The Chromatic Index of an Abelian Cayley Graph

Izak Broere and Johannes H. Hattingh

Department of Mathematics
Rand Afrikaans University
JOHANNESBURG
2000 South Africa

Abstract. Suppose Γ is a finite multiplicative group and $S \subseteq \Gamma$ satisfies $1 \notin S$ and $S^{-1} = \{x^{-1} | x \in S\} = S$. The abelian Cayley graph $G = G(\Gamma, S)$ is the simple graph having vertex set $V(G) = \Gamma$, an abelian group, and edge set $E(G) = \{\{x, y\} | x^{-1}y \in S\}$. We prove the following regarding the chromatic index of an abelian Cayley graph $G = G(\Gamma, S)$: if $\langle S \rangle$ denotes the subgroup generated by S, then $\chi'(G) = \Delta(G)$ if and only if $|\langle S \rangle|$ is even.

1. Introduction

All graphs considered in this paper are finite and simple.

Suppose throughout Γ is a finite multiplicative group and $S \subseteq \Gamma$ satisfies $1 \notin S$ and $S^{-1} = \{x^{-1} | x \in S\} = S$. The *Cayley graph* $G = G(\Gamma, S)$ is the simple graph having vertex set $V(G) = \Gamma$ and edge set $E(G) = \{\{x, y\} | x^{-1}y \in S\}$. If Γ is an abelian group, then the Cayley graph is called an *abelian Cayley graph*. Every Cayley graph is regular of degree $|S|$.

In this paper we prove that the chromatic index $\chi'(G)$ of the abelian Cayley graph $G = G(\Gamma, S)$ is $\Delta(G)$ if and only if $|\langle S \rangle|$ is even. This is accomplished in Section 3 by analysing, in Section 2, the structure of disconnected abelian Cayley graphs. This result extends the main result of [5] and the analysis of Section 2 is based on the corresponding analysis in [3].

2. The Structure of Disconnected Abelian Cayley Graphs

If Γ is a group and $S \subseteq \Gamma$, then we use the notation $\langle S \rangle$ to denote the subgroup of Γ generated by S. The same notation is used for induced subgraphs: if $G = (V, E)$ is a graph and $W \subseteq V$, then $\langle W \rangle$ denotes the subgraph of G induced by W.

Proposition 1. *If $G = G(\Gamma, S)$ is an abelian Cayley graph with $\Gamma_1 = \langle S \rangle$, then the number of components of G is equal to $|\Gamma|/|\Gamma_1|$.*

Proof: Let $n = |\Gamma|/|\Gamma_1|$. Then $\Gamma = a_1\Gamma_1 \cup a_2\Gamma_1 \cup \cdots \cup a_n\Gamma_1$ and the components of G are the subgraphs of G induced by $a_i\Gamma_1, i = 1, 2, \ldots, n$. ∎

Proposition 2. *Every component of the abelian Cayley graph $G = G(\Gamma, S)$ is isomorphic to the abelian Cayley graph $H = G(\Gamma_1, S)$ where $\Gamma_1 = \langle S \rangle$.*

Proof: The mapping $f : a_i\Gamma_1 \to \Gamma_1$ defined by $f(x) = a_i^{-1}x$ for all $x \in a_i\Gamma_1$ is an isomorphism between $\langle a_i\Gamma_1 \rangle$ and H. ∎

Proposition 3. *Suppose the abelian Cayley graph $G = G(\Gamma, S)$ is connected. Let $g \in S, T = S - \{g, g^{-1}\}$ and $\Gamma_1 = \langle T \rangle$. If the abelian Cayley graph $H = G(\Gamma, T)$ is disconnected, then*

(a) *there is a path in G with exactly one vertex in each component of H*
(b) *the number of components of H is equal to the smallest positive integer n such that $g^n \in \Gamma_1$.*

Proof: We start by noting two preliminary facts:

(1) If $g \in \Gamma_1$, then $\Gamma = \Gamma_1$, and
(2) H is disconnected iff $\Gamma_1 \subset \Gamma$.

