On the Packing of Pairs by Quintuples With Index 2

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Abstract. A packing design (briefly packing) of order v, block size k, and index λ is a pair $(X, \mathcal{D})$ where X is a v-set (of points) and $\mathcal{D}$ is a collection of k-subsets of X (called blocks) with cardinality b such that every 2-subset of X is contained in at most λ blocks of $\mathcal{D}$. We denote it by $\mathrm{SD}(k, \lambda; v, b)$. If no other such packing has more blocks, the packing is said to be maximum, and the number of blocks in $\mathcal{D}$ is the packing number $\mathrm{D}(k, \lambda; v)$. For fixed k, λ and v, the packing problem is to determine the packing number. In this paper, the values of $\mathrm{D}(5, 2; v)$ are determined for all $v \geq 5$ except 48 values of v.

1. Introduction.

A packing design (briefly packing) of order v, block size k, and index λ is a pair $(X, \mathcal{D})$ where X is a v-set (of points) and $\mathcal{D}$ is a collection of k-subsets of X (called blocks) with cardinality b such that every 2-subset of X is contained in at most λ blocks of $\mathcal{D}$. We denote it by $\mathrm{SD}(k, \lambda; v, b)$. If no other such packing has more blocks, the packing is said to be maximum, and the number of blocks in $\mathcal{D}$ is the packing number $\mathrm{D}(k, \lambda; v)$. For fixed k, λ and v, the packing problem is to determine the packing number.

The following is a special case of the Johnson bound (see [6] or [10])

$$\mathrm{D}(k, \lambda; v) \leq \big[v/k[\lambda(v-1)/k-1]\big] = \psi(k, \lambda; v) \tag{1.1}$$

where $[x]$ is the largest integer satisfying $[x] \leq x$.

The values of $\mathrm{D}(3, \lambda; v)$ are determined in [4,10]; $\mathrm{D}(4, 1; v)$ are determined in [3]. Recently, A. M. Assaf [1] and Alan Hartman [5] have studied the packing number $\mathrm{D}(4, \lambda; v)$ and determined them all. In this paper we are concerned with the number $\mathrm{D}(5, 2; v)$. We shall prove the following.

Theorem 1.1. *The equation* (1.2) *holds for each* $v \geq 5$ *except for the values of* v *shown in Table 1.*

$$\mathrm{D}(5, 2; v) = \begin{cases} \psi(5, 2; v) - 1 & \text{if } v \equiv 7 \text{ or } 9 \pmod{10} \\ \psi(5, 2; v) & \text{otherwise} \end{cases} \tag{1.2}$$

Table 1

13	15	16	17	18	19	26	27	28	29	33	36	38	39
44	48	66	67	69	73	76	77	78	79	80	84	86	88
93	97	99	167	169	197	199	226	228	236	238	276	278	288
326	338	339	438	439	526								

2. Preliminaries.

In this section, we shall define some terminology and state some fundamental results which will be used later. First, we define a useful generalization of an almost packing design (see, for example, [4]), called a maximum almost packing design. Let v, w be positive integers. A maximum almost packing design $MSD(k, \lambda; v(w), b)$ is a triple $(X, Y, \mathcal{D})$, where X is a v-set (of points), Y is a w-subset of X and $\mathcal{D}$ is a collection of b k-subsets of X (called blocks) which has the following properties:

(1) $b = \psi(k, \lambda; v) - \psi(k, \lambda; w)$;
(2) for any $B \in \mathcal{D}$, $|B \cap Y| \leq 1$; and
(3) each pair of distinct points x and y from X where at least one of x and y does not lie in Y, occurs in at most λ blocks of $\mathcal{D}$.

We adopt the convention that $\psi(k, \lambda; w) = 0$ when w is a nonnegative integer less than k, and we admit $Y = \phi$. The set Y is referred to as the hole of the design. Clearly a $MSD(k, \lambda; v(w), b)$ is a SD$(k, \lambda; v, \psi(k, \lambda; v))$ whenever $w=0$ or 1.

The next two lemmas are fairly obvious.

Lemma 2.1. *If there exists a $MSD(k, \lambda; v(w), b)$ and $D(k, \lambda; w) = \psi(k, \lambda; w)$, then $D(k, \lambda; v) = \psi(k, \lambda; v)$.*

Lemma 2.2. *If there are both $MSD(k, \lambda; v(w), b)$ and $MSD(k, \lambda; w(u), b')$, then there is a $MSD(k, \lambda; v(u), b + b')$.*

We assume that the reader is familiar with Wilson's "Fundamental Construction". For the details the reader is referred to [13]. For the definition of pairwise balanced design (PBD) and balanced incomplete block design (BIBD), group divisible design (GDD), transversal design (TD), resolvable BIBD (RBIBD) and resolvable GDD (RGDD), see [2,4,13]. Sometimes we shall use the "exponential" notation to indicate the group-type (or type) of a GDD. If $e \notin K$, then $(v, K \cup \{e^*\}, 1)$-PBD denotes a $(v, K \cup \{e\}, 1)$-PBD which contains a unique block of size e and if $e \in K$, then $(v, K \cup \{e^*\}, 1)$-PBD is a $(v, K, 1)$-PBD containing at least one block of size e. We need the following lemmas, the first two are taken from [4] and the last two are taken from [2,9,11,12,14] respectively.

Lemma 2.3. *A $(v, 5, 1)$-BIBD exists for any integer $v \geq 5$ satisfying $v \equiv 1$ or*

5 (mod 20) *and a* $(v, 5, 2)$*-BIBD exists for any integer* $v \geq 5$ *satisfying* $v \equiv 1$ *or* 5 (mod 10) *and* $v \neq 15$.

Lemma 2.4. *There is a* $GD(5, 2, 2; 12)$. *There is a* $GD(7, 2, r; 7r)$ *for any positive integer* r.

Lemma 2.5. *There exists a* $TD(5, n)$ *if* $n \geq 4$, $n \neq 6, 10$. *There exists a* $TD(6, n)$ *if* $n \geq 5, n \neq 6, 10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44$, *or* 52.

