graph G with n vertices is carried on by treating the cycle(s) of length 1 and length > 2 in the same way as above. For the cycle of length 2, one of the two vertices is treated like a cycle of length 1 and the other vertex is joined to the same vertices of at least one cycle of length > 2

which are already joined to the first one and for other cycles of length > 2 one only needs to maintain the complementarity (cf. remark on page 3). If there is a cycle of length 1 in addition to the cycle of length 2 then exactly one of the latter is joined to the vertex of the former. Here the missing edge is the edge joining the vertices in a unique 2-cycle or the two 1-cycles of the given strong c.p.

Now in both cases above it can be easily checked that the resulting graph G is, in fact, an a.s.c. graph. For illustration of the above construction method take a weak c.p. $\tau = \tau_1\tau_2$, where $\tau_1 = (1\ 2\ 3\ 4)$ & $\tau_2 = (5\ 6\ 7\ 8\ 9\ 10)$. Then $S = \{2,3,5,6,7,8\}$. Take the unordered pair $(1,j)$ as an edge, for each $j \in S$. By the construction method (1,2), (3,4), (1,3), (1,5), (1,6), (1,7), (1,8), (3,7), (1,9), (3,5), (3,9), (3,8), (1,10), (3,6), (3,10) are edges in $G(\tau_1)$ and edges joining vertices in $G(\tau_1)$ with those in $G(\tau_2)$. In the second step of the construction we take case of $G(\tau_2)$ by taking the "new" $S = \{6,7,8\}$ with $(5,j) \in E(G(\tau_2))$ for each $j \in S$. So the edges in $G(\tau_2)$ are (5,6), (7,8), (9,10), (5,7), (7,9), (9,5) and (5,8). The resulting graph is given in Fig. 2. Here the missing edge is (7,10).

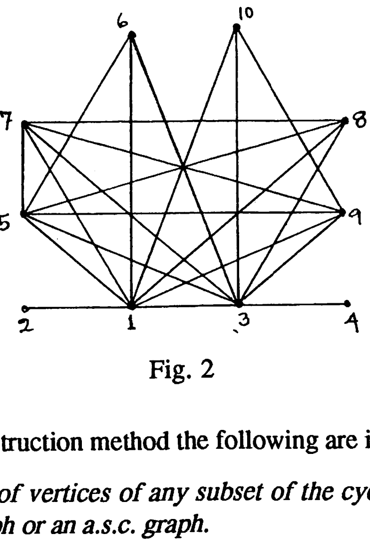


Fig. 2

From the above construction method the following are immediate.

Lemma 3.1. *The set of vertices of any subset of the cycles of a (strong/weak) c.p. is either a s.c. graph or an a.s.c. graph.*

(For a s.c. graph the induced subgraph on any subset of the cycles of a corresponding c.p. is always a s.c. graph).

Lemma 3.2. *The vertices in any cycle of length > 3 in a (weak/strong) c.p. alternate in degree and the sum of the degrees of the complementary vertices being $n-1$ except exactly one pair in the cycle of length $\ell \equiv 2 \pmod 4$ which have degree sum $n-2$.*

(In case of a c.p. of s.c. graph this holds without exception).

Lemma 3.3. *The adjacencies among the vertices of two cycles of lengths* $\ell \equiv 0 \pmod 4$ *and* $\ell' \equiv 2 \pmod 4$ *have exactly four possibilities, i.e., all the vertices of either cycle are adjacent to a fixed half of the other cycle.*

Now we prove a lemma and then complete the proof of the remark following Lemma 2.4.

Lemma 3.4. *Every a.s.c. graph with more than three " vertices and a weak c.p. consisting of a single cycle is connected.*

Proof: Suppose G is an a.s.c. graph with $n > 2$ vertices and a weak c.p. $\tau = (1\ 2 \ldots n)$, where $V(G) = \{1, 2, 3, \ldots, n\}$. Clearly n is even and τ partitions $V(G)$ into two disjoint parts, say, $V_1 = \{x, \tau^2(x), \ldots\}$ and $V_2 = \{\tau(x), \tau^3(x), \ldots\}$, for some $x \in V(G)$. Also one of $G[V_1]$ and $G[V_2]$ is complete while the other is totally disconnected and $G[V_1, V_2]$ contains at least one edge. Without loss of generality, take $G[V_1] = K_{n/2}$, $G[V_2] = \overline{K}_{n/2}$ and $(u, v) \in E(G)$ for some $u \in V_1$ and $v \in V_2$. Then $(\tau^{2r}(u),\ \tau^{2r}(v)) \in E(G)$ for every positive integer r. This implies that every vertex of V_2 is joined in G to at least one vertex in V_1. Thus G is connected.