From these facts we have that $\Gamma_1 \subset \Gamma$, and hence $g \notin \Gamma_1$. Let $n (\geq 2)$ be the smallest positive integer such that $g^n \in \Gamma_1$. Then one can prove that $\Gamma_1, g\Gamma_1, g^2\Gamma_1, \ldots, g^{n-1}\Gamma_1$ forms a partition of Γ. By the proof of Proposition 1, it follows that the components of H are the subgraphs of H induced by $g^i\Gamma_1$ for $i = 0, 1, \ldots, n-1$. Clearly, $1, g, g^2, \ldots, g^{n-1}$ is a path in G with exactly one vertex in each component of H, while the number of components of H is equal to the smallest positive integer n such that $g^n \in \Gamma_1$. ∎

We are now in a position to obtain a picture of the structure of a connected abelian Cayley graph $G = G(\Gamma, S)$ of which the abelian Cayley graph $H = G(\Gamma, T)$ is disconnected, where $T = S - \{g, g^{-1}\}$. If $\Gamma_1 = \langle T \rangle$, then $H_i = H\langle g^i\Gamma_1 \rangle$ are the components of $H, i = 0, 1, \ldots, n-1$, where n is the smallest positive integer such that $g^n \in \Gamma_1$. The mapping $f_g : \Gamma \to \Gamma$ defined by $f_g(x) = gx$ for each $x \in \Gamma$ is an automorphism of G that maps the vertex set $g^i\Gamma_1$ of each H_i to the vertex set $g^{i+1}\Gamma_1$ of $H_{i+1}, i = 0, \ldots, n-2$. This automorphism also has the property that each vertex x of G is adjacent to its image $f_g(x)$ in the abelian Cayley graph G.

3. The Chromatic Index of an Abelian Cayley Graph

We now determine the chromatic index of an abelian Cayley graph. In addition to the usual notation $C_n, n \geq 3$, for the cycle on n vertices we use the (non-standard) notation C_2 for the complete graph on two vertices.

Proposition 4. *Let $G = G(\Gamma, \{g, g^{-1}\})$ be an abelian Cayley graph with $|\langle g \rangle| = n$. Then G consists of $|\Gamma|/n$ disjoint copies of C_n.*

Proof: By Proposition 2, G consists of $|\Gamma|/n$ disjoint copies of the Cayley graph $G(\langle g \rangle, \{g, g^{-1}\})$ which is clearly the graph C_n. ∎

Proposition 5. *Suppose Γ is an even order abelian group which is generated by $S = \{g_1, \ldots, g_n\}$. Then there is a $g \in S$ such that $|\langle g \rangle|$ is even.*

Proof: We first prove that there is an element $h \in \Gamma$ with order 2: to see this, simply remove the identity 1 and all g for which $g \neq g^{-1}$ from Γ. Since $|\Gamma|$ is even, we eventually arrive at an $h \in \Gamma$ with $h = h^{-1}$, i.e. $h^2 = 1$.

We now prove that there is a $g \in S$ such that $|\langle g\rangle|$ is even: Suppose on the contrary that $|\langle g\rangle|$ is odd for all $g \in S$. Since $h \in \Gamma = \langle S\rangle$, we have integers $a_1, \dots a_n$ such that $h = g_1^{a_1} \dots g_n^{a_n}$. Let $t = \Pi_{i=1}^{n} |\langle g_i\rangle|$ which is clearly odd. Then $h^t = (g_1^{a_1} \dots g_n^{a_n})^t = 1$, so that 2, the order of h in Γ, divides t. Hence t is even, which is a contradiction. ∎

Remark 6. *If G_1 and G_2 are graphs with $V = V(G_1) = V(G_2)$ and $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$, then $\chi'(G_1 + G_2) \leq \chi'(G_1) + \chi'(G_2)$.* ∎

We now state Vizing's Theorem which appeared in 1964 [6,7]. This result was later proved independently by Gupta [4].

Theorem 7. *If G is a graph with maximum degree Δ, then either $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$.* ∎

The following result is due to Beineke and Wilson (see [1]).

Theorem 8. *If G is a regular graph with an odd number of vertices, then $\chi'(G) = \Delta(G) + 1$.* ∎

As an immediate consequence we have

Corollary 9. *If $G = G(\Gamma, S)$ is an abelian Cayley graph of odd order, then $\chi'(G) = \Delta(G) + 1$.* ∎

We are now ready to determine the chromatic index of abelian Cayley graphs in general.

Theorem 10. *If $G = G(\Gamma, S)$ is a connected abelian Cayley graph of even order, then $\chi'(G) = \Delta(G)$.*

Proof: If $|S| \leq 2$ and $G = G(\Gamma, S)$ is a connected abelian Cayley graph of even order, then $\Gamma = \langle g\rangle$. If $|\Gamma| = 2$, then $G = K_2$, so that $\chi'(G) = 1 = \Delta(G)$. If $|\Gamma| \geq 3$, then G is the cycle on $|\Gamma|$ vertices and hence $\chi'(G) = 2 = \Delta(G)$.

