# On Latin Triangles

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**Abstract.** Halberstam, Hoffman and Richter introduced the idea of a Latin triangle as an analogue of a Latin square, showed the existence or non-existence of Latin triangles for small orders, and used a multiplication technique to generate triangles of orders $3^n$ and $3^n - 1$. We generalize this multiplication theorem and provide a construction of Latin triangles of odd order $n$ for $n$ such that $n+2$ is prime. We also discuss scalar multiplication, orthogonal triangles, and results of computer searches

## I. Introduction

The concept of a Latin square has been generalized in many ways, such as by relaxing the Latin property (the requirement that each symbol must occur exactly once in each line) or by requiring that lines in each of three directions be Latin. In [1], Halberstam, Hoffman and Richter introduce the Latin triangle, in which the rows are parallel to the sides of an equilateral triangle and a natural pairing of the rows produces lines which are to be Latin.

A triangle of order $n$ is an array in the shape of an equilateral triangle having $n$ rows in each of the directions parallel to a side of the triangle with row $i$ ($1 \leq i \leq n$) having $n+1-i$ entries. The horizontal rows are called the $a$-rows. The rows in the \-direction are the $b$-rows, and the rows in the /-direction are the $c$-rows. For the definition of a Latin triangle of order $n$, or an LT($n$), we must distinguish between the even and the odd case.

For odd positive integer $n$ and for $x \in \{a, b, c\}$ the line given by $x = i$ (for $2 \leq i \leq \frac{n+1}{2}$) is the union of the $x$-rows $x = i$ and $x = n+2-i$. The line given by $x = 1$ is just the $x$-row 1. An LT($n$) is a triangle of order $n$ with entries among $n$ distinct symbols such that each symbol occurs exactly once in every line. For even positive integer $n$ and for $x \in \{a, b, c\}$ the line given by $x = i$ (for $1 \leq i \leq \frac{n}{2}$) is the union of the $x$-rows $x = i$ and $x = n+1-i$. An LT($n$) is a triangle of order $n$ with entries among $n+1$ distinct symbols such that each symbol occurs exactly once in every line.

```
                5
              6   4
            4   5   6
          8   6   4   2
        9   7   5   3   1
      7   8   9   1   2   3
    2   3   1   5   9   7   8
  3   1   2   6   4   8   9   7
1   2   3   4   5   6   7   8   9
```

Figure 1

```
              9
            4   5
          5   9   4
        6   4   5   3
      7   8   9   1   2
    2   3   1   8   6   7
  3   1   2   9   7   8   6
1   2   3   4   5   6   7   8
```

Figure 2

Figure 1 illustrates an LT (9) with fifth $a$-row (9 7 5 3 1), seventh $b$-row

```
2
  1
    3
```

and sixth $c$-row

```
      3
    7
  8
6
```

.

The line given by $a = 3$ is the union of the third $a$-row (2 3 1 5 9 7 8) and the eighth $a$-row (6 4).

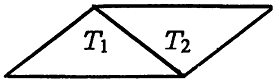
Figure 2 illustrates an LT (8). The line given by $a = 3$ is the union of the third $a$-row (2 3 1 8 6 7) and the sixth $a$-row (5 9 4).

An entry in the lines $a = i, b = j, c = k$ has coordinates $(i, j, k)$. In Figure 1 the five occurrences of the entry 2 have coordinates $(1,3,2), (2,5,3), (3,4,1), (4,2,5), (5,1,4)$. In Figure 2 the occurrences of 7 have coordinates $(1,2,2)$, $(2,3,4), (3,1,3)$, and $(4,4,1)$. With this system of coordinatizing, coordinates do not represent unique positions in triangles of even order.

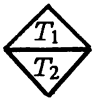
In [1] and [2] examples are given of LT($n$)'s for every order $n \leq 13$, except for $n = 4, 6, 10$ which are shown not to exist, and for $n = 15$. The first of these papers presents a multiplication theorem which implies the existence for every positive integer $n$ of an LT($3^n$) and an LT($3^n - 1$). We generalize this multiplication theorem and show restrictions on this type of multiplication result. We give a constructive proof of existence of LT($n$)'s for any $n$ such that $n + 2$ is prime and show a new example of an LT(17) constructed in this way. We also present an LT(14) and discuss multiplication of a Latin triangle by a scalar and results of some computer searches. Finally, the concept of orthogonal Latin triangles introduced in [2] is addressed.

