Concrete Graph Covering Projections

M. Hofmeister,
Cologne

Abstract. A graph covering projection is a local graph homeomorphism. Certain partitions of the vertex set of the preimage graph induce a notion of "concreteness". The concrete graph covering projections will be counted up to isomorphism.

1. Introduction

In this paper, all graphs are supposed to be simple. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. A graph H is called an *r-fold covering* of the graph G if there is an r-to-one homomorphism p from H onto G, called *r-fold covering projection*, which sends the neighbors of each vertex $x \in V(H)$ bijectively to the neighbors of $p(x) \in V(G)$. Topologically speaking, the covering projection is a local homeomorphism.

The *fiber* of the vertex $v \in V(G)$ is the set $p^{-1}(v)$. An r-fold covering projection $p : H \to G$ is said to be *concrete* if there is, in addition, a partition $\mathcal{P} = (P_1, \cdots, P_r)$ of the vertices of H such that every partition set P_i meets every vertex fiber exactly once; we write $(p, \mathcal{P})$ for short. The sets P_i of $\mathcal{P}$ are called the *sheets* of p.

There is a natural kind of isomorphism between covering projections of G. Let Γ be the automorphism group of G. An *isomorphism of the covering projections* p and $\tilde{p}$ of G is a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\psi} & \tilde{H} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ G & \xrightarrow[\gamma]{} & G \end{array} \tag{1}$$

with an isomorphism ψ and $\gamma \in \Gamma$. An *isomorphism of the concrete covering projections* $(p, \mathcal{P})$ and $(\tilde{p}, \tilde{\mathcal{P}})$ of G is an isomorphism of p and $\tilde{p}$ such that $\psi(P) \in \tilde{\mathcal{P}})$ for every $P \in \mathcal{P}$.

Graph coverings are useful in many areas of graph theory. A nice example is due to Mohar, who used covering constructions for the complete graph on 4 vertices to enumerate the akempic triangulations of the 2-sphere with exactly 4 vertices of degree 3 [6], [7]. A general theory of graph covering is developed in [2].

Although there is a good classification of the isomorphism classes of r-fold covering projections [5] by means of permutation voltage assignments (which will be defined in Chapter 2), counting formula are only known in two special cases, namely for 2-fold covering projections, where the enumeration can be done by commutative algebra arguments [4], and for identity graphs, i.e. graphs with trivial automorphism group [5]. Our purpose is to count the isomorphism classes of concrete r-fold covering projections of graphs.

2. Permutation voltage assignments

Permutation voltage assignments were introduced by Gross and Tucker [1] as a powerful tool to handle graph covering spaces. Let S_r denote the symmetric group on the set $\{1, \cdots, r\}$. For a graph G, let $A(G)$ be the arc set of the corresponding symmetric directed graph. A *permutation voltage assignment* in S_r for G is a mapping $f : A(G) \to S_r$ such that $f(v,w) = f(w,v)^{-1}$ whenever v and w are adjacent vertices of G. The pair (G,f) is called a *permutation voltage graph*.

Given such a permutation voltage graph (G,f), we construct the *derived graph* G_f as follows. Its vertex set is the cartesian product $V(G) \times \{1, \cdots, r\}$; two vertices $(v,i), (w,j)$ are adjacent in G_f iff v, w are adjacent in G and $f(v,w)(i) = j$. It is easy to see that G_f is an undirected simple graph.

Gross and Tucker showed that the natural projection $p_f : G_f \to G$ (sending vertex (v,i) of G_f to vertex v of G) associated with a permutation voltage graph (G,f) is an r-fold covering projection [1]. Moreover, considering the sets $P_i = \{(v,i) \mid v \in V(G)\}$ $(i = 1, \cdots, r)$ as sheets of p_f, we obtain a concrete r-fold covering projection $(p_f, \mathcal{P}_f)$.

The following theorem allows us to restrict attention to derived graphs of permutation voltage graphs.