To use an inductive proof, let $G = G(\Gamma, S)$ be a connected abelian Cayley graph of even order with $|S| \geq 3$ and suppose the result holds for every connected abelian Cayley graph $G(\Lambda, T)$ of even order with $|T| < |S|$. By Proposition 5, there is a $g \in S$ such that $|\langle g\rangle|$ is even. Let $H = G(\Gamma, T)$, where $T = S - \{g, g^{-1}\}$. If H is connected, it follows, by the induction hypothesis, that $\chi'(H) = \Delta(H)$. Consider two cases:

Case (i): $g = g^{-1}$. In this case $\Delta(H) = \Delta(G) - 1$ and $|\langle g\rangle| = 2$. By Proposition 4, it follows that $G(\Gamma, \{g, g^{-1}\})$ consists of $|\Gamma|/2$ disjoint copies of $C_2 = K_2$, so that $\chi'(G(\Gamma, \{g, g^{-1}\})) = 1$. From Remark 6, it follows that $\chi'(G) = \chi'(H + G(\Gamma, \{g, g^{-1}\})) \leq \chi'(H) + \chi'(G(\Gamma, \{g, g^{-1}\})) = \Delta(H) + 1 = \Delta(G) - 1 + 1 = \Delta(G)$.

Case (ii): $g \neq g^{-1}$. In this case $\Delta(H) = \Delta(G) - 2$ and $|\langle g\rangle| \geq 4$. By Proposition 4, it follows that $G(\Gamma, \{g, g^{-1}\})$ consists of $|\Gamma|/|\langle g\rangle|$ disjoint cycles of length $|\langle g\rangle|$, an even number. Hence $\chi'(G(\Gamma, \{g, g^{-1}\})) = 2$. From Remark 6, it follows that $\chi'(G) = \chi'(H + G(\Gamma, \{g, g^{-1}\})) \leq \chi'(H) + \chi'(G(\Gamma, \{g, g^{-1}\})) = \Delta(H) + 2 = \Delta(G) - 2 + 2 = \Delta(G)$.

By Theorem 7, we have, in both cases, that $\chi'(G) = \Delta(G)$.

Now consider the case where H is disconnected: let $\Gamma_1 = \langle T\rangle$ and let n be the number of components of H. By the proof of Proposition 3, the components of H are the induced subgraphs $H_i = H\langle g^i\Gamma_1\rangle, i = 0, 1, \ldots, n-1$. By Proposition 2, each component $H_i, i = 0, 1, \ldots, n-1$, of H is isomorphic to the connected abelian Cayley graph $G(\Gamma_1, T)$. If $|\Gamma_1|$ is even we have, by the induction hypothesis, $\chi'(G(\Gamma_1, T)) = \Delta(G(\Gamma_1, T))$, so that $\chi'(H) = \Delta(H)$. Therefore $\chi'(G) = \Delta(G)$, as before.

Consider the case where $|\Gamma_1|$ is odd. Note that, in this case, n is an even number. By Corollary 9, we have that $\chi'(H_0) = \Delta(H_0) + 1$. Consider any $(\Delta(H_0) + 1)$-edge-colouring of H_0. Then there is a colour available at each vertex of H_0. Let $i \in \{1, 2, \ldots, n-1\}$. Define $f : \Gamma_1 \to g^i\Gamma_1$ by $f(u) = g^i u$ for every $u \in \Gamma_1$. Then H_0 is isomorphic to H_i under the isomorphism f. Colour the edge $\{g^i u, g^i v\}$ of H_i with the same colour used to colour the edge $\{u, v\}$ of H_0 Then the same colour is available at the vertices $g^i u$ of H_i and u of H_0. Let $i \in \{0, 2, \ldots, n-2\}$. Consider the pair $\langle H_i, H_{i+1}\rangle$. The colour available at both $g^i u$ and $g^{i+1} u$ may now be used to colour all the edges $\{g^i u, g^{i+1} u\}$ of G, $u \in \Gamma_1$. If $g = g^{-1}$, then $n = 2$. Hence $G(\Gamma, \{g\})$ consists of $|\Gamma|/2$ disjoint copies of $C_2 = K_2$, so that in this case the above colouring colours all of the edges of $G(\Gamma, \{g\})$. If $g \neq g^{-1}$, then $|\langle g\rangle| \geq 4$. Hence $G(\Gamma, \{g, g^{-1}\})$ consists of $|\Gamma|/|\langle g\rangle|$ disjoint cycles of length $|\langle g\rangle|$, an even number. But then the above colouring colours half of the edges of $G(\Gamma, \{g, g^{-1}\})$. The other half can then be coloured with an additional colour to obtain the required $\Delta(G)$-edge-colouring of G. ∎

