Minimum Triangle-Free Graphs

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Abstract. We prove that $e(3,k+1,n) \geq 6n - 13k$ where $e(3,k+1,n)$ is the minimum number of edges in any triangle-free graph on n vertices with no independent set of size $k+1$. To achieve this we first characterize all such graphs with exactly $e(3,k+1,n)$ edges for $n \leq 3k$. These results yield some sharp lower bounds for the independence ratio for triangle-free graphs. In particular, the exact value of the minimal independence ratio for graphs with average degree 4 is shown to be 4/13. A slight improvement to the general upper bound for the classical Ramsey $R(3,k)$ numbers is also obtained.

1. Introduction

The study of the minimum number of edges in a triangle-free graph on n vertices with no independent set of size k, $e(3,k,n)$, and the construction of some related minimum graphs led, in particular, to the evaluation of the exact values of classical Ramsey numbers $R(3,6)$ (Kalbfleisch [5]), $R(3,7)$ (Graver and Yackel [3]) and $R(3,9)$ (Grinstead and Roberts [4]). In this paper we pursue further this approach by strengthening our results related to the function $e(3,k,n)$ obtained in [7].

The major progress in the investigation of asymptotics for the classical Ramsey numbers $R(3,k)$ was obtained by Ajtai, Komlós and Szemerédi [1], and later refined by Shearer [9] and Bollobás [2] by finding a good lower bound for the maximal size of independent set in a triangle-free graph with fixed average degree, which implies the best known so far general upper bound [2, p. 296]:

$$R(3,k+1) \leq \frac{(k-1)^2}{\log_e k + 1/k - c} + 1, \quad \text{for } k \geq 3, \text{ where } c = 1. \tag{1}$$

Our results imply that the bound (1) holds for $c = 0.9409\ldots$ for all $k \geq 3$. The technique used in this paper was originated by our previous work reported in [7],[8]. Recently we have learned that Shearer [10, and his private communication] showed that $R(3,k) \leq k^2/(\log_e k - c) + O(k/\log_e k)$ for $c = 0.7665\ldots$.

Section 2 introduces the notation and recalls some previous results. Section 3 develops properties of a class of minimum graphs, which is of particular importance for further sections. Section 4 completes the full characterization of all minimum triangle-free graphs with the average degree not exceeding 10/3. The main

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theorem stating that $e(3, k+1, n) \geq 6n - 13k$ is presented in Section 5. Section 6 relates these results to the independence ratio, in particular the exact value of minimum independence ratio of triangle-free graphs with average degree not exceeding 4 is calculated. Finally, the connection of the latter with Ramsey numbers $R(3, k)$ is presented.

2. Definitions and Previous Work

All the graphs considered in this paper are simple and triangle-free. If v is a vertex of a graph $G = (V, E)$ then $\deg_G(v)$ denotes the degree of vertex v in G, $N_G(v)$ is the neighborhood of v. $n(G)$ and $e(G)$ denote the number of vertices and edges, respectively, in G. Also if $G = (V, E)$ then we define $V(G) = V$ and $E(G) = E$. By a *component* of a graph we will always mean a connected component. If $A, B \subseteq V$ then the set of edges in E with one endpoint in A and the other in B will be denoted by $E_G(A, B)$. The *Z-sum* of vertex v is the number $Z_G(v) = \sum\{\deg_G(x) : x \in N_G(v)\}$. Sometimes we will write $Z_G(v) = d_1 + d_2 + \cdots + d_{\deg_G(v)}$ when the exact specification of the degrees of the vertices in $N(v)$ is needed (in some situations were no confusion arises the subscript denoting graph will be omitted). Any vertex of degree d will be called a d-vertex. $I(G)$ denotes the maximum size of independent set in G and the *independence ratio* $i(G)$ of G is defined as $i(G) = I(G)/n(G)$. If Θ is a class of graphs then the *minimum independence ratio* of Θ is defined by $i(\Theta) = \inf\{i(G) : G \in \Theta\}$. The minimal degree of vertices in a graph G is denoted by $\delta(G)$. We write $G \equiv H$ if graphs G and H are isomorphic. $G + H$ or $\sum\{H_i : i \in I\}$ will denote the disjoint union of graphs G and H or the disjoint union of a family of graphs $\{H_i\}_{i \in I}$, respectively.

A $(3, k, n, e)$ Ramsey graph is a triangle-free graph on n vertices with e edges and no independent set of size k. Similarly, $(3, k)$-, $(3, k, n)$- or $(3, k, n, e)$-graphs are $(3, k, n, e)$ Ramsey graphs for some n and e. $e(3, k, n)$ is the minimum number of edges in any $(3, k, n)$-graph and is defined to be ∞ if no such graph exists. A $(3, k, n, e)$-graph is a *minimum* graph if $e = e(3, k, n)$. We note that for any minimum $(3, k, n)$-graph G we have $I(G) = k - 1$ unless $n < k - 1$, in which case G is formed by n isolated points and $I(G) = n$. Observe that if H is a minimum graph and $H = S + P$, then obviously S and P are also minimum graphs, but the converse is not true in general. In the above context the classical *Ramsey number* $R(3, k)$ can be defined as the smallest nonnegative integer n such that $e(3, k, n) = \infty$. Since the maximal degree in a $(3, k, n)$-graph is at most $k - 1$, then in order to conclude that $R(3, k) \leq n$ it is sufficient to show that $e(3, k, n) > n(k-1)/2$.

If v is a vertex of graph H, then H^v is the graph induced in H by the set of vertices $V(H) - (N(v) \cup \{v\})$. Using the terminology of [3], [4], the graph H^v coincides with the so called graph $H_2(v)$ in H obtained by *preferring* vertex v in H. If v is a d-vertex of a $(3, k+1, n+d+1, e)$-graph H then the graph H^v is a $(3, k, n, e - Z_H(v))$-graph. A vertex v is called *full* in H if H^v is a minimum graph. An operation in some sense inverse to preferring a vertex is that of extension, which is formalized below.

Definition 2.1. *Graph H is a d-extension of a $(3,k)$-graph G, $I(G) = k-1$, if H has a d-vertex v such that $H^v \equiv G$ and H is a $(3,k+1)$-graph.*

To facilitate reading we also include in this section some of the previous results needed later in this paper. The following proposition is a condensation of Theorems 1, 2 and 4 appearing in [7]:

Proposition 2.2.

(a) *For $k \geq 2$*

$$e(3,k+1,n) = \begin{cases} 0 & \text{if } 0 \leq n \leq k, \\ n-k & \text{if } k < n \leq 2k, \\ 3n-5k & \text{if } 2k < n \leq 5k/2. \end{cases}$$

Furthermore, the corresponding minimum graphs are unique and are given by n isolated points for $0 \leq n \leq k$, $2k-n$ isolated points and $n-k$ isolated edges for $k < n \leq 2k$, and $5k-2n$ isolated edges with $n-2k$ pentagons for $2k < n \leq 5k/2$.

(b) *For all $k, n \geq 0$*

$$e(3,k+1,n) \geq 5n - 10k. \tag{2}$$

Furthermore, (2) becomes an equality for $k = 3, n = 8$ and for all n and k such that $k \geq 4$ and $5k/2 \leq n \leq 3k$.

Finally, let us mention an obvious consequence of a technical Lemma 3 in [7].

Proposition 2.3. *If v is a 2-vertex in a minimum graph H and $Z(v) = 2+2$ then the component of H containing v is a pentagon.*

3. Graphs G_k and F_k

The minimum graphs G_k introduced in [7] will be of a particular importance in further sections. They are defined below.

Definition 3.1. *For all $k \geq 4$ define the graph $G_k = (V_k, E_k)$ as follows:*

$$V_k = \{a_x, b_x, c_x : x \in \mathbf{Z}_k\},$$
$$E_k = \{\{c_x, c_{x+1}\}, \{c_x, a_x\}, \{c_x, b_x\}, \{a_x, b_{x+1}\}, \{a_x, b_{x+2}\} : x \in \mathbf{Z}_k\}.$$

The graph G_k has $2k$ vertices of degree 3 and k vertices of degree 4. G_k is drawn in Figure I for $k = 5$.

We have shown in [7] that for all $k \geq 4$ the graph G_k is a minimum $(3, k+1, 3k, 5k)$-graph, $I(G_k) = k$ and it's full automorphism group is isomorphic to the dihedral group D_k. It can be easily seen that all 3-vertices in G_k are equivalent up to symmetry, similarly as are all 4-vertices. This justifies the correctness of the next definition.

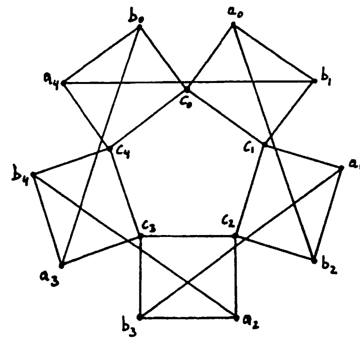

Figure I. Graph G_5.

Definition 3.2. *For all $k \geq 3$ define the graph F_k to be the graph G^v_{k+1}, where v is a 3-vertex of the graph G_{k+1}. Define the graph F_2 to be the pentagon.*

Note that if we would have defined also the graph G_3 by Definition 3.1 (containing one triangle $c_0c_1c_2$) then we will have $F_2 \equiv G_3^{a_2}$. Since $Z_{G_{k+1}}(v) = 10$ for any 3-vertex v in G_{k+1} hence by (2) the graph F_k is a minimum $(3, k+1, 3k-1, 5k-5)$ graph and $I(F_k) = k$ for each $k \geq 2$. In the following we will thus assume that $F_k = G^{a_k}_{k+1}$. Easy examination of the graph G_{k+1} reveals that the graph F_k has $4, 2k-2$ and $k-3$ vertices of degree 2, 3 and 4, respectively, for $k \geq 3$. Also note that F_3 is the well known unique $(3,4,8,10)$-graph [3], [5].

The graph F_k is drawn in Figure II for $k \geq 3$.

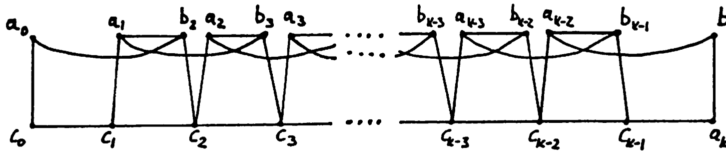

Figure II. Graph F_k.