Theorem 1. *Let $(p, \mathcal{P})$ be a concrete r-fold covering projection of G. Then there is an assignment f of voltages in the symmetric group S_r for G such that the diagram*

$$\begin{array}{ccc} H & \xrightarrow{\psi} & G_f \\ & p\searrow \qquad \swarrow p_f & \\ & G & \end{array}$$

is an isomorphism between $(p, \mathcal{P})$ and $(p_f, \mathcal{P}_f)$ for some ψ.

Proof: Let $P_1, \cdots, P_r$ be the sheets of P. For any two adjacent vertices v, w of G, set $f(v,w)(i) = j$ iff vertex $v_i \in p^{-1}(v) \cap P_i$ is adjacent to vertex $w_j \in p^{-1}(w) \cap P_j$ in H. Then the mapping $\psi : H \to G_f$, defined by $\psi(v_i) = (v,i)$, is the desired isomorphism. ∎

3. Classification of Concrete Covering Projections

Let $\mathcal{F}_r$ denote the set of permutation voltage assignments in S_r for G. Let Γ act on $A(G)$ via

$$\gamma(v,w) = (\gamma(v), \gamma(w)),$$

and let S_r act on itself via conjunction; denote these actions by Γ_a and S_r^c respectively. Now define $\alpha : A(G) \to A(G)$ by $\alpha(v,w) = (w,v)$ and $\beta : S_r \to S_r$ by $\beta(\rho) = \rho^{-1}$. Then $\mathcal{F}_r$ consists of all mappings $f : A(G) \to S_r$ such that

$f\alpha = \beta f$. Since every $\gamma \in \Gamma_a$ commutes with α and every $\varphi \in S_r^c$ commutes with β, the group $\Gamma_a \times S_r^c$ acts on $\mathcal{F}_r$ via exponentiation:

$$(\gamma, \varphi)(f) = \varphi f \gamma^{-1}. \tag{2}$$

Theorem 2. *The isomorphism classes of concrete r-fold covering projections are in one-to-one correpondence with the orbits of the exponentiation group $\Gamma_a \times S_r^c$ in $\mathcal{F}_r$.*

Proof: Assume that the permutation voltage assignments $f, \tilde{f}$ are in the same orbit of the exponentiation group $\Gamma_a \times S_r^c$. Then we have

$$\tilde{f}(v, w) = \varphi f(\gamma^{-1}(v), \gamma^{-1}(w))\varphi^{-1} \tag{3}$$

for some $\gamma \in \Gamma$, $\varphi \in S_r$ and every $(v, w) \in A(G)$. Define $\psi : G_f \to G_{\tilde{f}}$ by $\psi(u, i) = (\gamma(u), \varphi(i))$. Then ψ is an isomorphism, and the pair (ψ, γ) constitutes an isomorphism between the covering projections p_f and $p_{\tilde{f}}$. Since ψ preserves sheets, we have found an isomorphism between $(p_f, \mathcal{P}_f)$ and $(p_{\tilde{f}}, \mathcal{P}_{\tilde{f}})$.

Conversely, let $(p_f, \mathcal{P}_f)$ and $(p_{\tilde{f}}, \mathcal{P}_{\tilde{f}})$ be isomorphic for some $f, \tilde{f} \in \mathcal{F}_r$ with an isomorphism ψ and $\gamma \in \Gamma$. Since ψ preserves sheets and fibers, $\psi(u, i) = (\gamma(u), \varphi(i))$ for some $\varphi \in S_r$. Now it is easy to verify that equation (3) is satisfied for f and $\tilde{f}$ by φ and γ; hence both assignments are in the same orbit of exponentiation defined by (2). ∎

