Uniquely Pseudointersectable Graphs

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Abstract. Uniquely pseudointersectable graphs are defined; this is closely related to the uniquely intersectable graphs introduced by Alter and Wang [1]. The S-property is necessary but not sufficient for a graph to be uniquely pseudointersectable. This condition is also sufficient for graphs with unique minimum cover. Finally, we show that for supercompact graphs, unique pseudointersectability and unique intersectability are equivalent. Thus we generalized some of the results in [1] to a wider class of graphs.

1. Introduction

All graphs considered in this paper are finite, undirected, loopless, and without multiple edges. Undefined terms and notations can be found in [2]. If X is a nonempty set and $F = \{X_1, X_2, \cdots, X_n\}$ is a family of distinct nonempty subsets of X whose union is X, then the *intersection graph determined by X and F*, $I(X, F)$, is the graph G whose vertex set can be put in one to one correspondence with the elements of F such that two vertices of G are adjacent if and only if the corresponding elements of F have a nonempty intersection. A graph G is an *intersection graph* if there exists a set X and a family F of distinct nonempty subsets of X such that $G \cong I(X, F)$. Every graph G is an intersection graph on some finite set [2] and the *intersection number* $w(G)$ is the minimum number of elements in a set X such that G is an intersection graph on X. Some results on $w(G)$ were given by Erdös et al [3], M. Hall Jr. [4], Harary [2], Lovasz [6] and Alter and Wang [1].

For the complete graph K_3 we have $w(K_3) = 3$. If $X = \{1,2,3\}$ then we can choose, for example, $X_1 = \{1\}$, $X_2 = \{1,2\}$, and $X_3 = \{1,3\}$ or $X_1 = \{1,2\}$, $X_2 = \{1,3\}$, and $X_3 = \{2,3\}$. In the former case it is clear that the elements 2 and 3 are needed only to make the X_i's distinct and do nothing to indicate adjacency. As another example, the graph $K_4 - x$ is given in figure 1 as an intersection graph, and, in this case, element 3 of X is not necessary to indicate the adjacency of any two vertices. The size required for X can be reduced

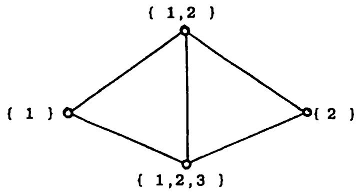

Figure 1

by eliminating these "fillers" used only to obtain distinct representation of each vertex.

Alter and Wang [1] introduccd the concept of unique intersectability of a graph. They defined a uniquely intersectable graph as follows:

Given a graph G and $|X| = w(G)$, let F_1 and F_2 be two families of nonempty distinct subsets of X. If $I(X,F_1) \cong I(X,F_2) \cong G$ implies that F_1 can be obtained from F_2 by a permutation of the elements of X (i.e. $F_1 \approx F_2$) then G is said to be a *uniquely intersectable (u.i) graph*.

Based on the argument that the size of X can be reduced we define pseudointersection graph and uniquely pseudointersectable graph by allowing subset repetitions in the family.

If X is a set and $F = \{X_1, X_2, \cdots, X_n\}$ is a family of nonempty subsets of X (not necessarily distinct) whose union is X, then the *pseudointersection graph*, denoted by $I^*(X,F)$, is the graph whose vertex set is F and X_i and X_j are adjacent if and only if $i \neq j$ and $X_i \cap X_j \neq \emptyset$. A graph G is a pseudointersection graph on X if there exists such a family F for which $G \cong I^*(X,F)$. The *pseudointersection number* of G, denoted by $w^*(G)$, is the minimum number of elements in a set X such that G is a pseudointersection graph on X. For the complete graph K_3, $w^*(K_3) = 1$ and for the graph $K_4 - x$, $w^*(K_4 - x) = 2$. Some results on $w^*(G)$ can be found in [7].

Given a graph G and $|X| = w^*(G)$, let F_1 and F_2 be any two families of nonempty subsets of X (not necessarily distinct). If $I^*(X,F_1) \cong I^*(X,F_2) \cong G$ implies that F_1 can be obtained from F_2 by a permutation of the elements of X (i.e. $F_1 \approx F_2$) then G is said to be a *uniquely pseudointersectable (u.p.i) graph*.