Proposition 3.3. *The full automorphism group of the graph F_k for $k \geq 3$ is isomorphic to the dihedral group D_4 and is generated by permutations:*

$$\alpha = (a_0 c_0)(c_1 b_2),$$
$$\beta = (a_0 b_k)(c_0 a_{k-1})(c_i c_{k-i})_{1 \leq i < k/2}(a_i b_{k-i})_{1 \leq i \leq k-2}.$$

Furthermore all 2-vertices in F_k are equivalent up to symmetry.

Proof: Let Γ be the full automorphism group of F_k. Observe that α and β are automorphisms of F_k of order 2. One can easily check that the order of $\alpha\beta$ is 4.

Hence Γ contains D_4 as a subgroup. By examination of graph F_k we can see that the pointwise stabilizer in Γ of the set of 2-vertices $\{a_0, c_0, a_{k-1}, b_k\}$ is the identity, so Γ is isomorphic to the restriction of Γ acting on 2-vertices, which is isomorphic to D_4. ∎

Proposition 3.4. *For all $k \geq 2$, F_{k+1} is a 2-extension of F_k.*

Proof: One can easily see that F_3 is a 2-extension of F_2. For $k \geq 3$ using Proposition 3.3 we can check that $F_k \equiv F_{k+1}^x$ if x is any 2-vertex in F_{k+1}. ∎

Proposition 3.5. *Let H be one of the graphs F_k or G_k. Then for any vertex $v \in V(H)$ there exists a k-independent set S in H not containing v.*

Proof: The pentagon F_2 obviously has the required property. If $H = G_k$ then let $S_1 = \{a_i : 0 \leq i < k\}$ and $S_2 = \{b_i : 0 \leq i < k\}$. Note that S_1 and S_2 are disjoint k-independent sets in G_k, hence at least one of them misses any given vertex v. Similarly, if $H = F_k$ for some $k \geq 3$ then consider S_1 as before and $S_2 = \{c_0\} \cup \{b_i : 2 \leq i \leq k\}$. Again S_1 and S_2 are disjoint k-independent sets in F_k, thus proposition follows. ∎

4. Minimum Graphs

4.1 Class Φ

Let Φ be the set of all nonempty minimum $(3, k+1, n, 5n-10k)$-graphs and observe that $F_i, G_j \in \Phi$ for $i \geq 2, j \geq 4$. Our first lemma below says, in particular, that any graph H, whose each component is formed by graph F_i or G_i for some i, is a member of Φ. The goal of Section 4 is to show that all the graphs in Φ have this property (Theorem 4.3.1) and consequently $n \leq 3k$ for any $(3, k+1, n)$-graph in Φ.

Lemma 4.1.1. *Let H be a disjoint union of nonempty graphs S and P. Then $H \in \Phi$ if and only if $S \in \Phi$ and $P \in \Phi$.*

Proof: First assume that $H \in \Phi$, so H is a $(3, k+1, n, 5n-10k)$-graph. Since any component of a minimum graph is a minimum graph, we can assume that S is a minimum $(3, k_1+1, n_1, e_1)$-graph and P is a minimum $(3, k_2+1, n_2, e_2)$-graph, and $k_1 + k_2 = k$. Then by (2) and by the fact that $H = S + P$ we have $5n - 10k = e_1 + e_2 \geq 5(n_1 + n_2) - 10(k_1 + k_2)$. Furthermore we must have $e_1 = 5n_1 - 10k_1$ and $e_2 = 5n_2 - 10k_2$, so $S, P \in \Phi$ as claimed. Conversely, assume that $S, P \in \Phi$. Then, using the same notation, we have $e_1 = 5n_1 - 10k_1$, $e_2 = 5n_2 - 10k_2$ and H is a $(3, k+1, n, 5n-10k)$-graph, where $k = k_1 + k_2$ and $n = n_1 + n_2$. Thus by (2) $H \in \Phi$. ∎

The restriction in Lemma 4.1.1 to the set Φ is essential since in general a disjoint union of minimum graphs is not a minimum graph. For example, if S is an isolated point and P is a pentagon then by Proposition 2.2(a) S and P are minimum graphs and $S + P$ is a (3,4,6,5)-graph, but $e(3,4,6)=3$.

Lemma 4.1.2. *If* $H = (V, E) \in \Phi$, *then*

(a) *for all* $v \in V, Z(v) \leq 5\deg(v) - 5$;

(b) *for all* $v \in V, \deg(v) \geq 2$;

(c) *if* $\deg(v) = 2$, *then* $Z(v) = 2+3$ *or the component of* H *containing* v *is a pentagon;*

(d) *if* $\delta(H) \geq 3$, *then for some* $v \in V, \deg(v) = 3$ *and* $Z(v) = 3+3+4$.

Proof: Let H be a minimum $(3, k+1, n, 5n-10k)$-graph and v be an r-vertex in H. Then H^v is a $(3, k, n-r-1)$-graph and by (2) $e(H^v) \geq 5n-10k-5r+5$. Counting the edges of H we have $e(H^v) + Z(v) = 5n-10k$ and consequently (a) holds. Now (b) follows since (a) and $Z(v) \geq r$ imply $r \geq 2$. For (c) assume that $r = 2$. Using (b) we note that $Z(v) \geq 4$, so by (a) $Z(v) = 4$ or 5. If $Z(v) = 4$ then by Proposition 2.3 the component of H containing v is a pentagon, otherwise $Z(v) = 2+3$ and (c) follows. To prove (d) suppose that $\delta(H) \geq 3$ and observe that $\delta(H) = 3$ since otherwise if we set $r = \delta(H) \geq 4$ then $Z(v) \geq r^2$ contradicts (a). Hence the graph H must have some 3-vertices, and by (a) for any 3-vertex v in H, $9 \leq Z(v) \leq 10$; so $Z(v) = 3+3+3$ or $Z(v) = 3+3+4$. Consequently, in order to complete the proof of (d) it is sufficient to show that H cannot have a cubic component. Assume that S is a cubic component of H. By Lemma 4.1.1, $S \in \Phi$; furthermore S is a $(3, k+1, n, 3n/2)$-graph and $3n/2 = 5n-10k$ for some integers k and n. Thus the independence ratio of S is $i(S) = k/n = 7/20$. On the other hand Staton [11] proved that $i(\Theta) = 5/14$ if Θ is the class of triangle-free cubic graph. Therefore S cannot exist since $7/20 < 5/14$. ∎

4.2. Minimum Extensions

The following two lemmas establish the minimum graphs in Φ which can be obtained as an extension of G_k or F_k.

Lemma 4.2.1. *No graph in* Φ *can be a d-extension of the graph* G_k *for* $d \geq 0$ *and* $k \geq 4$.

Proof: Assume that H is a d-extension of G_k, $H \in \Phi$ and $H^v = G_k$. If $x \in N(v)$ then by counting degrees in $N(x)$ and by Lemma 4.1.2(a) we have

$$d + (\deg(x) - 1)(\delta(G_k) + 1) \leq Z(x) \leq 5\deg(x) - 5. \qquad (3)$$

Note that $\delta(G_k) = 3$, hence (3) gives $\deg(x) \geq d+1$. Now similarly, by counting degrees in $N(v)$ and Lemma 4.1.2(a) we obtain

$$d(d+1) \leq Z(v) \leq 5d-5,$$

which is a contradiction. ∎

Lemma 4.2.2.

(a) *For all* $k \geq 2$ *the graph* F_k *has a unique* 2*-extension in* Φ *and it is isomorphic to* F_{k+1}.

(b) F_2 *has no* 3*-extension. For all* $k \geq 3$ *the graph* F_k *has a unique* 3*-extension in* Φ *and it is isomorphic to* G_{k+1}.

Proof: (a) F_3 is the unique (3,4,8,10)-graph, so by Proposition 3.4 it is the unique 2-extension of F_2. Let H be a 2-extension of F_k for some $k \geq 3$, $H \in \Phi$, $H^v = F_k$ and $\deg(v) = 2$. Note that H must be connected since by Lemma 4.1.2(b) $\deg(x) \geq 2$ for $x \in N(v)$ and F_k is connected. Thus by Lemma 4.1.2(c) we obtain $Z(x) = 2+3$ for any 2-vertex x in H. Let $N(v) = \{s,t\}$, $\deg(s) = 2$ and $\deg(t) = 3$. Since $Z(s) = 2+3$, s is connected to some 2-vertex in H^v and by Proposition 3.3 we can assume, without loss of generality, that $\{s, a_{k-1}\} \in E(H)$ (see Figure III).

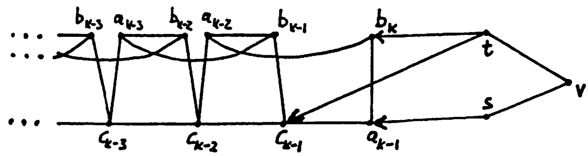

Figure III. Graph H from Lemma 4.2.2(a).

Now t must be connected to two vertices in H^v. One of them is b_k because otherwise b_k would be a 2-vertex in H with $Z(b_k) \geq 6$. One still missing edge connects t to some vertex u in $V(H^v) - \{a_{k-1}, b_k\}$. Let J be the graph H with the edge $\{t, u\}$ deleted. Observe that any vertex x in the set

$$S = \{a_0, c_0\} \cup \{a_i : 2 \leq i \leq k-2\} \cup \{b_i : 3 \leq i \leq k-2\}$$

satisfies $Z_J(x) = 5\deg_J(x) - 5$, so it is full in J, and this by Lemma 4.1.2(a) implies that u cannot be a neighbor of any vertex in S. Furthermore $u \neq a_{k-2}$ since we have to avoid the triangle $b_k a_{k-2} t$. The only possibilities left are $u = c_{k-1}$ for $k \geq 3$ and $u = a_1$ in the case $k = 4$. The latter case $u = a_1$ can be discarded since then the set $\{t, s, a_2, b_2, c_1, c_3\}$ would be a 6-independent set in H contradicting the fact that H is an extension of F_4. So $u = c_{k-1}$ and the graph H must be as drawn in Figure III. Finally observe that $H \equiv F_{k+1}$ since, after renaming $v \rightarrow a_k$, $s \rightarrow b_{k+1}$, $t \rightarrow c_k$, graph H is identical to F_{k+1}. This completes the proof of part (a).