## II. Multiplication Theorem

Definition. Let $n$ be an odd integer. A complementary pair of Latin triangles of order $n$ is an ordered pair $(T_1, T_2)$ such that $T_1$ is an LT($n$), $T_2$ is an LT($n-1$) and the arrays




and



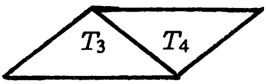

considered as squares, are Latin.

**Theorem 1.** *Let $(T_1, T_2)$, $(T_3, T_4)$ be complementary pairs of Latin triangles of orders $n$ and $m$ (both odd) respectively. Then there exists an $LT(nm)$ and an $LT(nm-1)$.*

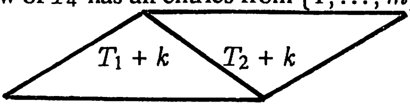
Proof: Without loss of generality, assume that the entries of $T_1$, $T_2$ belong to $\{1, 2, \ldots, n\}$, and the entries of $T_3, T_4$ belong to $\{1, 2, \ldots, m\}$. Construct a new triangle as in Figure 3, where $u_s$ is the entry in the corresponding position of $T_3$ ($s = (i, j, k)$) and $v_t$ is the corresponding entry in $T_4$. That is, $u_1$ is the (2,1,1) entry of $T_3, \ldots, u_{\frac{m(m+1)}{2}}$ is the (1,2,1) entry of $T_3$, and $v_1$ is the (1,1,1) entry of $T_4$ (top vertex),..., $v_{\frac{m(m-1)}{2}}$ is the (1,1,1) entry of $T_4$ (bottom right vertex). For $i = 1$ or 2, $T_i + k$ means the triangle with $(k-1)n$ added to each entry in $T_i$. We need only to check that each of the horizontal lines contains each element from $\{1, \ldots, nm\}$, since the argument is analogous for the other two directions.

The case $a = 1$ is clear. The line $a = n+1$ consists of the bottom rows of $T_1 + u_i$ for all $u_i$ in the second row of $T_3$ plus the bottom row of $T_1 + u_1$. Since $T_3$ is Latin, $u_1$ is not in its second row. Thus we obtain all the elements from $\{1, \ldots, nm\}$. Similar reasoning applies for the lines $a = in + 1$, for $i = 2, \ldots, \frac{m-1}{2}$.

Consider lines $a = j$ with $2 \leq j \leq n$. Since




is Latin the bottom row of $T_4$ has all entries from $\{1, \ldots, m\}$ except $u_1$. Clearly




will be Latin as well, and so line $j$ has all entries from $\{1, \ldots, nm\}$ except for the entries $(u_1 - 1)n + 1, \ldots, u_1 n$, but these appear by matching the correct rows of $T_1 + u_1$ at the bottom and top of the constructed triangle.

For $a = j$ with $n + 2 \leq j \leq 2n$ we note that $u_2, u_3$ must be included in the second row of $T_3$ and so the entries $(u_2 - 1)n + 1, \ldots, u_2 n, (u_3 - 1)n + 1, \ldots, u_3 n$ are covered. Since $T_3, T_4$ are complementary we get $v_1 = u_1$ and since $v_1$ does not appear in the second row of $T_3$ we obtain all the entries $(u_1 - 1)n + 1, \ldots, u_1 n$

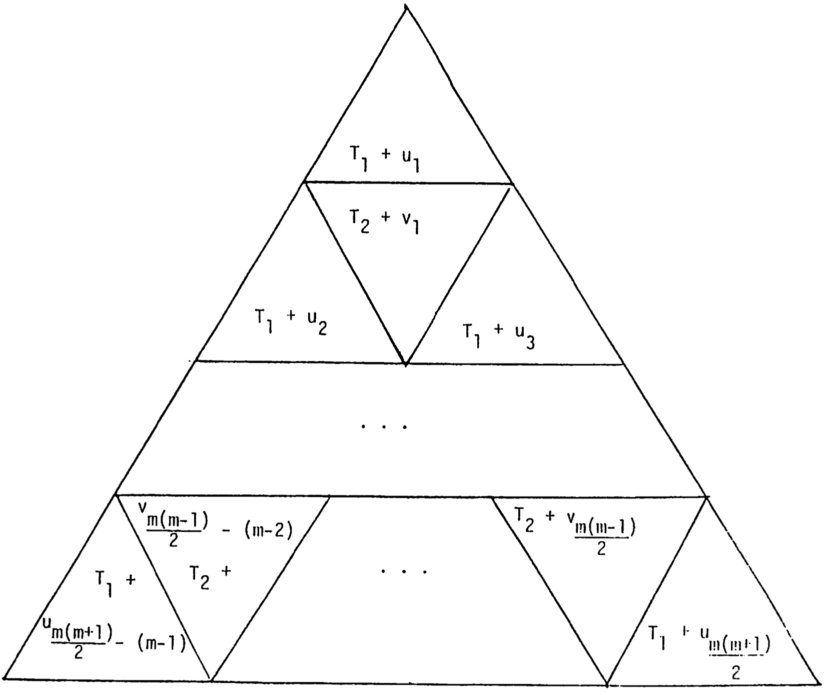
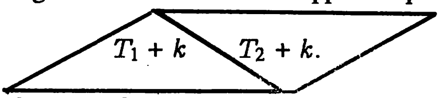
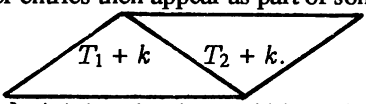