Proof of the Remark: Sufficiency is immediate. For necessity, suppose G is a disconnected a.s.c graph. By above Lemma, any c.p. of G has at least two cycles as every strong c.p. of an a.s.c. graph contains at least two cycles. Let G' be the maximal s.c. subgraph of G (using Lemma 3.1). Then the c.p. of G restricted to $G \backslash G'$ has exactly one cycle and is of length $4k' + 2$ (k' being a nonnegative integer). If $k' \neq 0$ then $G \backslash G'$ is a connected a.s.c. graph. Also $E(G[V(G'), V(G \backslash G')]) \neq \emptyset$. This means that G is connected—a contradiction. So $k' = 0$, i.e., $G \backslash G'$ has exactly two vertices. Again G being disconnected, only one of the two vertices of $G \backslash G'$ is joined to all the vertices of G' while the other vertex of $G \backslash G'$ is an isolated vertex of G.

Now the fact that every s.c. graph, and hence G', has a hamilton path, the component of G containing G', by above observation, is clearly pancyclic.

Theorem 3.1. *Suppose* $\tau(G) = \widetilde{G}$ *and* $\tau = (1\ 2\ 3 \ldots n)$ *is a weak c.p., where* $n = 4k + 2$. *Then (a) there is a set of exactly* k *consecutive odd (even) labeled vertices each of which is adjacent to exactly* $k + 1$ *even (odd) labeled vertices and the other set of consecutive* $k + 1$ *odd (even) labeled vertices are each adjacent to exactly* k *even (odd) labeled vertices. (b)* G *has vertices of four degrees: for some* r, $k \leq r \leq 2k$, *there are* $k + 1$ *vertices of degree* r; k *vertices of degree* $r + 1$; $k + 1$ *vertices of degree* $4k - r$ *and* k *vertices of degree* $4k + 1 - r$.

Proof: (a) Suppose $(1, 2i) \in E(G)$ for $i \leq k$. Then $\tau^{4k+3-2i}(1, 2i) = (1, 4k + 4 - 2i) \notin E(G)$ for all $i \leq k$, as the image of an edge under every odd power of τ is a nonedge. This implies that the vertex labeled 1 is adjacent to at least k

consecutive even labeled vertices. So every odd labeled vertex is adjacent to at least k consecutive even labeled vertices. Now we have two cases:

Case(i). $(1, 2k+2) \in E(G)$. Then $\tau^{2j}(1, 2k+2) \notin E(G)$ for all $j \geq k$ due to our construction of G, i.e., $2k+1$ is the first odd labeled vertex which is not adjacent to the same number of even labeled vertices as the vertex 1. So each of the first k odd labeled vertices $1, 3, 5, \ldots, 2k-1$ is adjacent to exactly $k+1$ even labeled vertices and each of the rest $k+1$ odd labeled vertices is adjacent to exactly k even labeled vertices.

Case(ii). $(1, 2k+2) \notin E(G)$. Then $(2, 2k+4) \in E(G)$ and so, as above, $\tau^{2j}(2, 2k+3) \notin E(G)$ for all $j \geq k$. That is, each of the first $k+1$ odd labeled vertices $1, 3, \ldots, 2k+1$ is adjacent to exactly k even labeled vertices and each of the rest k odd labeled vertices is adjacent to exactly $k+1$ even labeled vertices.

(b) From (a) we have each odd (even) labeled vertex in G is adjacent to k or $k+1$ even (odd) labeled vertices. Now for the adjacencies among the vertices of the same parity, consider an edge $(i, i+2)$ in G with i odd or even. The images of $(i, i+2)$ under different even powers of τ contribute degree two to each odd or even vertex according as i is odd or even such that the sum of the degrees of an odd vertex and an even vertex due to these adjacencies is $2k$. Thus an odd (even) vertex is adjacent to either none, two, four,..., $2\{k/2\}$ other odd (even) vertices in G. Then the proof follows by taking $r = k + j$, where $j \in \{0, 2, 4, \ldots, 2\{k/2\}\}$.

Remark. If $r = 2k$ in the proof of (b) then G has vertices of only two degrees: $2k+2$ vertices of degree $2k$ each and $2k$ vertices of degree $2k+1$ each. However, this is possible only when k is even.

