On 3-(8,4,λ) Designs[1]

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Abstract. The set of all distinct blocks of a t-design is referred to as the support of the design and its cardinality is denoted by b^*. In this article (i) the set of all possible b^*'s for the case of 3-$(8,4,\lambda)$ designs is determined and for each feasible b^* a design with a minimum b is produced; (ii) it is shown that a 2-$(8,4,3\lambda)$ design is a 3-$(8,4,\lambda)$ design iff it is self-comlementary; (iii) it is shown that there are at least 63 pairwise non-isomorphic 3-$(8,4,5)$ designs.

1. Introduction

A t-(v,k,λ) design is a pair (V,D) where V is a v-set and D is a collection of k-subsets of V (called blocks) such that each t-subset of V occurs in exactly λ blocks of D. (We do not require blocks to be distinct.) The set of all distinct blocks of a t-design is referred to as the support of the design and its cardinality is denoted by b^*. We denote the total number of blocks in D by b. A design is called *fundamental* if it does not contain any other t-(v,k,μ) design with $\mu < \lambda$ properly [**Foody and Hedayat 1986**]. A design is called *simple* if $b = b^*$. A t-(v,k,λ) design is called *self-complementary* if and only if $v = 2k$ and the complement of each block is a block.

The intent of this article is to investigate the set of all possible b^*'s for the case of 3-$(8,4,\lambda)$ designs and to produce a design for each possible b^* with a minimum b, utilizing the result of Hedayat and Li (1979). The problem of determining of possible support sizes of a t-design is the content of the questions 23 and 24 of Kageyama and Hedayat (1983).

Hedayat and Hwang (1984) have studied 2-$(8,4,3m)$ designs and their table of designs contain some self-complementary designs and they asked if there are other self-complementary 2-$(8,4,3m)$ designs. In this paper we show that a 2-$(8,4,3m)$ design is self-comlementary if and only if it is a 3-design, hence all self-complementary designs are obtained.

Two fundamental designs are known [**Gronau and Prestin (1982)**]: they are simple designs with $b = 14$ and $b = 42$. All the designs in Table 1 are constructed by using isomorphic copies of these two designs.

Finally, we show that there are at least 63 non-isomorphic 3-$(8,4,5)$ designs.

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2. Preliminaries

In this section we discuss some basic properties of 3-$(8,4,\lambda)$ designs.

From the basic relations of t-designs,

$$\lambda_i = \lambda_t \binom{v-i}{t-i} / \binom{k-i}{t-i}, \qquad 0 \le i \le t,$$

it follows that any i points occur in exactly λ_i blocks. In this case we have $b = \lambda_0 = 14\lambda$, $r = \lambda_1 = 7\lambda$, $\lambda_2 = 3\lambda$, $\lambda_3 = \lambda$. Since there exists a 3-$(8,4,1)$ design, λ can be any positve integer.

Proposition 1. *Every 3-$(8,4,\lambda)$ design is self-complementary.*

Proof: Let (V, D) be a 3-$(8,4,\lambda)$ design and B be a specified block in D. Let x_i, $i = 0,1,\ldots,4$, be the number of blocks apart from B itself which have exactly i points in common with the block B. From Mendelsohn (1971), the intersection numbers satisfy the following equations:

$$\sum_{i=j}^{4} \binom{i}{j} x_i = \binom{k}{j}(\lambda_j - 1), \qquad j = 0,\ldots,3.$$

From these equations it follows easily that $x_0 = x_4 + 1$; this establishes the proposition.

Proposition 2 [Alltop (1975) and Sprott (1955)]. *Every self-complementary t-$(2k, k, \lambda)$ design with t even is a $(t+1) - (2k, k, \nu)$ design, where $\nu = \lambda(k-t)/(2k-t)$.*

Corollary 3. *(V, D) is a 3-$(8,4,\lambda)$ design if and only if it is a self-complementary 2-$(8,4,3\lambda)$ design.*

Proposition 4. *There exists a 2-$(7,3,\lambda)$ design with support size b^* if and only if there exists a 3-$(8,4,\lambda)$ design with support size $2b^*$.*

Proof: By a theorem of Alltop (1975), it follows that every 2-$(7,3,\lambda)$ design can be extended to a 3-$(8,4,\lambda)$ design by complementation; that is, by adding the same new point to each of the original blocks and then taking the complement of each block with respect to the new set of points to form further new blocks.

On the other hand, from every 3-$(8,4,\lambda)$ design one can obtain as a restriction a 2-$(7,3,\lambda)$ design.

Remark: The above correspondence between 2-$(7,3,\lambda)$ and 3-$(8,4,\lambda)$ designs is not one-to-one. Ivanov (1985) has shown that there are 35 non-isomorphic 2-$(7,3,4)$ designs while the number of non-isomorphic 3-$(8,4,4)$ designs is 31.

3. Construction

For fundamental 2-$(7, 3, \lambda)$ designs it is known that:

1. Among simple designs only the designs with $\lambda = 1$ and $\lambda = 3$ are fundamental and these are unique. For $\lambda = 2$, Street (1980) has established the nonexistence of simple fundamental designs. For $\lambda = 4$, it is also a simple combinatorial exercise to extract a design with $\lambda = 1$ from it.

2. Gronau and Prestin (1982) have established the nonexistence of fundamental designs with repeated blocks for $\lambda \leq 4$. For the case $\lambda > 4$ the problem of existence of fundamental designs is under investigation, but we conjecture that in this case there exist no such designs.

By composition of different copies of two fundamental designs 3-$(8, 4, 1)$ and 3-$(8, 4, 3)$, we have constructed 28 non-isomorphic designs for all possible b^*'s with minimum b's. These designs are listed in Table 1.

4. A remark on enumeration of non-isomorphic designs

For the number of pairwise non-isomorphic 3-$(8, 4, 5)$ designs Gronua (1985) has given 107 aş an upper bound. Via a computer program we are able to produce 63 non-isomorphic nonsimple 3-$(8, 4, 5)$ designs. Table 2 contains these designs. These designs were obtained by composition of different copies of two fundamental designs with $b = 14$ and $b = 42$.

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