Figure 3

without repetition by matching the correct rows of $T_2 + v_1$ at the bottom and top of the constructed triangle. All the other entries appear as part of some




For $a = j$ with $2n + 2 \leq j \leq 3n$ we note that since $T_3, T_4$ are complementary, $\{u_2, u_3\} = \{v_2, v_3\}$. If $\{u_4, u_5, u_6\} \cap \{u_2, u_3\} = \phi$ then $u_4, u_5, u_6$ must be included in the third row of $T_3$ and so we obtain the entries $(u_4 - 1)n + 1, \ldots, u_4 n$, $(u_5 - 1)n + 1, \ldots, u_5 n$, $(u_6 - 1)n + 1, \ldots, u_6 n$ by matching the correct rows of $T_2 + v_2$ at the bottom and top of the constructed triangle. The situation is similar with $T_2 + v_3$, and since $u_2 u_3$ do not appear in the third row of $T_3$ we have no repetitions. All the other entries then appear as part of some




If $\{u_4, u_5, u_6\} \cap \{u_2, u_3\} \neq \phi$ then the element(s) in the intersection do not appear in the third row of $T_3$ nor in the third row of $T_4$, so if $k \in \{u_4, u_5, u_6\} \cap \{u_2, u_3\}$ we obtain the entries $(k - 1)n + 1, \ldots, kn$ from the third "row" from the top of the constructed triangle. If $k \in \{u_4, u_5, u_6\} - \{u_2, u_3\}$ then $k$ is in the third row of $T_3$ and if $k \in \{u_2, u_3\} - \{u_4, u_5, u_6\}$ then $k$ is in the third row of $T_4$, thus yielding all the necessary entries exactly once, while the others are obtained from some $T_1 + k \quad T_2 + k$ the third "row" of the constructed triangle. Arguments similar to the last case clearly work for remaining lines.

Similarly we also conclude that the triangle shown in Figure 4 is an LT($nm - 1$) where $u_s$, $v_t$ have the same meaning as before.

**Corollary 1.** *Let $(T_1, T_2)$ be a complementary pair of Latin triangles of odd order $n$. Then there exists an $LT(n^2)$ and an $LT(n^2 - 1)$.*

**Corollary 2.** *Let $(T_1, T_2), (T_3, T_4)$ be complementary pairs of Latin triangles as in Theorem 1. Then for each nonnegative integer $k$ there is an $LT(m^k n)$ and an $LT(m^k n - 1)$.*

Proof: Arguments similar to those in the proof of Theorem 1 show that the Latin triangles so constructed are again complementary.