For the a.s.c. graph G and weak c.p. τ considered in Theorem 3.1, an edge or nonedge $(1, j)$ contribute $2k+1$ edges or nonedges for G through even powers of τ for $j \leq 2k+1$ whereas the edge or nonedge $(1, 2k+2)$ contribute only k edges or nonedges. We call the first category as full orbits and the second as the half orbit of τ. It may be noted that there are only full orbits for any cycle of length $\ell \equiv 0 \pmod 4$ in a c.p. Now we have

Corollary 3.1.1. *Suppose G and τ are as in Theorem 3.1. Further, if $d_1 \geq d_2 \geq \cdots \geq d_{4k+2}$ is the degree sequence of G then the ends of the missing edge have degrees*

(i) *d_{k+1} and d_{3k+2}, provided G has vertices of four degrees, or*
(ii) *d_{2k+1} and d_{2k+2}, provided G has vertices of two degrees.*

Proof: Consider the spanning subgraph G' of G containing only the edges generated by the full orbits of τ. Then G' is quasi-regular or regular according as G has vertices of four or two different degrees. In the first case the vertex set $V(G')$ is partitioned into two classes as odd and even labeled vertices with degree difference of at least two between any two vertices of different parity. But any edge due to the half orbit always joins two vertices of opposite parity and so exactly k vertices of each of the two classes of vertices of G' will have an additional degree in G while the rest have the same degree as in G'. With this the proofs in both cases follow immediately.

Corollary 3.1.2. *Suppose G and τ are as in Theorem 3.1. Then the ends of the missing edge are adjacent to exactly k vertices in common.*

However, a result regarding common adjacency of the ends of the missing edge in case of a stong c.p. is totally different. For completeness we state it below as a lemma.

Lemma 3.4. *If G is an a.s.c. graph with $n(= 4k + 2$ or $4k + 3)$ vertices and a strong c.p. then the ends of the missing edge are adjacent to some $j\,(0 \leq j \leq 2k)$ common vertices.*

Gibbs (Theorem 4 [2]) has proved a decomposition theorem for s.c. graphs in terms of the smallest nontrivial induced s.c. subgraphs. It can be readily checked that an a.s.c. graph with $4k + 2(k > 0)$ vertices and a weak c.p. consisting of a single cycle possesses a collection of k disjoint induced four-vertex s.c. subgraphs. Again withdrawal of the vertices of the cycle(s) of length $\not\equiv 0 \pmod 4$ from any a.s.c. graph results in a s.c. graph. So we state the following without a proof.

Theorem 3.2. *If G is an a.s.c. graph with $4k+2$ vertices and a (weak/strong) c.p. then G possesses a collection of k disjoint induced four-vertex s.c. subgraphs.*

4. Almost regular and quasi regular a.s.c. graphs

The remark at the end of Theorem 3.1 guarantees the existence of an a.s.c. graph with $4k + 2$ vertices and a weak c.p. $\tau = (1\ 2 \ldots 4k + 2)$ which has vertices of only two degrees provided k is even. However, the restriction of k being even is not necessary in case of a strong c.p. Such a.s.c. graphs with vertices of two degrees only are called *quasi regular a.s.c. graphs*. By Lemma 3.2, every a.s.c. graph with $n\,(= 4k + 2)$ vertices has exactly one pair of complementary vertices with degree sum $n - 2$ whereas this sum for all other complementary pairs is $n - 1$. If such a G is quasi regular then take a new vertex, say x, and let it be fixed by a corresponding (weak/strong) c.p. of G. Now joining x to precisely all the vertices in a complementary half of the vertices of G of lower degree we obtain an a.s.c. graph with $4k + 3$ vertices of which $4k + 2$ vertices have degree $2k + 1$ each and one has degree $2k$. Such a graph is called an *almost regular* a.s.c. graph. It may be noted that such a graph can be obtained through a different construction. For example, take a s.c. graph with 8 vertices and and a degree sequence $(5,5,5,5,2,2,2,2)$, and an a.s.c. graph with 3 vertices and exactly one edge. Then joining every vertex of the latter to all the vertices of minimum degree of the former, an almost regular a.s.c. graph with 11 vertices is obtained in which 10 vertices have degree 5 each and one has degree 4. The following are some results on the existence of such graphs.

