On a Class of Linear Spaces with 16 Points

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Introduction.

A class in a language school for recent immigrants consists of 16 girls, of whom 4 are Chinese, 4 are Czech, 4 are Cuban, and 4 are Congolese. The girls study in groups of three, no two of the same nationality. Can one arrange a schedule for 32 study groups in such a way that each girl will be together with every girl of nationality other than her own in exactly one study group? If yes, how many essentially different schedules can be made up?

This is how the problem we want to address would likely have been formulated in the last century. In fact, the resemblance to the famous Kirkman's problem of 15 schoolgirls is not purely accidental.

Of course, the above questions are really asking, in modern terminology, whether there exists a group divisible design with 16 elements having four groups of size four and all blocks of size three (alternatively, whether there exists a linear space on 16 points with a parallel class of lines of size four and all other lines of size three), and what is the number of nonisomorphic designs of this kind.

The answer to the first question is well known to be yes (see, e.g., [4]). It is the purpose of this note to provide an answer to the second question. It turns out that there are exactly 23 nonisomorphic designs of the above kind.

This result can be interpreted also as a result on the number of nonisomorphic twofold triple systems of order 16 having the maximum possible number of repeated blocks, namely 32. We also mention an application to a special linear space on 19 points.

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All group divisible designs $GD(16;4;3)$.

A *group divisible design* is a triple (X, G, B) where X is a set of elements, G is a partition of X into subsets called groups, and B is a collection of subsets of X called blocks, with the property that any two elements of X belonging to distinct groups (to the same group) are contained in exactly one block (in no block). We will denote by $GD(16;4;3)$ a group divisible design with 16 elements, 4 groups of 4 elements each, and all blocks of size 3; it follows that the number of blocks is 32.

A *linear space* is a pair (P, L) where P is a set of points and L is a collection of subsets of P called lines such that any two points are contained in a unique line, and any two distinct lines intersect in at most one point. (It is usually assumed that each line has at least two points.)

It is clear that by declaring in any $GD(16;4;3)$ the elements to be points, and both groups and blocks to be lines we obtain a linear space with 16 points, 4 (disjoint) lines of size 4 and all other lines of size 3. It is only slightly less obvious (but cf. [6]), that from any given linear space with 16 points, 4 lines of size 4 and all other lines of size 3 we similarly obtain a $GD(16;4;3)$.

A *twofold triple sytem* of order v ($TTS(v)$) is a pair (V, B) where V is a v-set, and B is a collection of subsets of V called triples such that any 2-subset of V is contained in exactly two triples. In general, it is possible for B to contain two triples $\{x, y, z\}$, $\{x, y, z\}$ identical as subsets of V; in this case, $\{x, y, z\}$ is said to be a repeated triple.

The maximum possible number of repeated triples in a $TTS(16)$ is known to be 32 [6]. Moreover, it is shown in [6] that in any $TTS(16)$ with 32 repeated triples, the 16 non-repeated triples must be the triples of 4 pairwise disjoint subsystems of order 4. Thus, again, any $GD(16;4;3)$ yields a $TTS(16)$ with 32 repeated triples: it suffices to replace each group with a $TTS(4)$, and duplicate each triple to obtain the 32 repeated triples. Conversely, any $TTS(16)$ with 32 repeated triples yields a $GD(16;4;3)$.

Consequently, each of the three structures defined above, namely a $GD(16;4;3)$, a linear space on 16 points with 4 lines of size 4 and all other lines of size 3, and $TTS(16)$ with 32 repeated triples, uniquely determines any other of the three.

It is well known that (each of) the above exists (cf. [4]). The purpose of this note is to exhibit all nonisomorphic $GD(16;4;3)$'s.

Using SUN 4-280 computer, we procceded as follows.

At first, we generated a set of designs guaranteed to contain representatives of all isomorphism classes. If we designate by A, B, C, and D the 4 groups of 4 elements each in $GD(16;4;3)$, we have 8 blocks of each of types ABC, ABD, ACD, and BCD, and each element is contained in 6 blocks; the 6 blocks containing an element of a group, say A, are two of each of types ABC, ABD, and ACD. Thus, we can choose without loss of generality the 6 blocks containing a given fixed element, say element 0 of group A. Next, the 8 blocks of type BCD, disjoint with respect to pairs with the already chosen 6 blocks, were generated in all possible ways. For any such set of 14 blocks, the cubic graph whose edges are 2-subsets not occurring in any of these blocks, was 3-edge-coloured in all possible ways. The three colour classes were then associated in an obvious way with the remaining three elements of group A to give the remaining 18 blocks of a $GD(16;4;3)$.