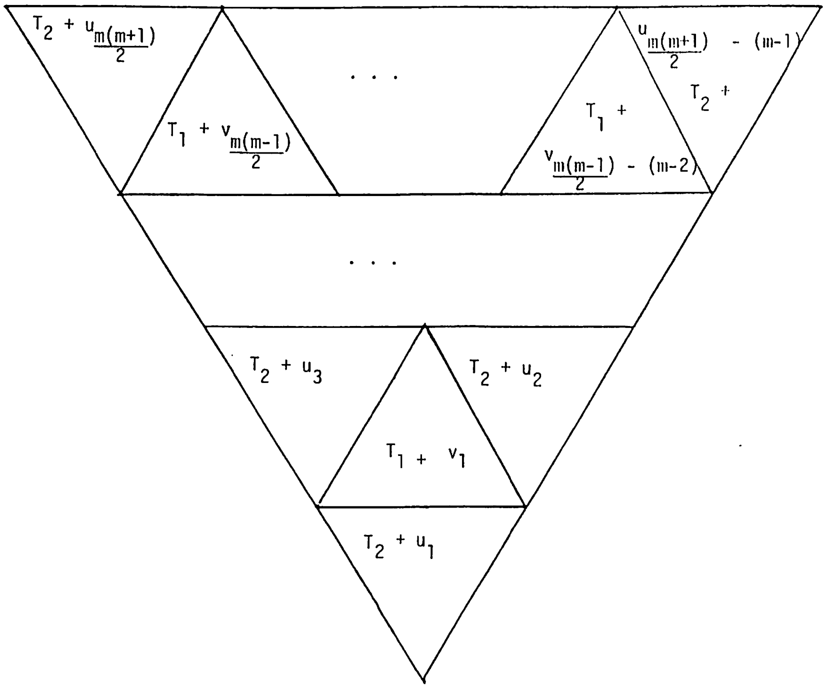



Figure 4

We provide two examples of complementary pairs of Latin triangles: one example of order 3 and one of order 15.

```
          1
   T1 =  3 2          1
        2 1 3   T2 = 2 3
```

```
                                 1
                                3 2
            1   3 2            2 1 3
           3 2   1              3 2
Clearly   2 1 3        and       1        are Latin.
```

```
                              1   E 4 7 5 2 3 6 C 8 B D 9 A F
                             B 2   D E C 8 7 4 1 A 9 3 6 F 5
                            3 1 8   C 7 E 4 5 2 B D A F 9 6
                           4 8 3 A   B 6 D E 9 1 7 F 5 C 2
                          7 D 1 5 9   A C B 3 E F 8 2 6 4
T1 =                     8 B C 6 2 3   9 D A F 1 E 4 5 7      = T2
                        2 9 A 1 4 7 B   8 F 5 C 6 D E 3
                       E C 5 3 8 D 6 F   7 2 4 1 A B 9
                      A 7 E D 1 5 F 9 4   6 3 2 B 8 C
                     6 3 B 4 E F A 2 8 C   5 4 1 7 D
                    C 5 2 9 F 1 E 7 B D 6   4 8 3 A
                   D 4 7 F 6 B 2 C E 9 A 5   3 1 8
                  9 6 F 8 D 4 1 A 5 3 E C 7   2 B
                 5 F 9 B A C 8 3 6 4 2 7 E D   1
                F A 6 2 3 9 5 1 D 7 8 B C 4 E
```

The first example together with Theorem 1 yields Theorem 2 of [1] and the same example with corollary 2 yields corollary 2.1 of [1]. The second example, however, is new. It is the result of a computer search [3] for an LT (15) that forms a complementary pair with a specified LT(14). (See section IV.) Applying the multiplication theorem and its corollaries, we deduce the existence of an LT($3^j \cdot 5^k$) and LT($3^j \cdot 5^k - 1$), where $j \geq k \geq 0$.

In [1] it was asked whether there are other multiplication theorems. It is clear that any result based on "bumping" two Latin triangles of orders $n$ and $(n-1)$ as in Theorem 1 produces a pair of complementary Latin triangles whose entries are the constants $u_s$ and $v_t$. Thus there are multiplication theorems of this type if and only if there are complementary Latin triangles of orders $n$ and $m$.

## III. An Algebraic Construction

Up to this point, the only known examples of Latin triangles have been those of small orders and those that result from the multiplication theorem of [1]. The following direct construction is based on properties of the finite fields.

**Theorem 2.** *If $n+2$ is prime, then there exists a Latin triangle of order $n$.*

Proof: We will show that the triangle above (or below) the cross diagonal of the multiplication table for the field $GF(n+2)$ is an LT($n$) when the headline and sideline of the table are properly ordered. The required ordering is $1, 2, 3 \ldots n+1$ across the headline and $1^{-1}, 2^{-2}, 3^{-2} \ldots (n+1)^{-1}$ down the sideline. (See the example following the proof.) The table has the form shown. The $a$-rows are those parallel to the cross diagonal. The $b$-rows are horizontal; the $c$-rows are vertical.