(b) A 3-extension of F_2 would be a (3,4,9)-graph, but such a graph does not exist since $R(3,4) = 9$. It is known [7] that there are unique $(3, k+1, 3k, 5k)$-graphs for $k = 4,5,6,7$ and they are isomorphic to the corresponding graph G_k. Hence, by Definition 3.2, Lemma 4.2.2(b) holds for $k \leq 6$. Let H be a 3-extension of F_k,

$H \in \Phi$ and $H^v = F_k$, for some $k \geq 7$. We have $Z(v) = e(H) - e(F_k) = 10$. Similarly as in (3), for $x \in N(v)$ we have

$$3 + (\deg(x) - 1)(\delta(F_k) + 1) \leq 5 \deg(x) - 5$$

and $\delta(F_k) = 2$, which implies $\deg(x) \geq 3$ and consequently $Z(v) = 3 + 3 + 4$. Let $N(v) = \{a, b, c\}$, $\deg(a) = 4$ and $\deg(b) = \deg(c) = 3$ (see Figure IV). In the graph H vertex a has three neighbors in H^v, say t_1, t_2, t_3, similarly let s_1, s_2 and u_1, u_2 be the vertices in H^v connected to b and c, respectively. Denote the sets of these vertices by T, S and U, respectively. The remaining portion of the proof consists of showing that the sets T, S and U are determined uniquely (up to symmetry) and that this situation yields $H \equiv G_{k+1}$.

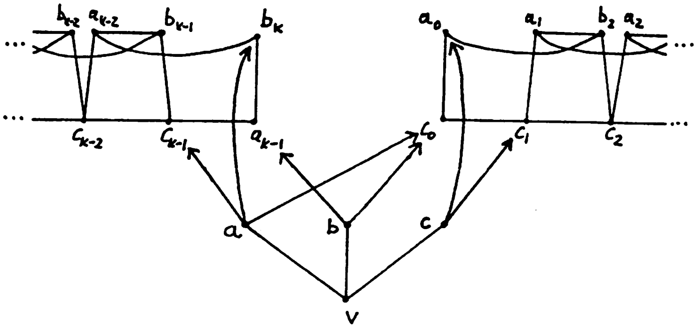

Figure IV. Graph H from Lemma 4.2.2(b).

Put $X = T \cup S \cup U$, so we have $|X| \leq 7$. Define Y to be the set of full vertices in $H^v = F_k$. Observe that $Y = Y_1 \cup Y_2$, where $Y_1 = \{a_0, c_0, a_{k-1}, b_k\}$ is the set of 2-vertices in F_k and $Y_2 = \{a_i : 2 \leq i \leq k-3\} \cup \{b_i : 3 \leq i \leq k-2\}$, hence we have $|Y_2| = 2(k-4) \geq 6$. Similarly as in part (a) we note that if $x \in X$ and $\{x, y\} \in E(H^v)$ for some full vertex y then $y \in X$. We will use repeatedly this property while determining the set X. First note that $Z(b) \leq 10$ implies that at least one vertex in S is a 2-vertex of F_k. Thus by Proposition 3.3, without loss of generality, we have $s_1 \in \{a_0, c_0\}$. But $\{a_0, c_0\} \subseteq Y$ and $\{a_0, c_0\} \in E(H^v)$, hence $\{a_0, c_0\} \subseteq X$ and $|X \cap Y_1| \geq 2$. The vertices in Y_2 form a path in F_k, consequently $X \cap Y_2 \neq \emptyset$ implies $Y_2 \subseteq X$, and therefore $X \cap Y_2 = \emptyset$ since $|Y_2| \geq 6$. Now similarly, for each $x \in X$ and $y \in Y_2$ we have $\{x, y\} \notin E(H)$. So

$$X \subseteq Y_1 \cup \{c_1, b_2, c_{k-1}, a_{k-2}\}.$$

Note that $\{c_1, b_2\} \subseteq X$ or $\{c_{k-1}, a_{k-2}\} \subseteq X$ implies that $a_1 \in X$ or $b_{k-1} \in X$, respectively, which as before implies that $Y_2 \subseteq X$. Thus using Proposition 3.3, without loss of generality, we can assume that

$$X \subseteq A = Y_1 \cup \{c_1, c_{k-1}\}.$$

We have $|A| = 6$, hence there exists $y \in A$ which is an endpoint of at least two edges in $E(N(v), E(H^v))$, and call any such y a repeated vertex. If c_1 or c_{k-1} is a repeated vertex then $a_1 \in X$ or $b_{k-1} \in X$, respectively. Similarly, if a_0 or b_k is a repeated vertex then $b_2 \in X$ or $a_{k-2} \in X$, respectively. Thus any repeated vertex is one of c_0, a_{k-1}. If both of them are repeated then we can conclude that $A \subseteq X$, which leads to a contradiction $|X| = |A| + 2 \leq 7$. Using once more the symmetry of F_k let c_0 be the only repeated vertex and $X = A$. Assume that c_0 is a common vertex of S and U. Consider the graph $J = (H^b)^c$. Observe that $n(J) + 6 = n(H)$ and $e(J) \leq e(H) - 12$ since we can easily count that there are at least 12 edges in $E(H) - E(J)$. On the other hand J must be a $(3, k)$-graph since any k-independent set in J could be extended by b and c to a $(k+2)$-independent set in H. Using (2) and $n(H) = 4 + n(F_k)$ we obtain a contradiction as follows

$$e(H) - 12 \geq e(J) \geq 5(n(H) - 6) - 10(k-1) = 5k - 5 = e(H) - 10.$$

Hence we have $S \cap U = \emptyset$, which implies that $c_0 \in T$. Finally, considering that T, S and U are independent sets in H we arrive, up to symmetry, to the unique possibility

$$T = \{c_0, c_k - 1, b_k\}, \quad S = \{c_0, a_{k-1}\}, \quad U = \{a_0, c_1\}$$

and the graph H must be as drawn in Figure IV. To complete the proof one can check that, after renaming $v \to a_k$, $a \to c_k$, $b \to b_0$, $c \to b_1$, the obtained graph H is identical to G_{k+1}. ∎

4.3. Characterization Theorem

Theorem 4.3.1. *Let H be a $(3, k+1, n)$-graph with $I(H) = k$. Then $H \in \Phi$ if and only if $H = \sum_{i \in I} F(i) + \sum_{j \in J} G(j)$, where I and J are multisets of integers satisfying:*

(a) *if $i \in I$ then $i \geq 2$, if $j \in J$ then $j \geq 4$,*
(b) $\sum_{i \in I} i + \sum_{j \in J} j = k$ *and* $3k - |I| = n$.

Proof: Assume first that H is as specified on the right side of the equivalence. Using properties of graphs F_i and G_j listed in Section 3 and simple arithmetic we conclude that $\sum_{i \in I}(3i - 1) + \sum_{j \in J} 3j = 3k - |I| = n$ and $\sum_{i \in I}(5i - 5) + \sum_{j \in J} 5j = 5n - 10k$, hence by (2) and Lemma 4.1.1 Φ contains all the graphs specified by the right side of Theorem 4.3.1. Conversely, assume that $H \in \Phi$. By Lemma 4.1.1 each component of H is a member of Φ. So it is sufficient to show that any connected graph in Φ is isomorphic to F_i or G_j, since simple calculation of parameters of graphs proves that (a) and (b) hold for any disjoint union of F_i's and G_j's. Let $H \in \Phi$ be a connected minimum $(3, k+1, n)$-graph. We will prove by induction on k that $H \equiv F_k$ or $H \equiv G_k$. For $k \leq 2$ the only graph of consideration is a pentagon, which is F_2. In the general case we consider two possibilities: $\delta(H) = 2$ or $\delta(H) = 3$. Note that by Lemma 4.1.2 one of them must occur.

If $\delta(H) = 2$ and v is a 2-vertex, then by Lemma 4.1.2(c) H is a pentagon or $Z(v) = 2 + 3$. In the first case we are done, so $Z(v) = 2 + 3$, v is a full vertex in H, $H^v \in \Phi$ and H is a 2-extension of H^v. Note also that there are exactly three edges in the set $Y = E_H(N(v), V(H^v))$. If H^v is connected then by induction and Lemmas 4.2.1 and 4.2.2 $H \equiv F_k$. Thus to complete the case $\delta(H) = 2$ it is sufficient to show that H^v must be connected. By induction we know that each component S of H^v is an F_i or G_j. If H^v has more than one component then, since $|Y| = 3$, there is a component S and exactly one edge $f = \{x, y\} \in Y$ such that $y \in V(S)$. Now by Proposition 3.5 there is a maximum independent set in S missing y, hence the removal of the edge f from H does not increase $I(H)$, and consequently H is not a minimum graph. This contradicts the fact that $H \in \Phi$.

If $\delta(H) = 3$, then by Lemma 4.1.2(d) there is a 3-vertex v in H such that $Z(v) = 3 + 3 + 4$, v is a full vertex in H, $H^v \in \Phi$ and H is a 3-extension of H^v. Note also that there are exactly 7 edges in the set $Y = E_H(N(v), V(H^v))$. If H^v is connected then by induction and Lemmas 4.2.1 and 4.2.2 $H \equiv G_k$. Note that any component $S \equiv F_i$ of H^v contributes at least 4 edges to Y since $\delta(H) = 3$ and F_i has 4 or 5 2-vertices. Observe that $2j$ 3-vertices of G_j form a cycle of full vertices in G_j and any vertex in G_j has some 3-vertex as a neighbor. Hence any edge in Y with an endpoint in G_j implies that there are at least $2j$ such edges in Y. Thus any component $S \equiv G_j$ of H^v contributes at least 8 edges to Y. But $|Y| = 7$, so H^v must be connected. This completes the proof of the theorem. ∎

Table I below presents all minimum $(3, k+1)$-graphs in Φ with $k \leq 8$. In Table I, $S_1 S_2 \ldots S_i$ denotes a disjoint union of graphs S_j, $1 \leq j \leq i$.