Corollary 11. *Let $G = G(\Gamma, S)$ be an abelian Cayley graph. Then $\chi'(G) = \Delta(G)$ if and only if $|\langle S\rangle|$ is even.*

Proof: Denote the components of G by $G_0, \ldots, G_{n-1}$ where $n = |\Gamma|/|\langle S\rangle|$. By Proposition 2, we have that $G_i \cong G(\langle S\rangle, S)$ for $i = 0, 1, \ldots, n-1$. If $|\langle S\rangle|$ is even, then by Theorem 10, $\chi'(G(\langle S\rangle, S)) = \Delta(G(\langle S\rangle, S))$ and hence $\chi'(G) = \Delta(G)$. If $|\langle S\rangle|$ is odd, then $\chi'(G(\langle S\rangle, S)) = \Delta(G(\langle S\rangle, S)) + 1$ by Corollary 9, so that $\chi'(G) = \Delta(G) + 1$. ∎

The chromatic index of the complete graph K_n has been determined by several authors using a variety of methods (see, for example, [2] and [7]). One can also use Corollary 11 to determine it.

The following result characterizes those abelian Cayley graphs which are 1-factorable.

Corollary 12. *The abelian Cayley graph* $G(\Gamma, S)$ *is 1-factorable if and only if* $|\langle S\rangle|$ *is even.* ∎

4. Concluding Remarks

In this section the main result of [5] is proven using the results of Section 3. Let $\{a_1, \ldots, a_k\} \subseteq \mathbf{Z}_p$ such that $0 < a_1 < \cdots < a_k < (p+1)/2$. Denote $\{a_1, \ldots, a_k, -a_1, \ldots, -a_k\}$ by S. The *circulant graph* $C_p\langle a_1, \ldots, a_k\rangle$ denotes the abelian Cayley graph $G(\mathbf{Z}_p, S)$.

Proposition 13. *Let* $d = \gcd(a_1, \ldots, a_k, p)$. *Then* $|\langle S\rangle| = p/d$.

Proof: Since $d = \gcd(a_1, \ldots, a_k, p)$, there are integers $\alpha_1, \ldots, \alpha_k, \alpha_{k+1}$ such that $d = \alpha_1 a_1 + \cdots + \alpha_k a_k + \alpha_{k+1} p$ and hence $d = \alpha_1 a_1 + \cdots + \alpha_k a_k (\bmod\ p)$ and $d \in \langle S\rangle$. Hence $\{0, d, 2d, \ldots, (p/d - 1)d\} \subseteq \langle S\rangle$.

Let $x \in \langle S\rangle$. Then there are integers $\alpha_1, \ldots, \alpha_k$ such that $x = \alpha_1 a_1 + \cdots + \alpha_k a_k (\bmod\ p)$ and hence $x = \alpha_1 a_1 + \cdots + \alpha_k a_k + np$ for some integer n. Since d divides a_i for $1 \leq i \leq k$ and d divides p, it follows that d divides x, so that $x \in \{0, d, 2d, \ldots, (p/d - 1)d\}$. Hence $\langle S\rangle \subseteq \{0, d, 2d, \ldots, (p/d-1)d\}$. ∎

Theorem 14. *If* $d = \gcd(a_1, \ldots, a_k, p)$, *then* $\chi'(C_p\langle a_1, \ldots, a_k\rangle) = \Delta(C_p\langle a_1, \ldots, a_k\rangle)$ *if and only if* p/d *is even.* ∎

References

1. L.W. Beineke and R.J. Wilson, *On the edge-chromatic number of a graph*, Discrete Math. **5** (1973), 15–20.
2. M. Behzad, G. Chartrand and J.D. Cooper, *The colour numbers of complete graphs*, J. London Math. Soc. **42** (1967), 226–228.
3. I. Broere, *Every connected circulant is Hamiltonian*, Verslagreeks van die Departement Wiskunde, RAU, May (1986).
4. R.P. Gupta, *Studies in the Theory of Graphs, Thesis*, Tata Inst. Fund. Res., Bombay (1967).
5. J.H. Hattingh, *The edge-chromatic number of a circulant*, Quaestiones Math. **11** (1988), 371–381.
6. V.G. Vizing, *On an estimate of the chromatic class of a p-graph (Russian)*, Diskret. Analiz **3** (1964), 25–30.
7. V.G. Vizing, *The chromatic class of a multigraph (Russian)*, Kibernetika, (Kiev) **3** (1965), 29–39.