For example, it is easy to see that the graph $K_4 - x$ and a triangle are uniquely pseudointersectable. But these same graphs are not uniquely intersectable. To see this, note that $w(K_4 - x) = 3$ and $w(K_3) = 3$ and let $X = \{1,2,3\}$, $F_1 = \{\{1\},\{2\},\{1,2\},\{123\}\}$, $F_2 = \{\{1\},\{1,2\},\{2,3\},\{1,2,3\}\}$, $F_1' = \{\{1\},\{1,2\},\{1,3\}\}$ and $F_2' = \{\{1,2\},\{1,3\},\{2,3\}\}$. Clearly $I(X,F_1) \cong I(X,F_2) \cong K_4 - x$ but $F_1 \not\approx F_2$ and $I(X,F_1') \cong I(X,F_2') \cong K_3$ but $F_1' \not\approx F_2'$. The graph given in figure 2 is not u.p.i or u.i.

Alter and Wang [1] established some results on unique intersectability of a graph and gave four families of graphs with triangles which are uniquely intersectable.

We denote the set of all cliques of a graph G by $C(G)$. A set of cliques $C'(G) \subseteq C(G)$ of a graph G, is called a *cover* of G if and only if every element (vertex and edge) of G belongs to some element of $C'(G)$. A cover $C'(G)$ of G is called

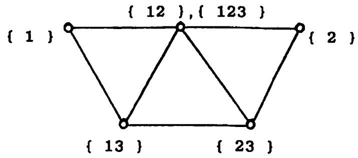

Figure 2

a *minimum cover*, denoted by $K(G)$, if and only if $|C'(G)| \leq |C''(G)|$ for any cover $C''(G)$ of G.

In section 2 we present a necessary condition for a graph to be uniquely pseudointersectable. The condition is also sufficient for graphs with a unique minimum cover. In general the condition is not sufficient for graphs with non-unique minimum cover, and we illustrate this by an example.

Let X be a nonempty finite set and F be a family of nonempty subsets of X. The family F is said to have the *Helly property* if for any subfamily $F' \subseteq F$ with $X_i \cap X_j \neq \emptyset$ for all X_i, X_j in F', we have $\cap\{X_i | X_i \in F'\} \neq \emptyset$. A graph G is called a *supercompact graph* if G is the intersection graph of some family F of nonempty subsets of a set X such that

(a) F satisfies the Helly Property and
(b) for any $x_1 \neq x_2$ in X, there exists $X_i \in F$ with $x_1 \in X_i, x_2 \notin X_i$

Various characterizations of supercompact graphs were given by Lim [5].

In section 3, we prove that for supercompact graphs, unique intersectability and unique pseudointersectability are equivalent. Thus we generalize some of the results in [1] to a wider class of graphs, namely supercompact graphs.

2. A necessary condition and other results

Let $Q \subseteq C(G)$. The edges which belong to exactly one clique in the set Q will be called *Q-unicliqual edges*. For each vertex v of a graph G, we define $C(v) = \{C \in C(G) | v \in C\}, T = \{C(v) | v \in V(G)\}$. For a given $K(G)$, let $K(v) = \{C \in K(G) | v \in C\}$ and $S' = \{K(v) | v \in V(G)\}$.

We say that $K(G)$ has the *S-property* if and only if every vertex of each clique C in the set $K(G)$ is incident to an $K(G)$-unicliqual edge of C. For example the graph G of figure 3 has a minimum cover, $K_1(G) = \{1,2,3,4,5,6,7,8,9\}$ (the numbers denote the cliques of G). But $K_1(G)$ does not have the S-property because the vertex v of the clique 8 in $K_1(G)$ is incident to edges of 8, all of which are not $K_1(G)$-unicliqual.