Corollary 4.3.2. *For $5k/2 \leq n \leq 3k$, H is a minimum $(3, k+1, n)$-graph if and only if H is a disjoint union of F_i's and G_j's.*

Proof: This is obvious from Proposition 2.2(b), Theorem 4.3.1 and from the properties of parameters of graphs F_i and G_j. ∎

Let H_{13} be the graph defined on the vertex set $\mathbf{Z}_{13}$ by joining with an edge vertices i and j if and only if $i - j$ is a cube in $\mathbf{Z}_{13}$. It is known that H_{13} is the unique up to isomorphism (3,5,13,26)-graph [5], [7]. We note that H_{13} is a 4-regular minimum graph and $H_{13}^v \equiv F_3$ for any vertex $v \in \mathbf{Z}_{13}$.

k	graph parameters	$n=3k-4$ $e=5k-20$	$n=3k-3$ $e=5k-15$	$n=3k-2$ $e=5k-10$	$n=3k-1$ $e=5k-5$	$n=3k$ $e=5k$
2	$(3,3,n)$				F_2	none
3	$(3,4,n)$				F_3	none
4	$(3,5,n)$			F_2F_2	F_4	G_4
5	$(3,6,n)$			F_2F_3	F_5	G_5
6	$(3,7,n)$		$F_2F_2F_2$	F_3F_3 F_2F_4	F_6 F_2G_4	G_6
7	$(3,8,n)$		$F_2F_2F_3$	F_3F_4 F_2F_5	F_7 F_3G_4 F_2G_5	G_7
8	$(3,9,n)$	$F_2F_2F_2F_2$	$F_2F_3F_3$ $F_2F_2F_4$	F_4F_4 F_3F_5 F_2F_6 $F_2F_2G_4$	F_8 F_4G_4 F_3G_5 F_2G_6	G_8 G_4G_4

Table I. Minimum $(3,k+1,n)$-graphs for $5k/2 \leq n \leq 3k, 2 \leq k \leq 8$.

Corollary 4.3.3.

(a) $e(3,k+1,3k+i) > 5(k+i)$ *for all* $i > 0$,

(b) $e(3,k+1,3k+1) = 5k+6$ *for all* $k \geq 8$.

Proof: Assume that $e(3,k+1,3k+i) \leq 5(k+i)$ for some $i > 0$. Then by (2) equality must hold and there exists a minimum $(3,k+1,3k+i,5k+5i)$-graph, which belongs to Φ. This contradicts with Theorem 4.3.1 and Corollary 4.3.2 considered simultaneously, hence (a) follows. The graph $H_{13} + G_{k-4}$ is a $(3,k+1,3k+1,5k+6)$-graph for $k \geq 8$, so (b) follows by (a) with $i = 1$. ∎

We note that Corollary 4.3.2 solves the characterization problem stated in [7] and Corollary 4.3.3(b) answers to a question given in [7] after Corollary 3 there. Another observation is that by Lemma 4.2.2 and Theorem 4.3.1 all connected graphs in Φ can be obtained by a sequence of 2-extensions and at most one 3-extension of an isolated edge, since the pentagon is a 2-extension of an edge.

Finally let us interpret Theorem 4.3.1 in terms of the average degree. Let G be a $(3,k+1,n,e)$-graph with average degree $d \leq 10/3$. Then $e = nd/2 \geq 5n-10k$ implies $n \leq 3k$ and consequently any minimum graph with an average degree not exceeding 10/3 is specified by Proposition 2.2(a) or Corollary 4.3.2. In particular, if $2 \leq d \leq 10/3$ then G is a minimum graph if and only if $i(G) = 1/2 - d/20$. Further discussion of the relation between average degree and independence ratio will be given in Section 6.

5. Main Theorem

5.1 The Theorem and Initial Properties

Theorem 5.1.1. *For all* $k, n \geq 0$

$$e(3, k+1, n) \geq 6n - 13k. \tag{4}$$

In order to present our proof of Theorem 5.1.1, which is technically complicated, we first develop some properties of graphs which meet exactly the bound (4). We start with introduction of a class of minimum graphs Ψ, whose idea is basically the same as of Φ. A general approach relating this kind of class to the independence ratio and some further properties of minimum triangle-free graphs will be presented in Section 6.

Definition 5.1.2. *Let* k_0 *be the largest integer (or* ∞*) such that (4) is true for all* $0 \leq k < k_0$ *and all* $n \geq 0$. *Define* Ψ *to be the class of minimum* $(3, k+1, n, 6n-13k)$*-graphs such that* $0 \leq k < k_0$.

Definition 5.1.2 clearly reflects the fact that we will use induction on k to prove (4). We say that a *smallest counterexample* to (4) is a minimum $(3, k_0+1, n, e)$-graph if we have $e < 6n-13k_0$. Denote by Λ the set of smallest counterexamples to (4) ($\Lambda = \emptyset$ if $k_0 = \infty$).

Assume for a moment that (4) holds for all $n, k \geq 0$, *i.e.* $k_0 = \infty$ is assumed to be true in this paragraph. Observe that by comparing (2) and (4) we can conclude that $\Psi \cap \Phi$ contains exactly all the $(3, k+1, 3k, 5k)$-graphs, thus by Theorem 4.3.1(b) these are the graphs whose all components are isomorphic to a G_j. Note also that $H_{13} \in \Psi$ and by Corollary 4.3.3(b) a disjoint union of H_{13} and any $(3, k+1, 3k, 5k)$-graph is a member of Ψ. Finally observe that when Theorem 5.1.1 is proved we will be able to say that Ψ is the class of all minimum $(3, k+1, n, 6n-13k)$-graphs.

The sequence of propositions and lemmas in Section 5 will impose various conditions which must be satisfied by graphs in Ψ or Λ. At the end we will be able to conclude that $\Lambda = \emptyset$ and $k_0 = \infty$, consequently proving Theorem 5.1.1.

Proposition 5.1.3.

(a) *If* G *is a minimum* $(3, k+1, n)$*-graph and* $G \in \Lambda$, *then* $k \geq 8$ *and* $n \geq 3k+2$.

(b) *If* $G \in \Lambda$, *then* G *is connected.*

(c) $H_{13}, G_4, G_5, G_6 \in \Psi$.

Proof: Check that the values and the bounds of $e(3, k+1, n)$ calculated in [3], [4], [7], [8] satisfy (4) for $k \leq 7$. From Corollary 4.3.3(b) follows that $n \geq 3k+2$, thus (a) holds. If G violates (4) and $G = S + P$, where S and P are nonempty graphs, then it is easy to see that S or P has to violate (4) for some $k < k_0$, so (b) follows. (c) is obvious from (a) and the fact that the specified graphs meet the bound (4) exactly. ∎

Proposition 5.1.4. *If $G \in \Psi \cup \Lambda$ is a minimum $(3, k+1, n)$-graph, then:*

(a) $n \geq 3k$,

(b) *if $H = G^v$ for some $v \in V(G)$, then $e(H) \geq 6n(H) - 13I(H)$, furthermore the equality holds if and only if $H \in \Psi$,*

(c) *if $G \in \Psi$, then for all $v \in V(G)$, $Z(v) \leq 6\deg(v) - 7$.*

Proof: (a) is obvious since $6n - 13k < 5n - 10k$ for $n < 3k$. To prove the inequality in (b) note that the parameters of H are smaller than those of G and use the definition of Ψ. Also by the definition of Ψ, for $G^v = H$ and $G \in \Lambda$, $H \in \Psi$ if and only if $e(H) = 6n(H) - 13I(H)$, hence (b) holds. Using (b) observe that $e(G^v) \geq 6n - 13k - 6\deg(v) + 7$. Counting the edges of G we have $e(G^v) + Z(v) = 6n - 13k$ and consequently (c) holds. ∎

Proposition 5.1.4 will be used many times in the remaining portion of this section, sometimes even without specific reference to it.

Lemma 5.1.5. *If $G \in \Psi$, then $\delta(G) \geq 3$.*

Proof: We use induction on $k = I(G)$. By examination of all the values of $e(3, k+1, n)$ for $k \leq 6$ [7], using Theorem 4.3.1 and Corollary 4.3.3 observe that only the graphs listed in Proposition 5.1.3(c) are members of Ψ with $k \leq 6$. Since $\delta(G_i) = 3$ and $\delta(H_{13}) = 4$ lemma holds for $k \leq 6$. Let G be a minimum $(3, k+1, n)$-graph in Ψ and let v be a d-vertex in G for some $k \geq 7$. Since $Z(v) \geq d$, by Proposition 5.1.4(c) we obtain $d \geq 2$ and $\delta(G) \geq 2$. Assume that $d = 2$. By applying once more Proposition 5.1.4(c) and using $Z(v) \geq d\delta(G)$ we have $Z(v) = 2 + 2$ or $Z(v) = 2 + 3$. If $Z(v) = 2 + 2$, then by Proposition 2.3 $G = S + P$ where S is a pentagon and P is a minimum $(3, k-1, n-5, 6n-13k-5)$-graph. But $k < k_0$, so by (4) we must have $6(n-5) - 13(k-2) \leq 6n - 13k - 5$, which is a contradiction. If $Z(v) = 2 + 3$, then there is a 2-vertex w in G such that $\{v, w\} \in E(G)$, furthermore by Proposition 5.1.4(b) $G^v \in \Psi$. By the same argument as before $Z(w) = 2 + 3$, hence the other neighbor of w must be some 3-vertex t in G. Note that t would be an i-vertex in G^v for some $i \leq 2$ and this contradicts the inductive assumption that $\delta(G^v) \geq 3$. ∎

Lemma 5.1.6. *If $G \in \Lambda$, then G is a connected 4-regular graph.*

Proof: Let G be a $(3, k+1, n, e)$-graph, $k = k_0$, $e < 6n - 13k$ and let v be an arbitrary d-vertex in G. We have $Z(v) \geq d$ and similarly as in the proof of Proposition 5.1.4(c) we obtain

$$Z(v) \leq 6d - 8, \tag{5}$$

which yields $d \geq 2$. If $d = 2$, then $Z(v) = 2 + 2$, and similarly as in the proof of Lemma 5.1.5, $G = F_2 + P$ which leads to a contradiction with (4) applied to graph P. Thus $\delta(G) \geq 3$. Recall that all the minimum graphs with average degree not exceeding 10/3 were characterized at the end of Section 4 and obviously G cannot be one of them. This implies that G must have a vertex of degree at least 4. Assume that $\delta(G) = 3$. Then (5) implies that $Z(x) = 3+3+3$ or $Z(x) = 3+3+4$ for any 3-vertex x in G. Note that by Proposition 5.1.3(b) G is connected, in particular G cannot have a cubic component. Consequently G must have a 3-vertex v with $Z(v) = 3+3+4$, furthermore v is full in G, so $G^v \in \Psi$. Let w be one of the two 3-vertices in G connected to v. Since $Z(w) \leq 10$ there must exist another 3-vertex t in G connected to w. By observing that t would be an i-vertex in G^v for some $i \leq 2$ we have a contradiction with Lemma 5.1.5 applied to G^v. Thus $\delta(G) \geq 4$. Choose a vertex v such that $d = \deg(v) = \delta(G)$. Then $Z(v) \geq d^2$ and (5) imply that $\delta(G) = 4$ and $Z(v) = 4+4+4+4$. Finally using again Proposition 5.1.3(b) we can conclude that G must be a connected regular graph of degree 4. ∎