Lemma 2.6. *There is a* $(v, \{5, 9^*\}, 1)$*-PBD if* $v \equiv 9$ *or* 17 (mod 20) *and* $v \notin E$. *Where* $E = \{17, 29, 49, 57, 69, 77, 97, 117, 129, 137, 157, 169, 197, 277, 397, 449, 497, 557, 577, 637, 717, 749, 777, 797, 897\}$.

Again, some known results on RBIBD, RGDD and $(v, 6, 1)$-BIBD will be used, our authority is [**8,15,16**] unless another reference is given. We shall also involve the notion of IPBD and IGDD, which we describe below.

An incomplete PBD (IPBD) of index λ, denoted by $(v, w; K, \lambda)$-IPBD is a triple $(X, Y, \mathcal{A})$ where X is a v-set (of points), $Y \subseteq X$, and $\mathcal{A}$ is a collection of subsets of X (called blocks) with sizes from K which satisfies the following properties:

(1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$, and
(2) each pair of distinct points x and y from X where at least one of x and y does not lie in Y, occurs in exactly λ blocks.

Hence, Y is the hole. If $K = \{k\}$, we briefly write $(v, w; k, \lambda)$-IPBD for $(v, w; \{k\}, \lambda)$-IPBD. Note that $(X, Y, \mathcal{A})$ is an $(v, w; k, 1)$-IPBD if and only if $(X, \mathcal{A} \cup \{Y\})$ is a $(v, \{k, w^*\}, 1)$-PBD. Moreover, a $(v, w; k, \lambda)$-IPBD can be produced from such a $(v, w; k, 1)$-IPBD by taking λ copies of each block in $\mathcal{A}$. And then as an immediate consequence of Lemma 2.3, 2.6, we have the following.

Lemma 2.7. *Let* λ *be a positive integer. Then*

(1) *there is a* $(v, 5; 5, \lambda)$*-IPBD if* $v \equiv 1$ *or* 5 (mod 20) *and* $v \geq 5$;
(2) *there is a* $(v, 9; 5, \lambda)$*-IPBD if* $v \equiv 9$ *or* 17 (mod 20) *and* $v \notin E$ *which is the same as in Lemma 2.6.*

Now we define the concept of IGDD. An incomplete GDD (IGDD) of index λ is a quadruple $(X, Y, \mathcal{G}, \mathcal{A})$ which satisfies the following properties:

(1) X is a set of points and $Y \subseteq X$ (called a hole),
(2) $\mathcal{G}$ is a partition of X into groups,
(3) $\mathcal{A}$ is a set of blocks, each of which intersect each group in at most one point,
(4) no block contains two members of Y, and
(5) every pair of points $\{x, y\}$ from distinct groups such that at least one of x, y is in $X \backslash Y$, occurs in exactly λ blocks of $\mathcal{A}$.

We say that an IGDD $(X, Y, \mathcal{G}, \mathcal{A})$ is a (K, λ)-IGDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The type of the IGDD is defined to be the multiset of ordered pairs $\{(|G|, |G \cap Y|) : G \in \mathcal{G}\}$. As with GDDs, we shall make use of an "exponential" notation to describe type. By (k, λ)-IGDD we mean a $(\{k\}, \lambda)$-IGDD.

For the convenience of notation, in what follows we respectively write $\psi(v)$, $MSD(5,2; v(w))$ for $\psi(5,2;v)$, $MSD(5,2;v(w),b)$ and define

$$\begin{aligned} B(k) &= \{v : \text{ a } (v,k,1)\text{-BIBD exists}\}, \\ RB(k) &= \{v : \text{ a } (v,k,1)\text{-RBIBD exists}\}, \\ IP_\lambda(w) &= \{v : \text{ a } (v,w;5,\lambda)\text{-IPBD exists}\}, \\ MSD(w) &= \{v : \text{ a } MSD(5,2;v(w)) \text{ exists}\}, \\ M &= \{v : \text{ a } SD(5,2;v,\psi(v)) \text{ exists or } v = 0,1,2,3,4\}. \end{aligned}$$

As mentioned earlier, we know that $v \in M$ is equivalent to $v \in MSD(0) \cap MSD(1)$ whenever $v \geq 5$.

3. Direct Constructions.

In order to establish our results, we shall employ both direct and recursive constructions. In this section we shall construct directly some designs for 16 values of v which form the base of the general solution of the packing under consideration.

Lemma 3.1. *There exists a SD(5,2;v, $\psi(v) - 1$) if $v = 7$ or 9.*

Proof: For $v = 7$, let $X = Z_7$. Then the blocks

$$\{0,1,2,4,6\} \quad \{0,2,3,4,5\} \quad \{0,1,3,5,6\}$$

form the required packing design.

For $v = 9$, let $X = Z_9$. Then the set of blocks

$$\begin{matrix} \{0,1,3,5,6\} & \{0,2,4,5,6\} & \{0,2,3,7,8\} \\ \{0,1,4,7,8\} & \{1,2,3,6,7\} & \{1,2,4,5,8\} \end{matrix}$$

form the required packing design.

Lemma 3.2. $\{6, 8, 14, 24\} \subset MSD(2)$.

Proof: For $v = 6$, let $X = Z_6$. Then the blocks

$$\begin{matrix} \{0,1,2,3,4\} \\ \{0,1,2,3,5\} \end{matrix}$$

form a $MSD(5,2;6(2))$ with the hole $\{5,4\}$.

For $v = 8$ let $X = Z_8$. Then the blocks

$$\{0,1,3,4,5\} \quad \{0,2,3,4,6\}$$
$$\{0,1,2,6,7\} \quad \{1,2,4,5,7\}$$

form the required MSD with the hole $\{5,6\}$.

For $v = 14$, let $X = Z_8 \cup \{A, B, C, D, R, Q\}$. Take the following blocks:

$$\begin{array}{lll} \{A,3,4,5,6\} & \{A,0,1,2,7\} & \{A,C,R,0,4\} \\ \{A,C,R,2,6\} & \{A,D,Q,1,5\} & \{A,D,Q,3,7\} \\ \{B,0,2,3,5\} & \{B,1,4,6,7\} & \{B,D,R,0,4\} \\ \{B,D,R,2,6\} & \{B,C,Q,1,5\} & \{B,C,Q,3,7\} \\ \{C,0,5,6,7\} & \{C,1,3,2,4\} & \{D,2,4,5,7\} \\ \{D,0,1,3,6\} & & \end{array}$$

Then these blocks form a $MSD(5,2;14(2))$ with the hole $\{R,Q\}$.