Lemma 4.1. *There exists no regular a.s.c. graph.*

(For every positive integer k, there exists a regular s.c. graph with $4k+1$ vertices).

Lemma 4.2. *If k is even then there exists a quasi regular a.s.c. graph with $4k+2$ vertices and a weak c.p. $\tau = (1\ 2 \ldots 4k+2)$.*

Lemma 4.3. *There exists a quasi regular a.s.c. graph with $4k+2$ vertices and an almost regular a.s.c. graph with $4k+3$ vertices for every positive integer k and a strong c.p.*

So, in general, we have

Theorem 4.1. *For every positive integer k, there exists at least one quasi regular a.s.c. graph with $4k+2$ vertices and at least one almost regular a.s.c. graph with $4k+3$ vertices.*

Proof: Consider a quasi regular s.c. graph G' with $4k$ vertices (which always exists). Then G' will have $2k$ vertices of degree $2k$ each and the rest $2k$ vertices of degree $2k-1$ each. Also the vertices of the two kinds are complementary of one another. Take two new vertices x and y, and join both to either all vertices of degree $2k$ only or $2k-1$ only of G'. The resulting graph is a quasi regular a.s.c. graph. Next, consider a regular s.c. graph G'' with $4k+1$ vertices. Then G'' has a vertex x fixed by some c.p. σ of G''. Take two new vertices u and v not in G'' and join u to some $2k$ vertices and v to the other $2k+1$ vertices of G'' such that the self complementarity is preserved, i.e., $(u,y) \in E(G) \iff (v,\sigma(y)) \in E(G)$, $x \neq y \in V(G'')$ and $(v,x) \in E(G)$, where G is the new graph. Notice that all the vertices except u of the new graph G with $V(G) = V(G'') \cup \{u,v\}$ are of degree $2k+1$ and u is of degree $2k$. Hence the graph G is an almost regular a.s.c. graph.

Corollary 4.1.1. *For every almost regular a.s.c. graph with $4k+3$ vertices there corresponds a quasi regular a.s.c. graph with $4k+2$ vertices but not conversely.*

A trivial result may be noted that there is no quasi-regular a.s.c. graph with an odd number of vertices and no almost regular a.s.c. graph with an even number of vertices.

5. Diameter

For any graph G, the diameter of G, denoted by $\text{diam}(G)$, is the maximum of the distances between pairs of vertices of G. The class of s.c. graphs and that of a.s.c. graphs share the same result regarding their diameters. Now we prove it for a.s.c. graphs.

Theorem 5.1. *Every nontrivial connected a.s.c. graph has diameter 2 or 3.*

Proof: (By contradiction) Suppose G is a nontrivial a.s.c. graph. Clearly diam $(G) \neq 1$, since G is not complete. Take diam$(G) \geq 4$. Then by definition of a.s.c. graph and diameter being a graphical invariant, diam$(\tilde{G}) \geq 4$. Let u and v be any pair of vertices in $\tilde{G}$ such that dist $(u, v) = 4$, i.e. there is a shortest path $uxyzv$ in $\tilde{G}$ joining u and v. Then none of the edges $(u, y), (u, z), (x, z), (x, v)$, (y, v) is in $\tilde{G}$. Note that at most one of these edges may be the missing edge e, where $\overline{G} = \tilde{G} + e$. In any case $\text{dist}_{\overline{G}}(u, v) > 2$, which is a contradiction because diam$(G) \geq 4$ implies diam $(\overline{G}) \leq 2$ [1]. Hence diam(G) is 2 or 3.

Now the following follows from [3] (pages 357–58).

Theorem 5.2. *There is at least one a.s.c. graph with n vertices and diameter 2 and at least another with diameter 3 for every admissible integer $n \geq 6$.*

In the passing it may be remarked that the general result of Gibbs (Theorem 6 [2]) regarding the $(0, 1, -1)$-adjacency matrices of s.c. garphs also holds for a.s.c. graphs.

6. Conclusion

Some other results concerning paths and cycles in an a.s.c. graph will be discussed in part II under the same title. From the construction and properties of a.s.c. graphs studied so far we have a feeling that most of the results of s.c. graphs will also hold for a.s.c. graphs with minor modifications. Besides the a.s.c. graphs defined and constructed here, there are other possibilities of defining s.c. like graphs, for example, one after deleting a 1-factor or a near 1-factor from K_n, for some suitable n. We propose to continue the work by defining such other graphs and studying their properties along with a few more properties of a.s.c. graphs.

References

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