Proposition 5.1.7. *If $G \in \Lambda$, then all the vertices in G are full, $G^v \in \Psi$ for any vertex $v \in V(G)$, and $e(G) = 6n(G) - 13k_0 - 1$. Furthermore $k_0 \geq 11$ and $n(G) \geq 36$.*

Proof: Let G be a minimum $(3, k+1, n, e)$-graph in Λ, $k = k_0$. By Lemma 5.1.6 G is 4-regular, thus for any vertex v, $Z(v) = 16$. Then G^v is a $(3, k, n-5, e-16)$-graph and, since $k-1 < k_0$, by (4) we obtain $e \geq 6n - 13k - 1$. On the other hand since $G \in \Lambda$ we have $e < 6n - 13k$, so consequently $e = 6n - 13k - 1$, $G^v \in \Psi$ and thus v is full in G. Since by Lemma 5.1.6 G is 4-regular we have $6n - 13k - 1 = 2n$ and the smallest integer solution to this equation with $k \geq 8$ (Proposition 5.1.3(a)) is $k = 11$ and $n = 36$. ∎

Proposition 5.1.8. *If $G \in \Lambda$, then G has no 4-cycle.*

Proof: Assume that $abcd$ is a 4-cycle in $G \in \Lambda$ and $\{a, c\} \notin E(G)$. By Proposition 5.1.7 $G^a \in \Psi$ and it is easy to see that c is an i-vertex in G^a for some $i \leq 2$ since by Lemma 5.1.6 G is 4-regular. This contradicts Lemma 5.1.5 applied to the graph G^a. ∎

5.2. Pentagons

In a few of the next lemmas we will prefer more than one vertex at a time, so we need a generalization of the technique used so far. Let S be an independent set in a graph G. If S has only one vertex v then define G^S as G^v, if $S = R \cup \{v\}$ then G^S is defined inductively by $G^S = (G^v)^R$. The *Z-sum* of the set S in G is the number of edges in G adjacent to a neighbor of some vertex in S. Note that for triangle free graphs this is a generalization of the Z-sum defined for vertices. By the neighborhood of the set S in G we will mean the set of vertices $N_G(S) = \bigcup\{N_G(v) : v \in S\}$. The *support* of the set S is defined as the graph induced in G by vertices in $N(S) \cup S$ and this induced graph will be denoted by $\sup_G(S)$. In all the definitions from this paragraph the subscript G will be omitted if no confusion arises. The following proposition gives some basic properties of the concepts introduced above. It's proof is omitted, but can be obtained directly from definitions.

Proposition 5.2.1. *If G is a $(3, k+1, n, e)$-graph and S is a t-independent set in G, then:*

(a) *G^S is a $(3, k-t+1, n-n(\sup(S)), e-Z(S))$-graph;*
(b) *$Z(S) = e-e(G^S) = \sum_{x \in N(S) \cup S} \deg_G(x) - e(\sup(S))$.*

We can now continue to investigate the class of smallest counterexamples to Theorem 5.1.1 using the tools just introduced.

Lemma 5.2.2. *If $G \in \Lambda$, then in G there are*

(a) *at least 1 pentagon passing through each path of length 2,*
(b) *at least 3 pentagons passing through each edge and*
(c) *at least 6 pentagons passing through each vertex.*

Proof: Let G be a $(3, k+1, n, e)$-graph in Λ. By Proposition 5.1.7 we have $e = 6n - 13k - 1$ and by Lemma 5.1.6 G is a 4-regular graph. Let v and u be any two vertices with a common neighbor, say t, and let R denote the graph $\sup_G(\{v, u\})$. Note that by Proposition 5.1.8 $N(v) \cap N(u) = \{t\}$, which implies that $n(R) = 9$. Let $N(v) = \{t, v_1, v_2, v_3\}$ and $N(u) = \{t, u_1, u_2, u_3\}$. Since G is a smallest counterexample to (4), we can apply (4) and Proposition 5.2.1(a) to the graph $G^{\{v,u\}}$, and this, using also Proposition 5.2.1(b), yields

$$e - Z(\{v, u\}) = (6n - 13k - 1) - (9 \cdot 4 - e(R)) \geq 6(n-9) - 13(k-2). \quad (6)$$

Whence $e(R) \geq 9$. The graph R has 8 edges adjacent to v or u, thus since there are no triangles, R must have at least one edge in the set $E_R(\{v_1, v_2, v_3\}, \{u_1, u_2, u_3\})$. Any such edge gives a pentagon $vtuu_iv_j$ and furthermore this reasoning is valid for any path of length 2 vtu in G, so (a) holds. (b) and (c) are easy consequences of (a) and the fact that G is 4-regular. ∎

The main goal of this section is to establish a result saying that in Lemma 5.2.2 "at least" can be substituted by "exactly". To achieve this we will investigate properties of graphs $H = G^v$ when $G \in \Lambda$. In Lemmas 5.2.[3–9] and 5.3.1 H will always denote such a graph and J will denote the subgraph of H induced by it's 3-vertices. By Lemma 5.1.6 and Proposition 5.1.8 H has 12 3-vertices and $(n(H) - 12)$ 4-vertices. By Proposition 5.1.7 $H \in \Psi$, so it is a minimum $(3, k+1, n, e)$-graph, where $e = 6n - 13k$. The symbols for parameters k, n and e of H will be also fixed in the same scope as H and J. Note that J is a graph on 12 vertices and the modification of Lemma 5.2.2 mentioned above is equivalent to Lemma 5.2.2 together with the statement that J is formed by 6 isolated edges.

Lemma 5.2.3.

(a) *J has no cycles of length* $i \leq 5$.
(b) *J has no isolated points.*
(c) *Any 1-vertex in* J *is an endpoint of an isolated edge in* J.
(d) *Any component* S *of* J *is an isolated edge or* S *has only 2- and/or 3-vertices.*

Proof: J is a subgraph of H, which is a triangle-free subgraph of a smallest counterexample $G \in \Lambda$. Let v be a vertex of G such that $H = G^v$. The graph J has no i-cycles for $i \leq 4$ by Proposition 5.1.8. Note that $V(J)$ is the set of endpoints in H of edges in $E_G(N(v), V(H))$. Since $|N_G(v)| = 4$ then if J has a 5-cycle then some two of its points, say s and t, are connected to the same vertex x, for some $x \in N_G(v)$. However this would imply a triangle or a 4-cycle in G passing through sxt since any two points on a 5-cycle are in distance at most 2. This contradicts Proposition 5.1.8 and hence (a) follows. If $\deg_J(s) = 0$ for some $s \in V(J)$ then $\{x, s\} \in E(G)$ for some $x \in N(v)$ and it is easy to see that in this situation G cannot have any pentagon passing through vxs, which contradicts Lemma 5.2.2(a). Thus (b) holds.

Let s be a 1-vertex in J, hence $Z_H(s) = 3+4+4 = 11$. By Proposition 5.1.4(c) $Z_H(s) \leq 11$, thus s is full in H and $H^s \in \Psi$. Let t be the only 3-vertex in H such that $\{s,t\} \in E(H)$ and note that $t \in V(J)$. Now if t is not a 1-vertex in J then there exists some other vertex y in J such that $\{t, y\} \in E(H)$ and y is an i-vertex in H^s for some $i \leq 2$. This contradicts Lemma 5.1.5 applied to H^s and proves (c). Finally, (d) is obvious from (b) and (c) and the definition of J. ∎

Lemma 5.2.4. *J has no 6-cycles.*

Proof: Assume that J has a 6-cycle $C = abcdef$ and let p, q and r be the other neighbors in H of a, c and e, respectively. Note that p, q and r are three different vertices not lying on C since the contrary would imply a 4-cycle in H (see Figure V). Let $f(x) = \deg_H(x) - 3$, so $f(x)$ is equal to 0 or 1 for $x \in V(H)$. Consider a 3-independent set $S = \{a, c, e\}$ and its support $R = \sup_H(S)$ in H. If we set $F = f(p) + f(q) + f(r)$ then F is the number of 4-vertices in H belonging to $V(R)$, so $0 \leq F \leq 3$. Now $n(R) = 9$ and similarly as in (6) in the proof of Lemma 5.2.2 we obtain

$$e - Z(S) = (6n - 13k) - (9 \cdot 3 + F - e(R)) \geq 6(n-9) - 13(k-3).$$

Whence $e(R) \geq 12 + F$. The graph R has 9 edges adjacent to some vertex in S. Thus at least $3 + F$ edges must have both endpoints in $N(S) = \{b, d, f, p, q, r\}$, so since we have to avoid triangles they are in the set

$$P = \{\{p,q\}, \{p,r\}, \{q,r\}, \{p,d\}, \{q,f\}, \{r,b\}\}.$$

If $F = 3$, then $P \subseteq E(H)$ since $|P| = 6$, but P has a triangle pqr, so $F \leq 2$. If $F = 2$, then we can assume that $f(p) = 0$, *i.e.* $p \in V(J)$, and consequently we cannot use the edge $\{p, d\}$ since this would form a 5-cycle in J, contrary to Lemma 5.2.3(a). Thus again we are forced to make triangle pqr, so $F \leq 1$. If $F = 1$, then assume that $f(q) = f(r) = 0$, so $q, r \in V(J)$.

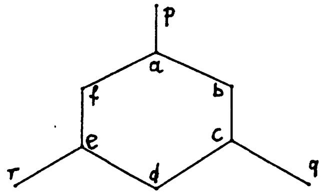

Figure V. 6-cycle in Lemma 5.2.4.

Similarly, by avoiding 5-cycles in J we can see that only 3 edges with an endpoint p from the set P can be used, but we need 4 of them. Hence the last possibility to consider is $F = 0$ and $p, q, r \in V(J)$. However in this case the addition of any edge from P forms a 5-cycle in J, thus we have a contradiction completing the proof of the lemma. ∎

Lemma 5.2.5. *Let x and y be 2-vertices in J with a common neighbor t in J. Furthermore let $N_H(x) = \{t, x_1, x_2\}$ and $N_H(y) = \{t, y_1, y_2\}$, where $x_1, y_1 \in V(J)$. Then $\{x_1, y_2\}, \{x_2, y_1\} \in E(H)$.*

Proof: Note that x_2 and y_2 are 4-vertices in H since x and y are 2-vertices in J. Furthermore by Lemmas 5.2.3(a,c) and 5.2.4 there are two other vertices $v, u \in V(J)$ such that $\{x_1, v\}, \{y_1, u\} \in E(J)$ (see Figure VI).