Lemma 2.1. *For any graph G and any minimum cover $K(G)$,*

$$G \cong I^*(K(G), S') \text{ and } w^*(G) = |K(G)|.$$

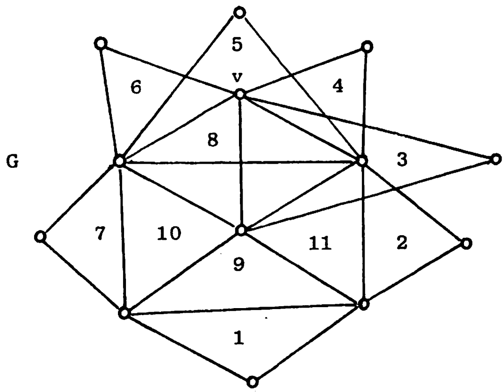

Figure 3

Proof: Recall that for each $v \in V(G)$, we write $K(v) = \{C \in K(G) | v \in C\}$ and $S' = \{K(v) | v \in V(G)\}$. The function

$$f : G \rightarrow I^*(K(G), S')$$

defined by $f(v) = K(v)$ is an isomorphism.

Let u and v be vertices of G. Then u and v are adjacent

$\Longleftrightarrow$ there exists $L \in K(G)$ such that $\{u, v\} \in E(L)$.

$\Longleftrightarrow K(u) \cap K(v) \neq \emptyset$.

Thus $G \cong I^*(K(G), S')$. It follows that $w^*(G) \leq |K(G)|$. Suppose that $G \cong I^*(X, F)$ with $w^*(G) = |X| < |K(G)|$. Let $X = \{x_1, \cdots, x_n\}$. Let C_i be a clique of G such that

$$C_i \supseteq \{D \in F | x_i \in D\}$$

where $x_i \in X$. Then $\{C_i | x_i \in X\}$ is a cover of G. But this is impossible because $K(G)$ is a minimum cover. Thus $w^*(G) \geq |K(G)|$ and hence $w^*(G) = |K(G)|$.

A necessary condition for a graph to be u.p.i. is formulated in the following theorem.

Theorem 2.2. *Let G be an u.p.i. graph. Then every minimum cover of G has the S-property.*

Proof: Let G be an u.p.i. graph. Suppose that to the contrary there exists a minimum cover $K(G)$ of G such that $K(G)$ does not have the S-property. Then there exists an element L of $K(G)$ and a vertex v of L such that the vertex v is incident to edges of L, all of which are not $K(G)$-unicliqual.

By lemma 2.1, $G \cong I^*(K(G), S')$ where $S' = \{K(x) | x \in V(G)\}$ and $K(x) = \{C \in K(G) | x \in C\}$. Now let $C^*(v) = K(v) - \{L\}$ and

$$S^* = [S' - \{K(v)\}] \cup \{C^*(v)\}.$$

Since all edges of L, incident to v are not $K(G)$-unicliqual; $K(v) \cap K(u) \neq \emptyset$ if and only if $C^*(v) \cap K(u) \neq \emptyset$ for any $u \in V(G)$. It is easy to see that $G \cong I^*(K(G), S') \cong I^*(K(G), S^*)$ but S' cannot be obtained from S^* by a permutation of the elements of $K(G)$. This would imply that G is not u.p.i., which is a contradication to our assumption.

The condition that every minimum cover has the S-property is not sufficient for a graph to be u.p.i. as can be seen by the following example.

Example 2.3: The graph G of figure 4 has two minimum covers and both have the S-property. The minimum covers are $K_1(G) = \{1,2,3,4,5,6,7\}$ and $K_2(G) = \{1,2,3,4,5,8,9\}$, where the numbers denote the cliques of G. Then $G \cong I^*(K_1(G), S_1') \cong I^*(K_2(G), S_2')$ but we will show that $S_2' \not\cong S_1'$.