For $v = 24$, proceed as follows. In a $(25,5,1)$-BIBD, we delete one point to obtain a GD(5,1,4;24) and denote it by $(Z_{23} \cup \{\infty\}, \mathcal{G}, \mathcal{A}_1)$. Let $G_0 = \{\infty,0,1,2\} \in \mathcal{G}$ and $\mathcal{A}_2 = \{G \cup \{\infty\} : G \in \mathcal{G} \text{ and } G \neq G_0\}$. Take $\mathcal{A}_3 = \{\{j, j+1, j+4, j+6, j+13\} : 0 \leq j \leq 22\}$. We then obtain a $MSD(5,2;24(2))(Z_{23} \cup \{\infty\}, \{0,\infty\}, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$.

Lemma 3.3. $\{34, 488\} \subset M$.

Proof: For $v = 34$, we have $\psi(34) = 108$. Take $Y = \{\infty_i : 1 \leq i \leq 9\}$ and $X = Z_{28} \cup Y$. Use Lemma 2.7 and construct a (37,9;5,1)-IPBD on X with the hole Y. Denote its blocks by $\mathcal{A}$. Let $\mathcal{A}_1$ denote the block set from $\mathcal{A}$ by replacing the symbol ∞_7, ∞_8 and ∞_9 by ∞_4, ∞_5 and ∞_6 respectively wherever they occur. In addition, it has been shown that there exists a RGD(4,1,3;24) (see [7]). This provides us a GD(5,1,{3,7};31) of type $3^8 7^1$. We construct such a GDD on $X \cup \{\infty_1, \infty_2, \infty_3\}$ such that $\{\infty_1, \infty_2, \infty_3\}$ and $\{1, 2 \ldots 7\}$ are its two groups. We write $\mathcal{A}_2$ for block set of the GDD. Take $S = \{1,2,3,4,5\}$. It is easy to check that $(X \cup \{\infty_i : 1 \leq i \leq 6\}, \{\infty_i : 1 \leq i \leq 6\}, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{S\})$ is a $MSD(5,2;34(6))$. This guarantees that $34 \in M$ by Lemma 2.1 and 3.2.

For $v = 488$, proceed as follows. Since $126 \in RB(6)$, there is a GD(6,1,6;126). Give weight 0 to every point of one group of the GDD and weight 4 to every point of the other groups and apply "the Fundamental Construction". We obtain then a GD(5,1,24;480) of type 24^{20}. Add a set T of 7 new points to it and break up each group together with T to form a GD(5,1,{3,7};31)such that the group of size 7 is T. This produces a GD(5,1,{3,7};487) of type $3^{160} 7^1$. Thus the result can be obtained in the same way as we did for $v = 34$. But, in this case, we take $X = Z_{480} \cup Y, \mathcal{A}$ is the block set of a (489,9;5,1)-IPBD on X with the hole Y. $\mathcal{A}_1$ is the block set obtained from $\mathcal{A}$ by replacing the symbol ∞_9 by ∞_8, and $\mathcal{A}_2$ is the block set of a GD(5,1,{3,7};487) on $X \setminus \{\infty_8, \infty_9\}$ which unique group of size 7 is $Y\backslash\{\infty_8,\infty_9\}$. It is easy to show that $(X\backslash\{\infty_9\}, Y\backslash\{\infty_9\}, A_1 \cup A_2)$ is a $MSD(5,2;488(8))$ which implies $488 \in$ M from Lemma 2.1 and 3.2. This completes the proof.

In the following constructions, we shall use difference sets (see, for example, [2]). Instead of listing all of the blocks of a design, it suffices to give the group G acting on a set of base block, we shall adapt the notation:

$$\text{dev}\mathcal{B} = \{B + g : B \in \mathcal{B} \text{ and } g \in G\}$$

where $\mathcal{B}$ is the collection of base blocks of the design.

Lemma 3.4. $\{10,12,20,30,40,70\} \subset MSD(2)$ *and* $\{22,32\} \subset M$.

Proof: For $v = 70$, we have $\psi(v) = 476$. Take $X = Z_{70}$. Let $\mathcal{B}_1$ be the set of the following block:

$\{j, j+14, j+21, j+28, j+42\}$ $\{j, j+7, j+14, j+56, j+63\}$
$\{j, j+7, j+21, j+28, j+49\}$ $\{j+7, j+14, j+21, j+35, j+63\}$
$\{j+7, j+28, j+35, j+49, j+56\}$ $\{j+21, j+35, j+42, j+49, j+63\}$
$\{j+14, j+28, j+35, j+42, j+56\}$ $\{j, j+42, j+49, j+56, j+63\}$

where $j = 0,1,2\ldots6$. Let $\mathcal{B}_2$ be the set of the following blocks:

$\{0,1,16,20,33\}$ $\{0,1,3,26,60\}$ $\{0,3,40,46,51\}$
$\{0,9,17,29,55\}$ $\{0,5,9,27,39\}$ $\{0,10,16,18,41\}$

It is readily checked that $(X,\{0,35\},\mathcal{B}_1\cup \text{dev}\mathcal{B}_2)$ is a $MSD(5,2,70(2))$.

For the others, we let G be an abelian group and $X = G$ or $X = G\cup(\{\infty\}\times Z_2)$. It is readily checked that $(X,Y,\text{dev}\mathcal{B})$ is the required MSD where $\mathcal{B}$ are listed below.

(1) $v = 10$, $G = Z_4 \times Z_2$, $Y = \{\infty\} \times Z_2$,
$\mathcal{B} = \{\{(\infty,0)(0,0)(0,1)(1,1)(3,0)\}\}$.

(2) $v = 12$, $G = Z_{12}$, $Y = \{0,6\}$,
$\mathcal{B} = \{\{0,1,2,5,10\}\}$.

(3) $v = 20$, $G = Z_9 \times Z_2$, $Y = \{\infty\} \times Z_2$,
$\mathcal{B} = \{\{(\infty,0)(0,0)(0,1)(1,0)(3,1)\},$
$\{(8,1)(0,0)(1,0)(3,0)(5,0)\}\}$.