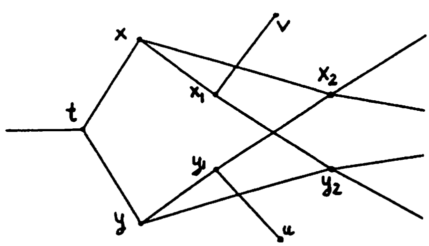

Figure VI. H in Lemma 5.2.5.

Let $S = \{x, y\}$ and $R = \sup_H(S) = \{x, y, t, x_1, x_2, y_1, y_2\}$, so $n(R) = 7$. Observe that by Lemma 5.2.3(d) the vertices v and u in the graph H^S have degree less than 3, hence by Lemma 5.1.5 H^S is not a minimum graph in Ψ, and $e(H^S)$ is at least 1 larger than the bound given by (4). Considering the latter, once more by the same argument as in (6) we obtain

$$e - Z(S) = (6n - 13k) - (5 \cdot 3 + 2 \cdot 4 - e(R)) \geq 6(n-7) - 13(k-2) + 1.$$

Whence $e(R) \geq 8$. The graph R has 6 edges adjacent to x or y, hence at least two additional edges must be in the set $E_H(\{x_1, x_2\}, \{y_1, y_2\})$. By Lemma 5.2.3(a) we cannot take $\{x_1, y_1\}$, so it is easy to see that the only possibility is to add edges $\{x_1, y_2\}$ and $\{x_2, y_1\}$, since by Lemma 5.2.5 H has no 4-cycle. ∎

Lemma 5.2.6. *If* $\deg_J(x) = 3$, *then* $Z_J(x) = 2 + 2 + 3$.

Proof: If $\deg_J(x) = 3$ then by Lemma 5.2.3(d) we have $6 \leq Z_J(x) \leq 9$. Define $s_j(x)$ to be the number of vertices in J in distance j from x. Assume first that $Z_J(x) \geq 8$, so x has at least two 3-vertices as neighbors in J. Using the facts that a component of J containing x has only 2- and 3-vertices (Lemma 5.2.3(d)), and J has no i-cycles for $i \leq 6$ (Lemmas 5.2.3(a) and 5.2.4), we can easily derive that $s_0(x) = 1$, $s_1(x) = 3$, $s_2(x) \geq 5$ and $s_3(x) \geq 5$. Hence we obtain a contradiction $14 \leq \sum_{j=0}^{3} s_j(x) \leq |V(J)| = 12$. Thus to complete the proof it is sufficient to show that $Z_J(x) \neq 6$. Let $N_J(x) = \{a, b, c\}$ and assume that a, b and c are 2-vertices in J. Then there are three other vertices p, q and r in J connected to a, b and c, respectively (see Figure VII).

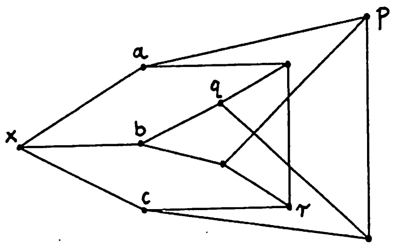

Figure VII. H in Lemma 5.2.6.

Note that the three unlabeled vertices in Figure VII are 4-vertices in H. If we apply three times Lemma 5.2.5 to pairs of vertices from $\{a, b, c\}$ with common neighbor x, then the six resulting edges are those in Figure VII which are nonadjacent to any of a, b or c. This contradicts Lemma 5.2.3(c), since p, q and r are 1-vertices in J, but their neighbors in J are 2-vertices. ∎

We are ready to put together properties of graph J. What are possible components S of J? If S is not an isolated edge, then by Lemma 5.2.3(d) $\delta(S) \geq 2$, so S must have cycles and by Lemmas 5.2.3(a) and 5.2.4 S has at least 7 vertices. Hence if S has only 2-vertices then S is a cycle of length 8, 10 or 12, since $V(J) = 12$. If S has some 3-vertex, then by Lemma 5.2.6 it has at least two of them and, in general, 3-vertices of J are grouped in disjoint pairs of adjacent vertices. Using the fact that S has no i-cycles for $i \leq 6$ one can easily derive that there are exactly two possible graphs S_1 and S_2, both of them on 12 vertices, say $\mathbf{Z}_{12}$, with edges $\{i, i+1\}$ for $i \in \mathbf{Z}_{12}$ and a diagonal edge $\{0, 6\}$, and S_2 has one additional edge $\{3, 9\}$. We summarize this as the next proposition.

Proposition 5.2.7. *Any component of graph J is one of the following: an isolated edge, 8-cycle, 10-cycle, 12-cycle, S_1 or S_2.*

The next two lemmas will eliminate all the above possibilities with the exception of an isolated edge as a component of the graph J.

Lemma 5.2.8. *A component S of graph J is not an i-cycle for $i = 8, 10, 12$.*

Proof: Assume that $S = (\mathbf{Z}_i, \{\{j, j+1\} : j \in \mathbf{Z}_i\})$ for $i = 8, 10$ or 12. Apply Lemma 5.2.5 i times for vertices j and $j+2$ with a common neighbor $j+1$, for $j \in \mathbf{Z}_i$. If $i = 8$, then there exists a vertex $x \notin \mathbf{Z}_8$ in H such that $012x$ is a 4-cycle in H contradicting Proposition 5.1.8. If $i = 10$, then there exists a vertex $x \notin \mathbf{Z}_{10}$ in H such that $01x$ is a triangle in H. If $i = 12$, then there exist three 4-vertices x_0, x_1 and x_2 in H such that x_j is connected to vertices $3p + j$ for $0 \leq j \leq 2$ and $0 \leq p \leq 3$. Note that in this case the component of H containing J has vertices $\mathbf{Z}_{12} \cup \{x_0, x_1, x_2\}$. Recall that $H = G^v$ for some $G \in \Lambda$ and G is connected by Proposition 5.1.3(a), thus here H has to be connected, and consequently $|V(G)| = 5 + |V(H)| = 20$, which contradicts Proposition 5.1.7. ∎

Lemma 5.2.9. *J is not isomorphic to S_1 nor to S_2.*

Proof: Assume that $J \equiv S_1$ or $J \equiv S_2$, where $S_1 = (\mathbf{Z}_{12}, \{\{i, i+1\} : i \in \mathbf{Z}_{12}\} \cup \{\{0, 6\}\})$ and S_2 is the same as S_1 but with the edge $\{3, 9\}$ added. Now similarly as in the last lemma by applying Lemma 5.2.5 whenever possible, there exist two 4-vertices x and y in H such that x is connected to $\{2, 5, 8, 11\}$ and y is connected to $\{1, 4, 7, 10\}$. In the case $J \equiv S_2$ note that as in the Lemma 5.2.8 H has vertices $\mathbf{Z}_{12} \cup \{x, y\}$, so $|V(G)| = 19$, which is impossible by Proposition 5.1.7. Hence $J \equiv S_1$ and the situation is as drawn in Figure VIII.

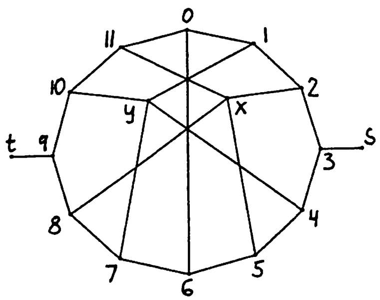

Figure VIII. H in Lemma 5.2.9.

Consider a 6-independent set $P = \{1,3,5,7,9,11\}$. Let $R = \sup_H(P)$ and note that $|V(R)| = 15$ or 16 depending whether 3 and 9 have a common vertex in H, *i.e.* whether s and t is the same vertex. Considering the latter and reasoning as in (6) we obtain

$$(6n-13k)-(12\cdot 3+3\cdot 4-e(R)) \geq e-Z(P) \geq 6(n-16)-13(k-6).$$

Whence $e(R) \geq 30$. Note that so far R has 23 edges in Figure VIII and that $\deg_H(i) = 3$ for $i \in \mathbf{Z}_{12}$. We need at least 7 additional edges, while only one can be added between s and t, furthermore only if $s \neq t$. ∎

Corollary 5.2.10. *If $G \in \Lambda$, then in G there are*

(a) *exactly 1 pentagon passing through each path of length 2,*
(b) *exactly 3 pentagons passing through each edge and*
(c) *exactly 6 pentagons passing through each vertex.*

Proof: By Proposition 5.2.7 and Lemmas 5.2.8 and 5.2.9 the graph J on 12 points is formed by 6 isolated edges, each of them yielding a pentagon passing through a vertex in G defining H and J. Thus Corollary 5.2.10 is a consequence of Lemma 5.2.2. ∎

Corollary 5.2.11. *If $G \in \Lambda$, then in G*

(a) *two pentagons can share zero or one edge,*
(b) *a pentagon and a hexagon can share at most two edges.*

Proof: (a) is implied by Corollary 5.2.10(a) and the fact that all the considered graphs have no triangles. A pentagon and a hexagon sharing more than 2 edges yield either a triangle or two pentagons sharing two edges, hence (b) follows. ∎

5.3. Hexagons and the Proof

Corollaries 5.2.10 and 5.2.11 give already quite strong conditions on pentagons in any possible counterexample to Theorem 5.1.1. However to conclude that no such graph can exist, we still need some more information about hexagons.

Lemma 5.3.1. *If $G \in \Lambda$, then two hexagons in G can share no more than two edges.*

Proof: If two hexagons in $G \in \Lambda$ share at least 4 edges or 3 nonconsecutive edges, then it is easy to see that G has a triangle or 4-cycle, which contradicts Proposition 5.1.8. Thus assume that two hexagons in G share 3 consecutive edges, *i.e.* there are two vertices x and y such that there are at least 3 disjoint paths of length 3 from x to y. Recall that G is 4-regular by Lemma 5.1.6 and let $N(x) = \{x_i\}_{1\leq i\leq 4}$, $N(y) = \{y_i\}_{1\leq i\leq 4}$. Note that $y \notin S = \cup_{i=1}^{4} N(x_i)$, but y is connected to at least 3 vertices in S, so $S \cap N(y) \geq 3$. Since we have to avoid 4-cycles

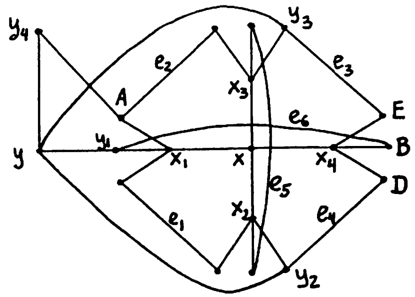

Figure IX. Lemma 5.3.1.

no two of them can belong to the same $N(x_i)$. Without loss of generality assume that $y_i \in N(x_i)$ for $1 \leq i \leq 3$ and consider 6 pentagons passing through x (Corollary 5.2.10(c), see Figure IX).