4. Enumeration of Concrete Covering Projections

Let R be the ring of rational polynomials in the variables $x_1, \cdots, x_r$; we write $\boldsymbol{x} = (x_1, \cdots, x_r)$ for short. A monomial in R is denoted by $\boldsymbol{x}^{\boldsymbol{\lambda}} = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r}$ where $\lambda_1, \cdots, \lambda_r$ are nonnegative integers. The *cap*-product on R, introduced by Redfield [9], is defined first for sequences $\boldsymbol{x}^{\boldsymbol{\lambda}}, \boldsymbol{x}^{\boldsymbol{\mu}}, \cdots$ of $q \geq 2$ monomials by

$$\boldsymbol{x}^{\boldsymbol{\lambda}} \cap \boldsymbol{x}^{\boldsymbol{\mu}} \cap \cdots = \left(\prod_{k=1}^{r} k^{\lambda_k} \lambda_k! \right)^{q-1} =: \Pi(\boldsymbol{\lambda})^{q-1}$$

if $\boldsymbol{\lambda} = \boldsymbol{\mu} = \cdots$, otherwise it is 0. Then the cap-product is linearly extended to arbitrary polynomials in these variables.

Now let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_r)$ be a partition of r, i.e.

$$r = \sum_{k=1}^{r} k\lambda_k.$$

The *partition polynomial* is the generating function of the partitions of r:

$$P_r(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}} \boldsymbol{x}^{\boldsymbol{\lambda}}.$$

Now let A be a group acting on a set X of order r. The *cycle type* of $\alpha \in A$ is $\lambda(\alpha) = (\lambda_1(\alpha), \cdots, \lambda_r(\alpha))$ where $\lambda_k(\alpha)$ is the number of cycles of α of length k for $k = 1, \cdots, r$. For a natural number p, the *p-th order cycle index* of A is the polynomial

$$Z_p(A; \mathbf{x}) = \frac{1}{|A|} \sum_{\alpha \in A} \mathbf{x}^{\lambda(\alpha^p)}.$$

For $p = 1$, this coincides with the (ordinary) cycle index introduced by Pólya [8].

Note that the automorphism group Γ of the graph G acts on the edge set $E(G)$ in a natural way; denote this action by Γ_e. An edge $[v, w]$ of G (or the corresponding edge cycle π) is said to be *diagonal* for $\gamma \in \Gamma_e$, if v, w are in the same (vertex) cycle of γ of length t and $\gamma^{\frac{t}{2}}(v) = w$. Every edge cycle π of $\gamma \in \Gamma_e$ of length $l(\pi)$ corresponds to two arc cycles of $\gamma \in \Gamma_a$ of the same length if π is not diagonal, while it corresponds to one arc cycle of γ of length $2\,l(\pi)$ if π is diagonal.

Now let $\gamma \in \Gamma_e$ and define, for any edge cycle π of γ, the polynomial

$$Q_r(\pi; \mathbf{x}) = \begin{cases} Z_2(S_r; \mathbf{x}) & \text{if } \pi \text{ is diagonal,} \\ P_r(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Theorem 3. *The number of isomorphism classes of concrete r-fold covering projections of G is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma_e} \frac{1}{r!} \sum_{\varphi \in S_r} \prod_{\pi \in C(\gamma)} Q_r(\pi; \mathbf{x}) \cap \mathbf{x}^{\lambda(\varphi^{s(\pi)})},$$

where $C(\gamma)$ the set of edge cycles of γ and $s(\pi)$ is the length of an arc cycle of γ corresponding to π.

Proof: By Theorem 2 and Burnside's lemma, we have to count the permutation voltage assignments f in S_r for G which are fixed under exponentiation (2) for $\gamma \in \Gamma_a$ and $\varphi \in S_r^c$, i.e. the assignments f such that

$$f(\gamma(v), \gamma(w)) = \varphi f(v, w) \varphi^{-1} \tag{4}$$

for every arc (v, w) of G. Let π be the edge cycle of γ containing $[v, w] \in E(G)$. We distinguish two cases.

Case 1: The edge cycle π is not diagonal. The length of a corresponding arc cycle π' containing (v, w) is $s(\pi)$. By induction, if the assignment f satisfies equation (4) for every arc in π', then

$$f(v, w) = \varphi^{s(\pi)} f(v, w) \varphi^{-s(\pi)}. \tag{5}$$

On the other hand, if we assign $f(v,w)$ to the arc (v,w) such that (5) holds, the assignments for the other arcs in π' are given recursively by equation (4). It is well known that there are

$$\Pi(\lambda(\varphi^{s(\pi)})) = Q_r(\pi;\boldsymbol{x}) \cap \boldsymbol{x}^{\lambda(\varphi^{s(\pi)})}$$

possible choices for $f(v,w)$ in S_r satisfying equation (5).