(4) $v = 22$, $G = Z_{22}$, $Y = \phi$,
$\mathcal{B} = \{\{0,5,11,12,14\}, \{0,2,6,7,10\}\}$.

(5) $v = 30$, $G = Z_{14} \times Z_2$, $Y = \{\infty\} \times Z_2$,
$\mathcal{B} = \{\{(\infty,0)(0,0)(2,0)(0,1)(3,1)\},$
$\{(0,0)(4,0)(3,1)(8,1)(9,1)\},$
$\{(0,0)(2,1)(7,1)(4,1)(8,1)\}\}$.

(6) $v = 32$, $G = Z_{16} \times Z_2$, $Y = \phi$,
$\mathcal{B} = \{\{(0,0)(2,0)(5,0)(0,1)(12,1)\},$
$\{(0,0)(2,0)(5,0)(1,1)(13,1)\},$
$\{(0,0)(1,0)(7,0)(8,0)(10,1)\}\}.$

(7) $v = 40$, $G = Z_{19} \times Z_2$, $Y = \{\infty\} \times Z_2$,
$\mathcal{B} = \{\{(\infty,0)(0,0)(4,0)(4,1)(11,1)\},$
$\{(6,0)(0,1)(3,1)(1,1)(9,1)\},$
$\{(13,0)(0,1)(4,1)(5,1)(11,1)\},$
$\{(0,0)(9,0)(2,1)(4,1)(18,1)\}\}.$

4. Constructions using IPBD.

In this section, we shall give some recursive constructions for IPBD so as to determine the values of D(5,2;v) for $v \equiv 1 \pmod 2$. Our method of construction will be based upon the following lemmas.

Lemma 4.1. *If* $v \equiv 7$ *or* 9 (mod 10), *then D(5,2;v)*$\leq \psi(v) - 1$.

Proof: Assume that there exists a SD(5,2;$v, \psi(v)$). We let r_x be the number of times that point x occurs in the design. Then $r_x \leq [2(v-1)/4] = (v-1)/2$ for each x. Since $v(v-1)/2 - 5\psi(v) = 1$, however, there must be one point which occurs $v(v-1)/2 - 1$ times, and all others appear $(v-1)/2$ times. Thus the number of pairs which occur less than twice in the blocks of the packing is 4 (counting multiplicities). This contradicts the fact that $2(v-1)v/2 - 10\psi(v) = 2$. Therefore the conclusion follows.

Lemma 4.2. *If* $v \equiv 7$ *or* 9 (mod 10), *then D(5,2;v)=* $\psi(v) - 1$ *provided that* $v \in IP_2(9)$.

Proof: Let $(X, Y, \mathcal{A})$ be a (v,9;5,2)-IPBD. From Lemma 3.1 we can replace the hole Y with $\psi(9) - 1$ blocks of a packing design of its pairs by quintuples. These blocks together with all blocks in $\mathcal{A}$ form a SD(5,2;$v, \psi(v) - 1$) on X. Then the conclusion holds by Lemma 4.1.

Similarly, we have

Lemma 4.3. *If* $v \equiv 7$ *or* 9 (mod 10), *then D(5,2;v)=* $\psi(v) - 1$ *provided* $v \in IP_2(7)$.

Proof: The proof is similar to that above. In this case, we use Lemma 3.1 and replace the hole with $\psi(7) - 1$ blocks of a packing design of its pairs by quintuples.

Lemma 4.4. *If* $v \equiv 3$ (mod 10), *then D(5,2;v)=* $\psi(v)$ *provided* $v \in IP_2(3)$.

Proof: Simple calculation shows that the number of blocks in a (v,3;5,2)-IPBD is $\psi(v)$ whenever $v \equiv 3$ (mod 10). The conclusion then follows from (1.1).

In view of Lemma 4.2, 4.3, and 4.4, it will be necessary for us to build families of IPBD. We shall employ some recursive constructions below.

A very powerful recursive construction for IPBD is given as follows.

Construction 4.5. *Let e and m be positive integers satisfying $e \equiv 0 \pmod{m}$ and let $q \geq 0$. Suppose that the following designs exist:*

(1) *a $(u+e+q, e+q; K, \lambda)$-IPBD,*
(2) *a $(u+q, q; K, (m-1)\lambda)$-IPBD.*

Then there exists a $(u+w, w; K, m\lambda)$-IPBD, where $w = q + e/m$.

Proof: Let X and Y be disjoint sets of cardinality u and $e+q$, and let $(X \cup Y, Y, \mathcal{A})$ be a $(u+e+q, e+q; K, \lambda)$-IPBD. Take $F \subset Y$ such that $|F| = q$ and divide the points of $Y \backslash F$ into e/m groups. We replace the j-th group by single new symbol θ_j $(1 \leq j \leq e/m)$ and put $M = \{\theta_j : 1 \leq j \leq e/m\}$. Let $\mathcal{A}_1$ be a set of blocks obtained from $\mathcal{A}$ by the above replacement. Then all pairs of distinct points of $X \cup M$, not both in M or X, occur exactly in $m\lambda$ blocks of $\mathcal{A}_1$, whereas all others of $X \cup F$, not both in F, occur exactly in λ blocks of $\mathcal{A}_1$. Now using hypothesis we form a $(u+q, q; K, (m-1)\lambda)$-IPBD on $X \cup F$ with the hole F and block set $\mathcal{A}_2$. It is clear that $(X \cup F \cup M, F \cup M, \mathcal{A}_1 \cup \mathcal{A}_2)$ is the required IPBD.

Corollary 4.6. *Suppose that $v = 20n + 3$ where n is a positive integer. Then $v \in IP_2(3)$.*

Proof: Apply Construction 4.5 with $K = \{5\}, u = 20n, e = 4, \lambda = 1, q = 1$ and $m = 2$. The required IPBDs come from Lemma 2.7.

Corollary 4.7. $47 \in IP_2(7)$ *and* $49 \in IP_2(9)$.

Proof: From $40 \in RB(4)$, we have $53 \in IP_1(13)$. And the result then follows by taking $K = \{5\}, u = 40, \lambda = 1, m = 2, q + e = 13$ and $e = 8$ or 12 in Construction 4.5.