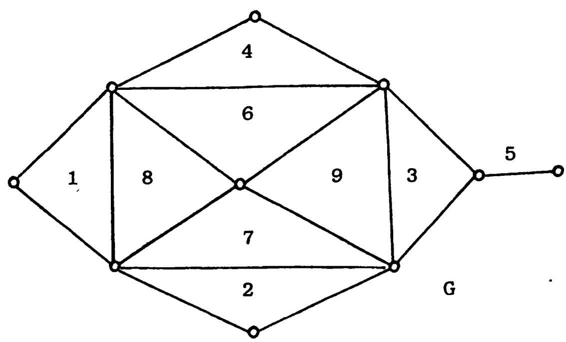

Figure 4

Note that $S_1' = \{\{1\},\{2\},\{5\},\{4\},\{3,5\},\{1,4,6\},\{3,4,6\},\{2,3,7\},\{1,2,7,\},\{6,7\}\}$ and $S_2' = \{\{1\},\{2\},\{5\},\{4\},\{3,5\},\{1,4,8\},\{3,4,9\},\{2,3,9\},\{1,2,8\},\{8,9\}\}$. Suppose that there is an isomorphism from $K_1(G)$ onto $K_2(G)$ such that S_2' can be obtained from S_1' by f. Then $\{f(1), f(2), f(4), f(5)\} = \{1,2,4,5\}$. Since $\{3,5\} \in S_1'$ and $\{3,5\} \in S_2'$ and $f(5) \neq 3$, we must have $f(3) = 3$. Thus $f(5) = 5$. It is easy to see that $\{f(6), f(7)\} = \{8,9\}$. Suppose that $f(6) = 8$. Then $\{3,4,6\} \in S_1'$ but $\{f(3), f(4), f(6)\} = \{3, f(4), 8\} \notin S_2'$. Therefore $f(6) \neq 8$ and hence $f(6) = 9$, $f(7) = 8$. This would imply that $\{f(2), f(3), f(7)\} = \{f(2), 3, 8\} \in S_2'$ because $\{2,3,7\} \in S_1'$. But this is impossible because $\{f(2), 3, 8\} \notin S_2'$. Thus $S_1' \not\cong S_2'$. So G is not u.p.i.

Lemma 2.4. *Let G be a graph. Then every clique ($\ncong K_1$) in a minimum cover $K(G)$ contains an $K(G)$-uniclíqual edge.*

Proof: Without loss of generality let G be a connected graph and $G \ncong K_1$. Suppose that there exists a clique C in a minimum cover $K(G)$ such that C is not $K(G)$-unicliqual. Then every edge of C belongs to some other clique in $K(G)$. Thus $K(G) - \{C\}$ also cover G. But this is impossible because $K(G)$ is a minimum cover of G.

Theorem 2.5. *Let $G \cong I^*(X, F)$ where $|X| = w^*(G)$ and F be an arbitrary family of non-empty subsets of X. Suppose that G has a unique minimum cover, say $K(G)$ that has the S-property. Then C is an element of $K(G)$ if and only if*

$$C = \{D \in F | x \in D\}$$

for some $x \in X$.

Proof: Let $G \cong I^*(X, F)$ where $X = \{x_1, \cdots, x_n\}$ and F be an arbitrary family of non-empty subsets of X whose union is X. Suppose that $K(G)$ is the unique minimum cover of G and has the S-property. Then $|K(G)| = n$.

For each $x_i \in X$, let L_i be a clique of G such that

$$L_i \supseteq \{D \in F | x_i \in D\}.$$

Then the set $\{L_1, L_2, \cdots, L_n\}$ is a minimum cover of G. Thus the uniqueness of $K(G)$ implies that for each $C \in K(G)$,

$$C \supseteq \{D \in F | x_i \in D\}$$

for some $x_i \in X$.

Necessity: Let $C \in K(G)$. Then $C \supseteq \{D \in F | x_i \in D\}$ for some $x_i \in X$. Let $\{D_1, D_2\}$ be an edge of C which is $K(G)$-unicliqual (this edge exists by Lemma 2.4). We assert that $D_1 \cap D_2 = \{x_i\}$. For if not, then there exists $x' \in D_1 \cap D_2$ with $x' \neq x_i$, $x' \in X$. Hence there exists $C' \in K(G)$ such that

$$C' \supseteq \{D \in F | x' \in D\}.$$

Note that $C' \neq C_i$ by the minimality of $K(G)$. But this would imply that the edge $\{D_1, D_2\}$ belongs to C. This is impossible because that edge $\{D_1, D_2\}$ is $K(G)$-unicliqual. Thus $D_1 \cap D_2 = \{x_i\}$.

Sufficiency: Let $T = \{D \in F | x_i \in D\}$, $x_i \in X$. Since $K(G)$ is unique, there exists $C \in K(G)$ with

$$C \supseteq \{D \in F | x_i \in D\}.$$

We claim that $C = \{D \in F | x_i \in D\}$. The proof is similar to the proof of the necessity. Hence $T = C \in K(G)$.