Case 2: If the edge cycle π is diagonal, the same arguments as in Case 1 lead to

$$f(v,w)^{-1} = \varphi^{\frac{s(\pi)}{2}} f(v,w) \varphi^{-\frac{s(\pi)}{2}} \tag{6}$$

as a necessary and sufficient condition for the choice of $f(v,w)$.

Lemma 4. *Let $\alpha \in S_r$. Then the equation $\beta^{-1} = \alpha\beta\alpha^{-1}$ has*

$$Z_2(S_r;\boldsymbol{x}) \cap \boldsymbol{x}^{\lambda(\alpha^2)}$$

solutions in S_r.

Proof of the lemma: It is easy to see that the solutions of $\beta^{-1} = \alpha\beta\alpha^{-1}$ correspond bijectively to the solutions of $\xi^2 = \alpha^2$ by setting $\xi = \beta\alpha$.

Now let $\delta = \alpha^2$ and let $\delta_1, \cdots, \delta_w$ be the conjugates of δ in S_r. It is well known that

$$\omega = \frac{r!}{\Pi(\lambda(\delta))}.$$

The sets $X_i := \{\xi \in S_r | \xi^2 = \delta_i\}$ are pairwise disjoint and of the same cardinality $c(\delta)$; hence

$$\begin{aligned} Z_2(S_r;\boldsymbol{x}) \cap \boldsymbol{x}^{\lambda(\delta)} &= \frac{1}{r!}\sum_{\varphi\in S_r} \boldsymbol{x}^{\lambda(\varphi^2)} \cap \boldsymbol{x}^{\lambda(\delta)} \\ &= \frac{1}{r!}\sum_{i=1}^{\omega}\sum_{\xi\in X_i} \boldsymbol{x}^{\lambda(\xi^2)} \cap \boldsymbol{x}^{\lambda(\delta)} \\ &= \frac{1}{r!}\omega\left(c(\delta)\frac{r!}{\omega}\right) \\ &= c(\delta). \end{aligned}$$

∎

To complete the proof of Theorem 3, note that equation (6) amounts to

$$Q_r(\pi;\boldsymbol{x}) \cap \boldsymbol{x}^{\lambda(\varphi^{s(\pi)})}$$

possible choices for $f(v,w)$ by Lemma 4. ∎

Corollary 1. *The number of isomorphism classes of concrete 2-fold covering projections of G is*

$$Z(\Gamma_e;\mathbf{2}),$$

where $\mathbf{2} = (2,\cdots,2)$.

Proof: We have $Z_2(S_2;\boldsymbol{x}) = x_1^2$ and $P_2(\boldsymbol{x}) = x_1^2 + x_2$. If $\boldsymbol{x}^{\lambda(\varphi^{s(\pi)})} = x_1^2$, then

$$Z_2(S_2;\boldsymbol{x}) \cap x_1^2 = P_2(\boldsymbol{x}) \cap x_1^2 = 2.$$

If $\boldsymbol{x}^{\lambda(\varphi^{s(\pi)})} = x_2$ then $s(\pi)$ is odd, hence π is not diagonal, and

$$P_2(\boldsymbol{x}) \cap x_2 = 2.$$

The assertion now follows from Theorem 3. ∎

We continue with some useful remarks for application of the counting formula of Theorem 3.

1. If π is a diagonal edge cycle for the (vertex) cycle σ of length t, then $s(\pi) = t$. Now assume that π is not diagonal and contains an edge $[v, w]$ such that v, w are contained in the (vertex) cycles σ_v, σ_w of lengths t_v, t_w respectively; then $s(\pi) = lcm(t_v, t_w)$.
2. The cycle type of φ^s is

$$\lambda_i(\varphi^2) = \sum_k \lambda_k(\varphi)\, gcd(s,k),$$

where k ranges over all nonnegative integers such that $lcm(s,k) = si$ ($i = 1,\cdots,r$).
3. The second order cycle index $Z_2(S_r;\boldsymbol{x})$ can easily be computed from the cycle index $Z(S_r;\boldsymbol{x})$. The cycle indices of symmetric groups S_r are tabulated in [3] for $r \leq 10$.