Corollary 4.8. *Suppose that $v = 20n + 7$ where n is a positive integer and $n \neq 1, 2, 3, 6, 8, 22, 37$. Then $v \in IP_2(7)$.*

Proof: For these values of n, $20n + 9 \in IP_1(9)$ and $20n + 5 \in IP_1(5)$. Take $K = \{5\}, u = 20n, \lambda = 1, m = 2, q = 5$, and $e = 4$ in Construction 4.5, we have $v \in IP_2(7)$.

Next, we start with a GDD to produce another construction for IPBD. The proof is immediate and omitted here.

Construction 4.9. *If there exists a $GD(5, 2, \{t_1, t_2 \ldots t_n\}; v)$ and $t_i + w \in IP_2(w)$ for $1 \leq i \leq n-1$, then $v + w \in IP_2(t_n + w)$. Moreover, $v + w \in IP_2(e)$ if $t_n + w \in IP_2(e)$.*

As an immediate corollary of Construction 4.9 we have the following useful constructions.

Construction 4.10. *Let t be an integer satisfying $t \equiv 0$ or $2 \pmod 5$ and $t \geq 5, t \neq 10, 30, 7, 22, 42, 52$. Then $10t + 2s + 1 \in IP_2(2s + 1)$, furthermore $10t + 2s + 1 \in IP_2(w)$ if $2s + 1 \in IP_2(w)$, where $0 \leq s \leq t$.*

Proof: For these values of t, there are both TD(6,t) and a $(2t + 1, 5, 2)$-BIBD by Lemma 2.3 and 2.5. Delete some points and leave s points in a group of a TD(6,t). Give weight 2 to every point and use "the Fundamental Construction". The required ingredient GD(5,2,2;10) and GD(5,2,2;12) come from Lemma 2.4. Then add one point to the resultant GDD. The conclusion then follows by taking $n = 6, t_1 = t_2 = \cdots = t_5 = 2t, t_6 = 2s$ and $w = 1$ in Construction 4.9.

Construction 4.11. *Let r be a positive integer and $0 \leq s \leq r$. Then $20r + 4s + w \in IP_2(4s + w)$ if $4r + w \in IP_2(w)$. Moreover, $20r + 4s + w \in IP_2(e)$ if $4s + w \in IP_2(e)$.*

Proof: From Lemma 2.4 we can know that there exists a GD(6,2,r;6r) for any positive integer r. Delete some points and leave s points in a group of a GD(6,2,6;6r). Give weight 4 to every point and use "the Fundamental Construction". We obtain a GD(5,2,4r, 4s; $20r + 4s$) of type $(4r)^5(4s)^1$. The required ingredient GDDs come from the fact that $\{21, 25\} \subset B(5)$. Then the conclusion follows from Construction 4.9

Construction 4.12. *Let a, b and v be integers satisfying $0 \leq a \leq 5, 0 \leq b < (v-1)/4$. Then*

(1) $2(v - 6) + 2a + 1 \in IP_2(2a + 1)$ *if* $v \in B(6)$,
(2) $2v + 2b + 1 \in IP_2(2b + 1)$ *if* $v \in RB(5)$, *furthermore* $2v + 2b + 1 \in IP_2(w)$ *if* $2b + 1 \in IP_2(w)$

Proof: Noticing that there is a GD(6,1,5,$v-1$) of type $5^{(v-1)/5}$ whenever $v \in B(6)$ and a GD(6,1,5;$v + ((v-1)/4) - 1$) of type $5^{v/5}((v-1/4)-1)^1$ whenever $v \in RB(5)$, the proof is similar to that of Construction 4.10.

We are now in a position to establish the main result of this section.

Lemma 4.13. $110 + q \in IP_2(q)$ *if* $q = 3, 7$ *or* 9.

Proof: In a TD(6,11), we delete $11 - s$ points in a group and a block of size 5 to obtain a $(\{5, 6\}, 1)$-IGDD of type $(11, 1)^5(s, 0)^1$. Give weight 2 to every point of the IGDD and use "the Fundamental Construction". The resulting design is a $(5, 2)$-IGDD, of type $(22, 2)^5(2s, 0)^1$. Adding a new point to the IGDD, we get $110 + 2s + 1 \in IP_2(2s + 1)$ by filling in each group of size 22 together with the new point with a $(23, 3; 5, 2)$-IPBD and filling in the hole by $(11, 5, 2)$-BIBD. Take $s = 1, 3$ or 4 and the result follows.

Lemma 4.14. *If $v \equiv 3 \pmod{10}, v \geq 23$ and $v \neq 33, 73, 93$, then $v \in IP_2(3)$.*

Proof: By Corollary 4.6 and Lemma 4.13, we need only to show that $v \in IP_2(3)$ for $v \equiv 13 \pmod{20}$ and $v \neq 13, 33, 73, 93, 113$. Take $t = 10n + 5$ ($n \geq 3$), $2s + 1 \in \{3, 23, 43, 63\}$ and $t = 10n + 7$ ($n \geq 3$), $2s + 1 = 63$ in Construction 4.10. Since $\{23, 43, 63\} \subseteq IP_2(3)$ from Corollary 4.6, we get $v \in IP_2(3)$ for all $v \geq 353$. We still make use of Construction 4.10, but taking $t = 5, 15, 17, 25, 27$ and $2s+1 \in \{3, 23, 43\}$, to obtain $\{53, 153, 173, 193, 253, 273, 293, 313\} \subset IP_2$. Apply Construction 4.11 with $r = 14$, $w = 5$ and $s = 12$, we have $333 \in IP_2(3)$. Since $65 \in RB(5)$ and $\{111, 121\} \subset B(6)$, we get $\{133, 213, 233\} \subset IP_2(3)$. This completes the proof.

Lemma 4.15. *If* $v \equiv 7 \pmod{10}$, $v \geq 37$ *and* $v \neq 67, 77, 97, 167, 197$, *then* $v \in IP_2(7) \cup IP_2(9)$

Proof: We need only consider the exceptions of listed in Lemma 2.7 and Corollary 4.8. By the foregoing, $\{47, 117\} \subset IP_2(7)$. $\{127, 137, 497\} \subset IP_2(7)$ follows from Construction 4.12, since $\{65, 205\} \subset RB(5), 66 \in B(6)$ and $87 \in IP_2(7)$. For $v \in \{57, 157, 277, 397, 447, 557, 577, 637, 717, 747, 777, 797, 897\}$, apply Construction 4.10 with $t \in \{5, 15, 27, 35, 40, 55, 57, 60, 67, 70, 77\}$, $2s + 1 \in \{7, 37, 47, 127\}$.