Corollary 2.6. *Let $G \cong I^*(X,F)$ where $|X| = w^*(G)$ and F is an arbitrary family of non-empty subsets of X. Suppose that $K(G)$ is the unique minimum cover of G and has the S-property. Then for any $C \in K(G)$*

$$|\cap\{D|D \in C\}| = 1.$$

Theorem 2.7. *Let G be a graph. Suppose that G has a unique minimum cover, $K(G)$ that has the S-property. Then G is uniquely pseudointersectable.*

Proof: By lemma 2.1, $G \cong I^*(K(G), S')$. Let F be an arbitrary family of non-empty subsets of X, $|X| = w^*(G)$ and $G \cong I^*(X,F)$. For each v_i of G, let X_i be the corresponding set in F.

By theorem 2.5 the following function is well-defined:

$$f : K(G) \to X$$
$$C \mapsto x$$

where $\{x\} = \cap\{D|D \in C\}$ (by corollary 2.6). f is onto by theorem 2.5 and corollary 2.6. By the definition of clique, f is one-one. Thus f is an isomorphism.

Claim: $X_i = \{f(C)|C \in K(v_i)\}$.

Proof of the Claim: Let $z \in X_i$. Then there exists a unique element C of $K(G)$ such that $f(C) = z$ and $\{z\} = \cap\{D|D \in C\}$. Theorem 2.5 implies that $X_i \in C$, i.e., $C \in K(v_i)$. Thus $z \in \{f(C)|C \in K(v_i)\}$.

Conversely let $z \in \{f(C)|C \in K(v_i)\}$. Then there exists $C \in K(v_i)$ such that $f(C) = z$ and $\{z\} = \cap\{D|D \in C\}$. But $X_i \in C$ and hence $z \in X_i$.

Since X_i is an arbitrary element of F and $X_i = \{f(C)|C \in K(v_i)\}$, F can be obtained from S' by an isomorphism f from $K(G)$ onto X; i.e., $F \approx S'$. Therefore G is u.p.i.

Corollary 2.8.

(a) *Every triangle-free graph is u.p.i.*
(b) *Every star n-gon is u.p.i.*
(c) *Let G be a graph such that $|C_i \cap C_j| \leq 1$ for any two cliques C_i and C_j of G. Then G is u.p.i.*

Theorem 2.9. *Let G be a graph with a unique minimum cover. Then G is an u.p.i. graph if and only if $K(G)$ has the S-property.*

Proof: Follows from Theorems 2.2 and 2.7.

3. Relation with uniquely intersectable graphs

Sumner [8] defines a graph G to be point distinguishing whenever distinct vertices of G have distinct closed neighbourhoods. i.e. $\overline{N(x)} \neq \overline{N(y)}$ for any $x \neq y$ in $V(G)$ where $\overline{N(x)} = \{x\} \cup \{z \in V(G)|z$ is adjacent to $x\}$. In [5], Lim shows that a graph is supercompact if and only if it is point distinguisuing. He further shows that for any graph G, G is supercompact if and only if $G \cong I(C(G),T)$.

Theorem 3.1. *A graph G is supercompact if and only if $G \cong I(K(G), S')$.*

Proof: Necessity: Let G be a supercompact graph. Then $G \cong I(C(G), T)$ where $T = \{C(x) | x \in V(G)\}$ and $C(x) = \{C \in C(G) | x \in C\}$.

We will show that all elements of S' are distinct. Recall that $S' = \{K(x) | x \in V(G)\}$ and $K(x) = \{C \in K(G) | x \in C\}$. Let $K(x)$ and $K(y)$ be any two elements of S'. Then we assert that $K(x) \neq K(y)$. To see this, we suppose that $K(x) = K(y)$. Let $v \neq y \in N(x)$ where $N(x) = \{z \in V(G) | z$ is adjacent with $x\}$. Then v is adjacent with x and there exists $L \in K(x)$ such that the edge $\{v, x\}$ belongs to L. Since $K(x) = K(y)$, $L \in K(y)$ and hence v is adjacent with y, i.e., $v \in N(y)$. Similarly if $v \neq x \in N(y)$ then $v \in N(x)$. Thus $N(x) - \{y\} = N(y) - \{x\}$. Since $K(x) = K(y)$, x and y are adjacent. Thus $\overline{N(x)} = \overline{N(y)}$ where $\overline{N(x)} = N(x) \cup \{x\}$. But this would imply that G is not supercompact which is impossible by assumption.