The crux in applications of Theorem 3 are diagonal edge cycles. If they are excluded, a more closed counting formula can be obtained.

Theorem 5. *If no automorphism of G contains diagonal edge cycles, then the number of isomorphism classes of concrete r-fold covering projections of G is*

$$\frac{1}{r!}\sum_{\varphi\in S_r} Z(\Gamma_e;\mathbf{\Pi}(\boldsymbol{\lambda}(\varphi))),$$

where $\mathbf{\Pi}(\boldsymbol{\lambda}(\varphi)) = (\Pi(\boldsymbol{\lambda}(\varphi)), \Pi(\boldsymbol{\lambda}(\varphi^2)), \cdots, \Pi(\boldsymbol{\lambda}(\varphi^r)))$.

Proof: Since $P_r(\boldsymbol{x}) \cap \boldsymbol{x}^{\lambda(\varphi^{s(\pi)})} = \Pi(\lambda(\varphi^{s(\pi)}))$, we obtain from Theorem 3

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma_e}\frac{1}{r!}\sum_{\varphi\in S_r}\Pi(\lambda(\varphi^{s(\pi)}))$$

$$= \frac{1}{r!}\sum_{\varphi\in S_r}\left(\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma_e}\Pi(\lambda(\varphi))^{\lambda_1(\gamma)}\Pi(\lambda(\varphi^2))^{\lambda_2(\gamma)}\cdot\ldots\cdot\Pi(\lambda(\varphi^r))^{\lambda_r(\gamma)}\right)$$

$$= \frac{1}{r!}\sum_{\varphi\in S_r} Z(\Gamma_e;\boldsymbol{\Pi}(\lambda(\varphi))).$$

∎

If we restrict attention to identity graphs, we conclude

Corollary 2. *If G is an identity graph with m edges, then the number of isomorphism classes of r-fold covering projections of G is*

$$P_r(\boldsymbol{x})^m_\cap;$$

the power is to be understood with respect to the cap-product, which is indicated by the $\cap$-index.

Proof: Since $Z(\Gamma_e;\boldsymbol{\Pi}(\lambda(\varphi))) = \Pi(\lambda(\varphi))^m$ if Γ_e is trivial, we obtain from Theorem 5

$$\frac{1}{r!}\sum_{\varphi\in S_r}\Pi(\lambda(\varphi))^m = \sum_{\lambda}{}'\,\Pi(\lambda)^{m-1} = P_r(\boldsymbol{x})^m_\cap,$$

which proves the corollary. ∎

Note that almost all graphs are identity graphs [3, p. 206].

References

1. J. L. Gross, T. W. Tucker, *Generating all Graph Coverings by Permutation Voltage Assignments*, Discrete Math. **18** (1977), 273–283.
2. J. L. Gross, T. W. Tucker, *Topological Graph Theory*, in "Wiley Interscience Series in Discrete Mathematics and Optimization", John Wiley & Sons, 1987.
3. F. Harary, E. M. Palmer, "Graphical Enumeration", Academic Press, New York and London, 1973.
4. M. Hofmeister, *Counting Double Covers of Graphs*, J. Graph Theory **12** (1988).
5. M. Hofmeister, *Isomorphisms and Automorphisms of Graph Coverings*, Discrete Math. (to appear).
6. B. Mohar, *Akempic Triangulations with 4 Odd Vertices*, Discrete Math. **54** (1985), 23–29.
7. B. Mohar, *The Enumeration of Akempic Triangulations*, J. Comb. Theory Ser. B **42** (1987), 14–23.
8. G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Math. **68** (1937), 145–254.
9. J. H. Redfield, *The Theory of Group-reduced Distributions*, Am. J. Math. **49** (1927), 433–455.