Lemma 4.16. *If* $v \equiv 9 \pmod{10}$, $v \geq 49$ *and* $v \neq 69, 79, 99, 169, 199, 339, 439$, *then* $v \in IP_2(9)$.

Proof: By Lemma 2.7 we need only consider the case $v \equiv 19 \pmod{20}$ and $v = 49, 129, 449, 749$. It is shown $\{49, 119\} \subset IP_2(9)$ in Lemma 4.13 and Corollary 4.7. Taking $v \in \{65, 205, 305, 405\} \subset RB(5), 2b+1 \in \{9, 89, 109\}$ and $v \in \{66, 111, 121, 151, 301\} \subset B(6), 2a + 1 = 9$ in Construction 4.12, we have $\{129, 139, 219, 239, 299, 419, 499, 599, 619, 699, 719, 819\} \subset IP_2(9)$. The remaining case in the interval $49 \leq v \leq 829$ are covered by using Construction 4.10 with $t \in \{5, 15, 17, 25, 27, 35, 37, 40, 45, 47, 55, 57, 65, 67, 70\}$ and $2s + 1 \in \{9, 49, 89, 109, 129\}$. For $v \geq 839$ and $v \equiv 19 \pmod{20}$, the result follows from Construction 4.10 in such a way that $t = 10n + 5$ ($n \geq 7$), $2s + 1 \in \{89, 109, 129, 149\}$ and $t = 10n + 7$ ($n \geq 7$), $2s + 1 = 149$. This completes the proof.

Combining Lemmas 4.14, 4.15, and 4.16 with Lemmas 4.2, 4.3, and 4.4, keeping in mind Lemmas 2.3 and 3.1, we are able to give the main result of this section. That is

Theorem 4.17. *If* $v \geq 5, v \equiv 1 \pmod 2$ *and* $v \notin F$, *then*

$$D(5,2;v) = \begin{cases} \psi(v) - 1 & \text{if } v = 7 \text{ or } 9 \pmod{10} \\ \psi(v) & \text{otherwise}. \end{cases}$$

Here $F = \{13, 15, 17, 19, 27, 29, 33, 39, 67, 69, 73, 77, 79, 93, 97, 99, 167, 169, 197, 199, 339, 439\}$.

5. Constructions using MSD

In this section we wish to determine the values of $D(5,2;v)$ for $v \equiv 0 \pmod 2$. Our constructions will mainly involve maximum almost packing design. We need the following notations:

$$M_1(w) = \{v : v, w \equiv 0 \text{ or } 2 \pmod{10} \text{ and } v \in MSD(w)\};$$
$$M_2(w) = \{v : v, w \equiv 6 \pmod{10} \text{ and } v \in MSD(w)\};$$
$$M_3(w) = \{v : v, w \equiv 4 \text{ or } 8 \pmod{10} \text{ and } v \in MSD(w)\}.$$

Here w is a nonnegative integer.

Construction 5.1. *Let t_1, t_2, u_1, u_2 and w be nonnegative even integers and let $1 \le j \le 3$. Suppose that*

(1) *there exists a $(5,2)$-IGDD of type $t_1 + u_1, u_1)^5 (t_2 + u_2, u_2)^1$*

(2) $t_1 + u_1 + w \in M_j(u_1 + w)$;

(3) $t_2 + u_2 + w \in MSD(u_2 + w)$; *and*

(4) $u_2 + w \equiv 0$ *or* $2 \pmod{10}$.

Then $5(t_1 + u_1) + t_2 + u_2 + w \in MSD(5u_1 + u_2 + w)$. *Furthermore,* $5(t_1 + u_1) + t_2 + u_2 + w \in M$, *if* $5u_1 + u_2 + w \in M$.

Proof: Let $(X, Y, \{G_i : 1 \le i \le 6\}, \mathcal{A})$ be a $(5,2)$-IGDD of type $(t_1 + u_1, u_1)^5 (t_2 + u_2, u_2)^1$, and let $Y \cap G_i = H_i$ for $1 \le i \le 6$. Take a set T of size w which is disjoint from the point set X. Add T to the IGDD. For $1 \le i \le 6$, we construct a MSD on $G_i \cup T$ with the hole $H_i \cup T$ by the conditions (2) and (3). Let corresponding block set be $\mathcal{A}_1 \mathcal{A}_2 \cdots \mathcal{A}_6$.

It is easy to show that $(X \cup T, \mathcal{A} \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_6)$ is the desired MSD with the hole $H_1 \cup \cdots \cup H_6 \cup T$ by counting the number of pairs which occur less than twice in these blocks. The last conclusion comes from Lemma 2.1. The proof is complete.

When we start with GDD, we can obtain the following constructions by the similar way as we did for IPBD. Here we consider MSD instead of IPBD.

Construction 5.2. *Let t_1, t_2, and w be nonnegative even integers. Suppose that a GD(5,2,$\{t_1, t_2\}$; v) of type $t_1^{n-1} t_2^1$ exists. Then*

(1) $v + w \in MSD(t_2 + w)$ *if* $t_1 + w \in M_1(w)$. *Moreover,* $v + w \in M$ *if* $t_2 + w \in M$;

(2) $v \in MSD(t_1)$ *if* $t_1 \in M_1(0)$ *and* $t_2 \in M$.

Construction 5.3. *Let t and s be integers satisfying $0 \le s \le t$. Suppose that a TD(6,t) exists. Then*

(1) $10t + 2s + w \in MSD(2s + w)$ *if* $2t + w \in M_1(w)$, *moveover,* $10t + 2s + w \in M$ *if* $2s + w \in M$;

(2) $10t + 2s \in MSD(2t)$ *if* $2t \in M_1(0)$ *and* $2s \in M$.

Construction 5.4. *Suppose that r and s are integers satisfying $r > 0$ and $0 \leq s \leq r$. Then*

(1) $20r + 4s + w \in MSD(4s + w)$ *if* $4r + w \in M_1(w)$, *moreover,* $20r + 4s + w \in M$ *if* $4s + w \in M$;

(2) $20r + 4s \in MSD(4r)$ *if* $4r \in M_1(0)$ *and* $4s \in M$.