Proceeding as above resulted in a total of 4543 distinct designs.

With each $GD(16;4;3)$ we may associate an invariant as follows. If a, b are two elements from the same group, let $G_{a,b}$ be the graph whose vertices are the 12 elements of the other three groups, and whose edges are the pairs of elements that occur together in a block with a or b. Then $G_{a,b}$ (the interlacing graph) is a 2-regular graph on 12 vertices whose each component has an even number (≥ 4) of vertices, and thus is one of 4 possible types: (1) C_{12}, (2) $C_8 \cup C_4$, (3) $C_6 \cup C_6$, and (4) $C_4 \cup C_4 \cup C_4$, where C_i denotes a cycle on i vertices. Next we form a 4-component vector (i_1, i_2, i_3, i_4), $\sum_{j=1}^{4} i_j = 24$, where i_j is the number of times the graph of type (j) occurs among the 24 interlacing graphs (6 for each group). This vector called the interlacing vector (cf. [3]) is an isomorphism invariant for $GD(16;4;3)$'s.

(A refinement of this invariant according to which one forms further 4 "interlacing vectors" associated with the individual groups, afforded no additional information in this case.)

The interlacing vector invariant partitioned the set of obtained designs into 22 equivalence classes, ranging in sizes from 8 to 768. This invariant also enables isomorphism testing within an equivalence class in a very simple manner; typically only a few potential isomorphisms needed to be checked. To our great surprise, this simple invariant turned out to be "almost complete" in the sense that, apart from one pair of nonisomorphic designs, it distinguishes all designs (the pair not distinguished by the interlacing vector is, however, distinguished, e.g., by the automorphism group order). Thus the number of nonisomorphic $GD(16;4;3)$'s is 23.

The procedure used was, to a degree, coloured by the fact that we expected the number of nonisomorphic designs to be much larger than what eventually turned out to be the case. This was suggested by previous experience, and also by comparison with the known number of nonisomorphic Steiner triple systems of order 15 (cf. [5]). That this expectation was not unreasonable was also confirmed in conversation with others [1].

As a partial independent check, 5000 designs were generated randomly by hill-climbing [7], and tested for isomorphism using the general purpose graph isomorphism algorithm of Brendan McKay - Nauty - (confirming, of course, the above result).

All 23 designs are listed in full in table 1.

Table 2 contains information about some properties of the 23 $GD(16;4;3)$'s:

order of the automorphism group, interlacing vector, total number of almost parallel classes, maximum number of almost parallel classes not containing a given element, and maximum number of disjoint almost parallel classes not containing a given element.

Only two designs (Nos. 19 and 22) have automorphism groups transitive on elements, while two designs (Nos. 2 and 8) are automorphism free.

Only one design (No. 22) is homogeneous in the sense that all its interlacing graphs are pairwise isomorphic.

The two transitive designs (Nos. 19 and 22), and only these two designs, have the following resolvability property: its 32 blocks can be partitioned, in a unique way, into 8 disjoint partial parallel classes, each containing 4 disjoint blocks; each such "holey" parallel class misses exactly one group, and for each group there are exactly two parallel classes missing this group.

An Application.

Finally, let us mention an interesting recent application of our result. A paper of J. C. Colbourn, D. R. Stinson and the third author [2] just submitted for publication deals with the determination of the spectrum of linear spaces with lines of size 3 and 4 only, with the additional condition that the number of lines of each size is prescribed (subject to simple arithmetic conditions, of course). The existence of $GD(16;4;3)$ with 3 disjoint almost parallel classes missing a given element (cf. Table 2) yielded, in a simple way, an otherwise hard-to-get linear space with 19 points, 17 lines of size 3 and 20 lines of size 4. If a is the element of a $GD(16;4;3)$ which is missed by 3 disjoint almost parallel classes of blocks, take as the set of points of our linear space the set of elements of the GD together with 3 new elements X, Y, and Z. Take as lines of size 4 (1) the blocks of the 3 almost parallel classes, with each block of the first, second, and third almost parallel class enlarged by X, Y, and Z, respectively, (2) the groups of the $GD(16;4;3)$, and (3) one new line $aXYZ$; take the remaining 17 blocks of the GD as lines of size 3. It is easily seen that this gives the required linear space.

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