Let $e_1, \ldots, e_6$ be the edges of J in $H = G^x$. Note that if $e_p = \{y_i, v\}$ and $v \in N(x_j)$, then $N(y) \cap N(x_j) = \emptyset$, since otherwise if $y_j \in N(x_j)$, then we have a triangle yy_iy_j if $y_j = v$, or we have two pentagons passing through y_ivx_j, namely $yy_ivx_jy_j$ and $x_iy_ivx_jx$, which contradicts Corollary 5.2.10(a). Hence, up to symmetry, y_1, y_2 and y_3 are located as in Figure IX, furthermore y_4 is some other vertex not in S. Now a pentagon guaranteed by Corollary 5.2.10(a) passing through yy_1x_1 cannot go through x by Corollary 5.2.11(a), so we can assume that y_4 is connected to A. The pentagon P passing through yy_1B by Corollary 5.2.11(a) cannot go through y_4, so it goes through y_2 or y_3, *i.e.* there exists a vertex C (not shown in Figure IX) such that $P = yy_1BCy_2$ or $P = yy_1BCy_3$. But then the pentagon $xx_2y_2Dx_4$ shares two edges with the pentagon BCy_2Dx_4 or the pentagon $xx_3y_3Ex_4$ shares two edges with the pentagon BCy_3Ex_4, respectively. This is again a contradiction with Corollary 5.2.11(a). ∎

Lemma 5.3.2. *If $G \in \Lambda$, then in G there are at least six hexagons passing through each edge.*

Proof: Fix an edge $f = \{v, u\}$ in $G \in \Lambda$. Let P_1, P_2 and P_3 be the three pentagons passing through f according to Corollary 5.2.10(b). Consider pentagons P_1, P_2 and a 4-independent set $S = \{x_1, x_2, x_3, x_4\}$ in G as in Figure X.

Observe that since G has no 4-cycles then by Corollary 5.2.10(a) vertices $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are all different, hence $R = \sup_G(S)$ has 16 vertices. By Corollary 5.2.10(a) there are two pentagons passing through x_1vx_2 and x_3ux_4, thus without loss of generality $\{2, 3\}$ and $\{6, 7\}$ are the edges in G. We estimate the number of edges in R by Propositions 5.1.7 and 5.2.1:

$$e(G) - Z(S) = (6n - 13k - 1) - (4 \cdot 16 - e(R)) \geq 6(n - 16) - 13(k - 4),$$

whence $e(R) \geq 21$. There are 19 edges already drawn in $E(R)$ in Figure X, hence we need at least two additional edges. Using repeatedly Corollary 5.2.11(a)

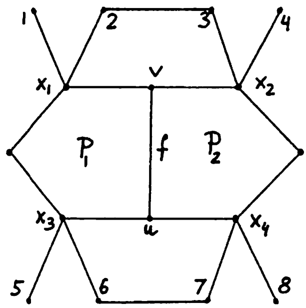

Figure X. R in Lemma 5.3.2.

we conclude that all the additional edges must be in the set $\{\{1,7\}, \{1,8\}, \{2,7\}, \{2,8\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}\}$, and furthermore each of them closes a different hexagon passing through f.

The same reasoning gives us two hexagons passing through the edge f for each pair of pentagons out of P_1, P_2 and P_3. Finally, observe that all of them are different by Lemma 5.3.1, so we have at least 6 hexagons passing through f. ∎

Corollary 5.3.3. *Let $G \in \Lambda$, f be a fixed edge in G and consider nine paths of length 3 in G with a center edge f. Then:*

(a) *through three of them passes a unique pentagon,*
(b) *through other six of them passes a unique hexagon.*

Proof: Obvious from Corollary 5.2.10, Lemma 5.3.1 and Lemma 5.3.2. ∎

This completes rather tedious derivation of properties of a possible smallest counterexample to (4). Using them we are finally ready to show that $\Lambda = \emptyset$, thus proving Theorem 5.1.1.

Proof of Theorem 5.1.1: Assume that $G \in \Lambda$. Let $C = x_0, \ldots, x_5$ be any hexagon in G guaranteed by Corollary 5.3.3. Recall that G is 4-regular and let $N(x_i) = \{x_{i-1}, x_{i+1}, a_i, b_i\}$ for $i \in \mathbf{Z}_6$. First observe that any common neighbor of two points on C must lie on C, since otherwise we would produce a triangle, 4-cycle or two pentagons sharing 2 edges, none of which can happen. By Corollary 5.2.10(a) there is a unique pentagon through $x_i x_{i+1} x_{i+2}$ for $i \in \mathbf{Z}_6$. Since no pair of them can share more than one edge, without loss of generality, the situation is as in Figure XI.

The above yields two pentagons through each edge $\{x_i, x_{i+1}\}$. The third pentagon through $\{x_i, x_{i+1}\}$ must go through a_i and b_{i+1} by Corollary 5.2.11(a). Using properties of pentagons in G easy check shows that the fifth missing point, say c_i, has to be some new vertex, furthermore all c_i's are different (see Figure XII).

By Corollary 5.3.3 there is a unique hexagon H passing through $a_0 x_0 x_1 x_2$ (it cannot be a pentagon since $x_0 x_1 x_2 a_2 b_0$ is a pentagon). Note also that H passes

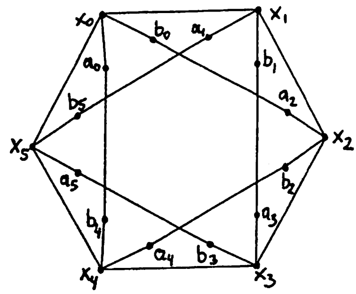

Figure XI. Theorem 5.1.1, cycle C.

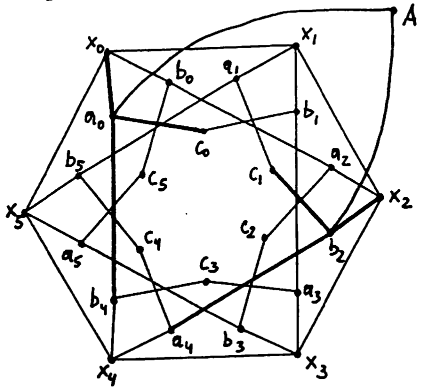

Figure XII. Theorem 5.1.1, final stage.

through b_2. As before, checking all possibilities we can deduce that the sixth point of H is a new vertex, say A, so $H = a_0 x_0 x_1 x_2 b_2 A$. Finally we will derive a contradiction by trying to find a unique pentagon P passing through $a_0 A b_2$. Observe that one missing point from P has to be in $\{x_0, c_0, b_4\}$ and the other one is in $\{x_2, c_1, a_4\}$. One can easily see that P is not passing through x_0 nor x_2, since all such pentagons have already been constructed. For each of the remaining four possibilities one can find without effort some configuration violating Lemma 5.1.6 or one of the Corollaries 5.2.10, 5.2.11 and 5.3.3. For example, if $P = a_0 A b_2 c_1 c_0$ then the pentagons $c_1 a_1 x_1 b_1 c_0$ and $a_0 x_0 x_1 b_1 c_0$ share two edges. Thus $\Lambda = \emptyset$, which completes the proof. ∎

The last corollary in this section extends Corollary 4.3.3(b).

Corollary 5.3.4. $e(3, k+1, n) = 6n - 13k$ *for all* $k \geq 0$ *and* $3k \leq n \leq 13k/4 - \text{sign}(k \bmod 4)$.

Proof: By Theorem 5.1.1 it is sufficient to prove the existence of graphs meeting the bound exactly in the ranges specified above. Consider the graph G formed by a disjoint union of $n-3k$ copies of the graph H_{13} (defined in Section 4.3) and the graph G_{13k-4n}, where G_0 is the empty graph. Observe that $I(G) = 4(n-3k) + (13k-4n) = k$, $n(G) = 13(n-3k) + 3(13k-4n) = n$ and $e(G) = 26(n-3k)+5(13k-4n) = 6n-13k$; hence G meets the bound exactly. To show that this construction covers the range $3k \leq n \leq 13k/4 - \text{sign}(k \bmod 4)$ first assume that $k \bmod 4 = 0$. Then $n \leq 13k/4$ implies that $13k-4n = 0$ or $13k-4n \geq 4$, hence G_{13k-4n} is defined and the above construction yields a desired graph. If $k \bmod 4 \neq 0$, then $n \leq 13k/4 - 1$ implies that $13k-4n \geq 5$ and consequently G is well defined. ∎

6. Independence Ratio

Proposition 2.2 and Theorem 5.1.1 provide sharp lower bound for the function $e(3, k+1, n)$ in the form of piecewise linear function. Furthermore, the classes of minimum graphs corresponding to these linear fragments seem to share some interesting properties. This observation prompts the following definition.

Definition 6.1. *If for some nonnegative reals x and y $e(3, k+1, n) \geq xn - yk$ for all $n, k \geq 0$, then define $\Omega(x, y)$ to be the class of minimum $(3, k{+}1, n, xn-yk)$-graphs.*

Proposition 6.2. *The class $\Omega(x, y)$ is closed under disjoint union of graphs and taking component of a graph.*

Proof: Obvious by simple arithmetic as in the proof of Lemma 4.1.1. ∎

After Lemma 4.1.1 we have already noted that a disjoint union of minimum graphs does not have to be a minimum graph. Here observe that even two copies of the same minimum graph can form a non-minimum graph; for example if G is a minimum (3,7,19)-graph then since $e(3,7,19) = 37$ [4] we see that $G+G$ is a (3,13,38,74)-graph, but by Corollary 5.3.4 $e(3,13,38) = 72 < 74$.

Using Proposition 6.2 we can see that each class $\Omega(x, y)$ can be characterized by it's connected members. In the next proposition, if C is a set of graphs then the symbol $\langle C \rangle$ denotes the class of graphs whose connected components are in C, and if $\Xi = \langle C \rangle$ then we say that Ξ is generated by C.