We shall also make use of the following constructions.

Construction 5.5. *Let a, b and v be integers satisfying $0 \leq a \leq 5, 0 \leq b < (v-1)/4$. Then*

(1) $2(v-6) + 2a + 2 \in MSD(2a + 2)$ *if* $v \in B(6)$,

(2) $2v + 2b + 2 \in MSD(2b + 2)$ *if* $v \in RB(5)$, *moreover,* $2v + 2b + 2 \in M$ *if* $2b + 2 \in M$;

(3) $2v \in M_1(2)$ *if* $v \in B(5)$.

Proof: The conslusion (1) and (2) come from Cosntruction 5.2. The verification of conditions are the same as in Construction 4.12, noticing that $12 \in M_1(2)$ mentioned in Lemma 3.4. For (3) we regard $(v, 5, 1)$-BIBD as a GD(5,1,1;v) and give weight 2 to every point of the GDD. This produces a GD(5,2,2;2v) which is the desired design.

Construction 5.6. *If $v \equiv 0$ or $2 \pmod{10}, v \geq 10$ and $v \in M$, then $5v-4 \in M$.*

Proof: By Lemma 2.5 for these values of v there exists a TD(5,v–1). A SD(5,2;5v –4, $\psi(5v-4)$) can easily be obtianed by adding a new point to such a TD(5,$v-1$) and forming a SD(5,2;v, $\psi(v)$) on each group together with the new point.

Next, we use the above constructions to establish our result of this section. As an aurhority for the existence of the required transversal designs, we use Lemma 2.5. For brevity we shall not mention them then.

Lemma 5.7. *Suppose that v is an integer satisfying $v \geq 10$ and $v \equiv 0$ or $2 \pmod{10}$. Then $v \in M_1(2)$ if $v \neq 22, 32, 80$; and $\{22, 32\} \subset M$.*

Proof: In view of Lemma 2.2, we need only to show that $v \in M_1(u)$ such that $u = 2$ or $u \in M_1(2)$ for every admissible v.

It has been shown that $\{22, 32\} \subset M$ and $\{10, 12, 20, 30, 40, 70\} \subset M_1(2)$ in Lemma 3.4.

$\{42, 82, 90, 132, 140, 142, 180, 182, 190, 192\} \subset M_1(2)$ follows from Construction 5.5. Suitable equations are $42 = 2 \cdot 21$, $82 = 2 \cdot 41$, $90 = 2 \cdot 45$, $132 = 2 \cdot 65 + 2$, $142 = 2 \cdot 65 + 12$, $140 = 2 \cdot 65 + 10$, $180 = 2(91 - 6) + 10$, $182 = 2 \cdot (91 - 6) + 12$, $190 = 2(96 - 6) + 10$, $192 = 2(96 - 6) + 12$.

For $v \in \{50, 52, 60, 62, 92, 100, 102, 110, 112, 120, 122, 130, 150, 152, 160, 162, 170, 172\}$, we take $t = 5, 9, 11, 15, 16$ in Construction 5.3 and select certain values of $2s + w \in \{0, 2, 10, 12, 20\}$ where $w = 0$ or 2.

For $v = 72$, the result follows from Construction 5.4 with $r = 3, s = 3$ and $w = 0$.

Now we further apply Construction 5.3 recursively for each $t \in \{n : n \equiv 0 \pmod 5 \text{ and } n \geq 35\} \cup \{20, 25, 29, 31\}$ in such a fashion that $2s + w \in \{n : n \equiv 0 \text{ or } 2 \pmod{10} \text{ and } 0 \leq n \leq 52\}$ and $w = 0$ or 2. This guarantees that $v \in M_1(2)$ whenever $v \geq 200$ and $v \equiv 0$ or $2 \pmod{10}$.

Lemma 5.8. *If* $v \equiv 4 \pmod{10}$ *and* $v \geq 14$, *then* $v \in M$ *provided* $v \neq 44, 84$.

Proof: Applying Construction 5.3 for each $t \in \{n : n = 0 \text{ or } 1 \pmod 5) \text{ and } n \geq 11, n \neq 26, 30\}$ and $2s + 0 \in \{4, 14, 24, 34\}$ covers all cases except $v \in$ {54, 64, 74, 94, 104, 134, 144, 294, 304}. The necessary ingredient $\{4, 14, 24, 34\} \subset M$ and $2t \in M_1(0)$ come from Lemma 3.2, 3.3 and 5.7. Here we use the fact that $M_1(2) \subset M$. For $v = 74$, we use Construction 5.2 as follows. In a TD(6,7), we delete 5 points from one block to produce a GD($\{5, 6\}, 1, \{6, 7\}$; 37) of type $6^5 7^1$. Give weight 2 to the GDD and use "the Fundamental Construction". This guarantees $74 \in M_3(14)$ which implies $74 \in M$. The remaining values of v can be taken care of by Construction 5.3–5.5 with the equations $54 = 10 \cdot 5 + 2 + 2$, $64 = 20 \cdot 3 + 4 + 0$, $94 = 10 \cdot 9 + 2 + 2$, $104 = 20 \cdot 5 + 4 + 0$, $134 = 2 \cdot 65 + 2 + 2$, $144 = 2 \cdot 65 + 2 \cdot 6 + 2$, $294 = 10 \cdot 29 + 2 + 2$, $304 = 10 \cdot 29 + 2 \cdot 6 + 2$.

Lemma 5.9. $\{328, 376, 378, 388\} \subset M$.