Since $K(G)$ covers G, it is easy to see that $C(x) \cap C(y) \neq \emptyset$ if and only if $K(x) \cap K(y) \neq \emptyset$ for any $x, y \in V(G)$. Thus $G \cong I(C(G), T) \cong I(K(G), S')$.

Sufficiency: Let G be a graph and $G \cong I(K(G), S')$. All elements of S' are distinct. This implies that all elements of T are also distinct. To see this, let $C(x)$ and $C(y)$ be two arbitrary elements of T. But $K(x) \neq K(y)$. So without loss of generality we assume that there exists $L \in K(x)$ and $L \notin K(y)$. Thus $L \in C(X)$ but $L \notin C(y)$ and hence $C(x) \neq C(y)$. Since $K(G)$ covers G, $K(x) \cap K(y) \neq \emptyset$ if and only if $C(x) \cap C(y) \neq \emptyset$ for any $x, y \in V(G)$. Thus $G \cong I(K(G), S') \cong I(C(G), T)$ which implies that G is supercompact.

Theorem 3.2. *For any supercompact graph G, $w(G) = |K(G)|$.*

Proof: By theorem 3.1, $G \cong I(K(G), S')$ and hence $w(G) \leq |K(G)|$. Suppose that $w(G) < |K(G)|$. Let $w(G) = |X|$ and $G \cong I(X, F)$ where F is a family of non-empty subsets of a set X. Let L_i be a clique of G such that

$$L_i \supseteq \{D \in F | x_i \in D\}$$

where $x_i \in X$. Then $\{L_i | x_i \in X\}$ is a cover of G and $|\{L_i | x_i \in X\}| < |K(G)|$. This is impossible because $K(G)$ is a minimum cover of G. Thus $w(G) \geq |K(G)|$ and hence $w(G) = |K(G)|$.

Theorem 3.3. *Let G be a supercompact graph. Then G is uniquely intersectable if and only if G is uniquely psuedointersectable.*

Proof: Necessity: Let G be an u.i. supercompact graph. By theorem 3.1 $G \cong I(K(G), S')$. Let $G = I(X, F)$ where $|X| = w^*(G)$ and F is an arbitrary family of non-empty subsets of a set X. We assert that all elements of F are distinct. To see this, we assume that $X_i = X_j$ for two subsets X_i and X_j of F. Then $X_i \cap X_k \neq \emptyset$ if and only if $X_j \cap X_k \neq \emptyset$ for any $X_k \in F$. But this would

imply that G is not supercompact, which contradicts our assumption. Since G is supercompact, theorem 3.2 implies that $w(G) = |K(G)|$. So $|X| = w^*(G) = |K(G)| = w(G)$. Thus $F \approx S'$ because G is u.i. and hence G is u.p.i.
Sufficiency: Let G be an u.p.i. supercompact graph. Theorem 3.2 implies that $w(G) = |K(G)|$. Let $G \cong I(X, F)$ where $|X| = w(G)$ and F is an arbitrary family of non-empty subsets of a set X. By theorem 3.1, $G \cong I(K(G), S')$. Then $F \approx S'$ because $w^*(G) = |K(G)| = |X|$ and G is u.p.i. Thus G is u.i.

Corollary 3.4 ([1, theorems 1.2 and 2.4]).

(a) *Every triangle-free graph is u.i.*
(b) *Every star n-gon is u.i.*

Proof: (a) Without loss of generality let G be a connected triangle-free graph. If $G \cong K_2$ then it is easy to verify that G is u.i. If $G \ncong K_2$ then G is supercompact ([5, corollary 2.4]). The result then follows from theorems 2.7 and 3.3.

(b) Star n-gon is supercompact and its minimum cover is unique and has the S-property. The result follows from Theorems 2.7 and 3.3.

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