Proposition 6.3.

(a) $\Omega(0,0) = \langle$*isolated point*$\rangle$,
(b) $\Omega(1,1) = \langle$*isolated point, isolated edge*$\rangle$,
(c) $\Omega(3,5) = \langle$*isolated edge, pentagon*$\rangle$,
(d) $\Phi = \Omega(5,10) = \langle \{F_i\}_{i \geq 2}, \{G_j\}_{j \geq 4} \rangle$,
(e) $\Psi = \Omega(6,13) \supseteq \langle \{G_j\}_{j \geq 4}, H_{13} \rangle$.

Proof: (a) through (d) follow directly from Proposition 2.2 and Corollary 4.3.2. To show (e) note that $G_j \in \Psi$ for all $j \geq 4$, $H_{13} \in \Psi$ and use Proposition 6.2. ∎

A natural open question is whether the containment in (e) can be changed to an equality. The discovery of any new connected generator G of $\Omega(6,13)$ would give a negative answer to this question, besides that such G would be a quite interesting graph. On the other hand the equality in (e) would constitute a good starting point to find parameters x and y of the next nontrivial class $\Omega(x,y)$. We feel that any extension of Proposition 6.3 by some nonempty class $\Omega(x,y)$ for $x > 6$ would be of considerable interest. We note that in Proposition 6.3 each two consecutive classes Ω share some generators, furthermore their intersection is also a class of the same type; for example one easily notes that $\Omega(3,5) \cap \Omega(5,10) = \Omega(4,15/2) = \langle$ pentagon $\rangle$. Another observation is that the minimal degree of vertices in consecutive classes increases and is equal to 0,0,1,2,3 in (a) through (e), respectively.

We are now able to relate the previous results to the independence ratio. Let Θ_d be the class of triangle-free graphs with average degree d. The difficult task of finding the minimal independence ratio $i(\Theta_d)$ is now possible for all $d \leq 4$.

Lemma 6.4. *If G is a graph with average degree d and $G \in \Omega(x,y)$ for some $y \neq 0$ then $i(\Theta_d) = (x-d/2)/y$.*

Proof: Let $G \in \Omega(x,y)$ be a graph with average degree d. By the definition of $\Omega(x,y)$ we have $e(3,k+1,n) \geq xn - yk$ for all $n,k \geq 0$, so for any $(3,k+1,n,nd/2)$-graph H with average degree d and such that $I(H) = k$, we have $nd/2 \geq xn - yk$, which implies that $i(H) = k/n \geq (x-d/2)/y$. Note also that graph G meets the last bound exactly, hence the lemma follows. ∎

Theorem 6.5. *For any rational* $0 \leq d \leq 4$

$$i(\Theta_d) = \begin{cases} 1 - d/2 & \text{if } 0 \leq d \leq 1, \\ 3/5 - d/10 & \text{if } 1 \leq d \leq 2, \\ 1/2 - d/20 & \text{if } 2 \leq d \leq 10/3, \\ 6/13 - d/26 & \text{if } 10/3 \leq d \leq 4. \end{cases} \tag{7}$$

Proof: Let $d = p/q$ for some nonnegative integers p and q. For each d in the ranges specified in (7) we will define a graph $P(d)$ with average degree d, such that $P(d) \in \Omega(x,y)$, where the parameters x and y are as in (b)–(e) of Proposition 6.3, respectively. Then the application of Lemma 6.4 proves the corresponding part of (7). In each case the graph $P(d)$ is defined as a disjoint union of s copies of some graph S and t copies of another graph T, and we will denote this by $P(d) = sS + tT$.

if $0 \leq d \leq 1$ then $P(d) = (2q-2p)$(isolated point) $+p$(isolated edge),
if $1 \leq d \leq 2$ then $P(d) = (10q-5p)$(isolated edge) $+(2p-2q)F_2$,
if $2 \leq d \leq 10/3$ then $P(d) = (40q-12p)F_2 + (5p-10q)G_4$,
if $10/3 \leq d \leq 4$ then $P(d) = (52q-13p)G_4 + (12p-40q)H_{13}$.

For each of the above cases one can easily check that the coefficients s and t are nonnegative, hence the definitions are correct. Similarly, it is easy to confirm that the average degree of $P(d)$ is p/q. We illustrate this in the case $2 \leq d \leq 10/3$. Note that $d = p/q \geq 2$ implies $5p-10q \geq 0$ and $d \leq 10/3$ implies $40q-12p \geq 0$. Also $n(P(d)) = 5(40q-12p) + 12(5p-10q) = 80q$ and $e(P(d)) = 5(40q-12p) + 20(5p-10q) = 40p$, hence the average degree of $P(d)$ is equal to $2e(P(d))/n(P(d)) = p/q$ as claimed. Note that in each case the graphs S and T are connected generators of the corresponding classes $\Omega(x,y)$, so by Proposition 6.3 $P(d) \in \Omega(x,y)$. ∎

Theorem 6.5 gives, in particular, two values of interest: $i(\Theta_3) = 7/20$ and $i(\Theta_4) = 4/13$. The value 7/20 for graphs with average degree 3 was established by Locke [6]. Using Theorem 5.1.1 one could even easily characterize all triangle-free graphs with average degree 3 achieving minimal independence ratio. In particular, note that the graph F_7 is the unique connected triangle-free graph with average degree 3 such that $i(F_7) = 7/20$. Finally, observe that if Ψ_d denotes the class of d-regular triangle-free graphs then we have $i(\Psi_d) = i(\Theta_d)$ for $d = 0,1,2,4$ since isolated point, isolated edge, pentagon and H_{13}, respectively, are d-regular generators of some class $\Omega(x,y)$. For $d = 3$ we have $7/20 = i(\Theta_3) < i(\Psi_3) = 5/14$, where the last equality is a result obtained by Staton [11].

7. A Bound for Ramsey Numbers

We close this paper with two theorems: first of them establishing a general lower bound for the independence ratio $i(\Theta_d)$ and the second one completing the proof of (1). Let for real $0 \leq x \leq 4$ the function $i^*(x)$ be a continuous extension of $i(\Theta_d)$ defined for rational $d \geq 0$.

Theorem 7.1. *Let*

$$h(x) = \begin{cases} i^*(x) & \textit{if } 0 \leq x \leq 4, \\ 6/13 - x/26 & \textit{if } 4 < x \leq 3+\sqrt{2}, \\ \frac{1}{(x-1)^2}(x \log_e x - cx + 1) & \textit{if } 3+\sqrt{2} < x, \end{cases}$$

where $c = \log_e(3+\sqrt{2}) - \frac{5\sqrt{2}}{13} = 0.9409\ldots$. *Then* $i(\Theta_d) \geq h(d)$ *for all rational* $d \geq 0$.

Proof: By the definition of $i^*(x)$ theorem holds for $0 \leq d \leq 4$. By a reasoning similar to that in the proof of Lemma 6.4 it also holds for $4 < d < 3+\sqrt{2}$. For the values of $d > 3+\sqrt{2}$ we will adapt the proof of Theorem 13 from Bollobás [2, pages 294–295]. He defines there (after Shearer [9]) a real function f by $f(0) = 1$, $f(1) = 1/2$ and $f(x) = (x\log_e x - x + 1)/(x-1)^2$ for other $x \geq 0$. In our notation

Theorem 13 there is even stronger than $f(d) \leq i(\Theta_d)$ for all $d \geq 0$. It's proof relies on the following properties of function f:

(P1) $f(0) = 1$,

(P2) f is strictly decreasing, continuous and convex for $x \geq 0$,

(P3) f solves the differential equation $(x^2 - x)g'(x) = 1 - (x+1)g(x)$.

The general solution g to P3 passing through point (x_0, y_0) obtained by a standard method for solving linear differential equations is

$$g(x) = \frac{x}{(x-1)^2}\left[\frac{y_0}{x_0}(1-x_0)^2 - \log_e \frac{x_0}{x} - \frac{x-x_0}{x x_0}\right].$$

To enforce condition P2 for function g we search for a solution which is tangent to the line $6/13 - x/26$ by solving $g'(x_0) = -1/26$ and $y_0 = 6/13 - x_0/26$, which gives $(x_0, y_0) = (3+\sqrt{2}, (9-\sqrt{2})/26)$. One can easily check that, for these values of x_0 and y_0, h is such a solution for $x > x_0 = 3+\sqrt{2}$, where $c = \log_e x_0 + 1/x_0 - y_0(1-x_0)^2/x_0$. Therefore h satisfies properties P1, P2 and P3 for $x > x_0$, and the method of [2] proves our theorem. We note that the above construction gives the best result by this method. ∎

Theorem 7.2. *For all* $k \geq 3$

$$R(3, k+1) \leq \frac{(k-1)^2}{\log_e k + 1/k - c} + 1, \quad \textit{for } k \geq 3,$$

where c *is as in Theorem 7.1.*

Proof: First check that the theorem is true for $k = 3$ and $k = 4$. Then suppose that, contrary to the assertion of the theorem, there exists a $(3, k+1, n)$-graph G, $I(G) = k$, for some $k \geq 5 > 3+\sqrt{2}$ such that

$$n = \lfloor (k-1)^2/(\log_e k + 1/k - c) \rfloor + 1.$$

Observe that the maximal degree in G is at most k, so the average degree d of G satisfies $d \leq k$. Now, since $h(x)$ is decreasing, by Theorem 7.1

$$i(G) = k/n \geq \frac{1}{(k-1)^2}(k \log_e k - ck + 1)$$

contradicts the choice of n. ∎

Recently Shearer [10] obtained the following result: Define the function f by a difference equation $f(0) = 1$, $f(d+1) = [1 + (d^2 - d)f(d)]/(d^2+1)$ for nonnegative integers d. Then for any triangle-free graph G we have $I(G) \geq \sum_{i=1}^{n} f(d_i)$, where $d_1, \ldots, d_n$ is the degree sequence of G. Shearer (private communication) also found asymptotics for the above difference equation $f(d) \approx (\log_e d - c)/d + O(\log_e d/d^2)$, which in turn yields a bound $R(3,k) \leq k^2/(\log_e k - c) + O(k/\log_e k)$ for $c = 0.7665\ldots$.

Finally, let us mention that an extension of Proposition 6.3 can also extend Theorem 6.5, decrease the constant c and consequently improve further the upper bound for $R(3,k)$.

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