Proof: Checking the proof of Lemma 5.8 we have $74 \in M_3(14)$. Using Construction 5.3(2) and 5.4 with the parameters $(t, s) = (5, 3), (5, 4)$ and $(r, s) = (3, 1), (3, 2), (3, 3)$ and the fact that $\{4, 6, 8, 12\} \subset M$, we have $56 \in MSD(0)$, $58 \in MSD(10)$, $64 \in M_3(4)$, $68 \in MSD(12)$, $72 \in M_1(12)$. From a TD(6,16) we can obtain a $(5, 2)$-IGDD of type $(60 + 4, 4)^5(4s + 4q, 4q)^1$ satisfying $0 \leq s \leq 16$ and $q = 0$ or 1, which proof is similar to those in Lemma 4.13. Take pair $(s, q) = (2, 0), (14, 0), (12, 0), (14, 1)$ we get 4 $(5, 2)$-IGDDs which types are seperately $(60 + 4, 4)^5(8, 0)^1$, $(60 + 4, 4)^5(56, 0)^1$, $(60 + 4, 4)^5(48, 0)^1$, $(60 + 4, 4)^5(60, 4)^1$. Thus the result follows by taking (t_1, u_1, t_2, u_2, w) = (60,4,8,0,0), (60,4,56,0,0), (60,4,48,0,10), (60,4,56,4,8) in Construction 5.1.

Lemma 5.10. *Let* v *be a positive integer satisfying* $v \equiv 6, 8, 16, 18, 46$ *or* $48 \pmod{50}$. *Then* $v \in M$ *if* $v \neq \{16, 18, 48, 66\}$.

Proof: It has been shown that $\{6, 8\} \subset M$ in Lemma 3.2. Applying Construction 5.3 for each $t \in \{n : n \equiv 0, 1 \text{ or } 4 \pmod 5 \text{ and } n \geq 5, n \neq 6, 10, 14, 26, 30, 34, 44\}$ such that $2s + w = 6, 8$ and $w = 0$ or 2, covers all cases except $v \in$ {46, 68, 106, 108, 146, 148, 266, 268, 306, 308, 346, 348, 446, 448} . The conditions $2t \in M_1(0)$ or $2t + 2 \in M_1(2)$ come from Lemma 5.7. Since {76, 136, 156}$\subset B(6)$, we can use Construction 5.5 to show that {146, 148, 266, 268, 306, 308}$\subset M$. For $v \in \{46, 106\}$, apply Construction 5.6 with the equations $46 = 5 \cdot 10 - 4$, $106 = 5 \cdot 22 - 4$. The remaining values of v are covered by Construction 5.3 and 5.4 with the equations $346 = 10 \cdot 29 + 2 \cdot 27 + 2$, $348 = 10 \cdot 29 + 2 \cdot 28 + 2$, $446 = 10 \cdot 39 + 2 \cdot 27 + 2$, $448 = 10 \cdot 39 + 2 \cdot 28 + 2$, $68 = 20 \cdot 3 + 4 \cdot 2 + 0$, $108 = 20 \cdot 5 + 4 \cdot 2 + 0$.

Lemma 5.11. *If $v \equiv 6$ or $8 \pmod{10}$, $v \geq 6$ and $v \notin H$, then $v \in M$. Where $H = \{16, 18, 26, 28, 36, 38, 48, 66, 76, 78, 86, 88, 226, 228, 236, 238, 276, 278, 288, 326, 338, 438, 526\}$.*

Proof: From our previous results in Lemma 3.3, 5.9 and 5.10, we need only consider the case $v \equiv 26, 28, 36, 38 \pmod{50}$ and $v \neq 328, 376, 378, 388, 488$.

For $v \in \{126, 128, 136, 138, 176, 178, 186, 188, 426, 428, 476, 478, 536, 538\}$, Construction 5.5 gives us the required result because of $\{66, 91, 96, 216, 241, 271\} \subset B(6)$ and $65 \in RB(5)$.

For $v \in \{386, 486\}$, the result can be obtained by taking $r = 17, 22$, $w = 2$ and $4s + w = 46$ in Construction 5.4.

For $v \in \{676, 678\}$, proceed as follows. First we use Construction 5.3 and the equation $130 = 10 \cdot 11 + 2 \cdot 10 + 0$ to obtain $130 \in M_1(20)$. Then we further make use of Construction 5.3 with $t = 55$, $s = 53, 54$ and $w = 20$.

For $v \in \{576, 626\}$, proceed as follows. We can easily show that there is a $(331, 6, 1)$-BIBD with a flat of order 66, since $66 \in B(6)$. This implies that there is a GD(6,1,{5,65} ;330) of type $5^{53}65^1$. Delete $65 - s$ points from the group of size 65 of such a GDD and give weight 2 to the resultant design. "The Fundamental Construction" gives us a GD(5,2,{10,2s};530+2s) of type $10^{53}(2s)^1$. Then the result follows from Construction 5.2 by taking $t_1 = 10$, $t_2 = 2s = 46, 96$ and $n = 54$.

For $v \in \{586, 588, 636, 638, 686, 688, 736, 738, 786, 788\}$, the result follows from Construction 5.3 with $t \in \{49, 54, 59, 64, 69\}$, $w = 2$ and $2s + w = 96, 98$.

For all admissible others for $v \leq 788$, the result can be obtained using Construction 5.3 and values of already determined. Suitable equations are $286 = 10 \cdot 24 + 2 \cdot 22 + 2$, $336 = 10 \cdot 29 + 2 \cdot 22 + 2$, $436 = 10 \cdot 39 + 2 \cdot 22 + 2$, $528 = 10 \cdot 46 + 2 \cdot 34 + 0$, $578 = 10 \cdot 51 + 2 \cdot 34 + 0$, $628 = 10 \cdot 56 + 2 \cdot 34 + 0$, $726 = 10 \cdot 61 + 2 \cdot 58 + 0$, $728 = 10 \cdot 61 + 2 \cdot 59 + 0$, $776 = 10 \cdot 65 + 2 \cdot 63 + 0$, $778 = 10 \cdot 65 + 2 \cdot 64 + 0$.

Finally, the case of $v \geq 826$ can be covered by applying Construction 5.3 for each $t \in \{n : n \equiv 0 \pmod 5 \text{ and } n \geq 70\}$ such that $w = 0$ and $2s \in \{126, 128, 136, 138\}$. The proof is complete.

Combining Lemma 5.7, 5.8 and 5.11 with (1.1), we establish the main theorem of this section.

Theorem 5.12. *If $v \equiv 0 \pmod 2$, $v \geq 6$ and $v \notin H \cup \{44, 80, 84\}$, then $D(5, 2; v) = \psi(v)$, where H as mentioned in Lemma 5.11.*

6. Proof of Theorem 1.1.

The conclusion follows immediately from Theorem 4.17 and 5.12.

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