| | $1$ | $2$ | $3$ | . | . | $j$ | . | . | . | $n+1$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $1^{-1}$ | $1$ | $2$ | $3$ | . | . | $j$ | . | . | . | $n+1$ |
| $2^{-1}$ | $2^{-1}$ | | | | | | | | | |
| $3^{-1}$ | $3^{-1}$ | | | | | | | | | |
| $\vdots$ | | | | | | | | | | |
| $i^{-1}$ | $i^{-1}$ | | | | | $i^{-1}j$ | | | | |
| $\vdots$ | | | | | | | | | | |
| $\vdots$ | | | | | | | | | | |
| $(n+1)^{-1}$ | $(n+1)^{-1}$ | | | | | | | | | |

The entry in the $(i, j)$ position is $i^{-1}j$. If the $(i, j)$ entry is in the partial cross diagonal whose upper right entry is $k$, then $i + j = k + 1$. Note that if $i^{-1}j$ is in the main cross diagonal, $i + j = n + 2 = 0$, so $j = -i$ and $i^{-1} = i^{-1}(-i) = -1 = n + 1$. Therefore all the entries above the main cross diagonal are from the set $\{1, 2, 3, \ldots, n\}$. The table has rotational symmetry; that is, the $(i, j)$ entry is the same as the $(-i, -j)$ entry. Since the table is a multiplication table with rotational symmetry, the triangle is Latin along the $b$ and $c$ lines.

Now consider lines in the $a$ direction. The first $a$-line is the partial cross diagonal headed by $n$. The $k$th line is made up of the two partial cross diagonals headed by $n - k + 1$ and $k - 1$ for $k = 2, \ldots, \frac{n+1}{2}$. Suppose that two entries in the $k$th $a$-line are the same and that they occur in the $(r, s)$ and $(t, u)$ positions. First, assume that $(r, s)$ and $(t, u)$ are in the same $a$-row. Let $h$ be the heading of that row. Then $r + s = h + 1 = t + u$; $r^{-1}s = t^{-1}u$; so $r + s = rus^{-1} + u$; $s(r + s) = u(r + s)$. So $s = u$ or $r + s - 0$. But $r + s = h + 1 < n + 2$, so $r + s \neq 0$; consequently, $s = u$ and $r = t$.

Now suppose $r^{-1}s = t^{-1}u$ are in the same $a$-line but in different $a$-rows where $r^{-1}s$ is in the diagonal row headed by $n - k + 1$ and $t^{-1}u$ is in the diagonal row headed by $k - 1$. Then $r + s = n - k + 2$ and $t + u = k$, so $r + s + t + u = 0$. Thus $s(r + s) + s(t + u) = 0$, and since $t = rus^{-1}$, $(s + u)(r + s) = 0$. Therefore $r + s = 0$ or $s + u = 0$. But $r + s = n - k + 2 < n + 2$, since $1 \leq k \leq n$, so $r + s \neq 0$. Furthermore $r + s = n - k + 2$ and $t + u = k$ imply that $s \leq n - k + 1$

| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1^{-1}$ | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $2^{-1}$ | 10 | 10 | 1 | 11 | 2 | 12 | 3 | 13 | 4 | 14 | 5 | 15 | 6 | 16 | 7 | 17 | 8 | 18 | 9 |
| $3^{-1}$ | 13 | 13 | 7 | 1 | 14 | 8 | 2 | 15 | 9 | 3 | 16 | 10 | 4 | 17 | 1 | 5 | 18 | 12 | 6 |
| $4^{-1}$ | 5 | 5 | 10 | 15 | 1 | 6 | 11 | 16 | 2 | 7 | 12 | 17 | 3 | 8 | 13 | 18 | 4 | 9 | 14 |
| $5^{-1}$ | 4 | 4 | 8 | 12 | 16 | 1 | 5 | 9 | 13 | 17 | 2 | 6 | 10 | 14 | 18 | 3 | 7 | 11 | 15 |
| $6^{-1}$ | 16 | 16 | 13 | 10 | 7 | 4 | 1 | 17 | 14 | 11 | 8 | 5 | 2 | 18 | 15 | 12 | 9 | 6 | 3 |
| $7^{-1}$ | 11 | 11 | 3 | 14 | 6 | 17 | 9 | 1 | 12 | 4 | 15 | 7 | 18 | 10 | 2 | 13 | 5 | 16 | 8 |
| $8^{-1}$ | 12 | 12 | 5 | 17 | 10 | 3 | 15 | 8 | 1 | 13 | 6 | 18 | 11 | 4 | 16 | 9 | 2 | 14 | 7 |
| $9^{-1}$ | 17 | 17 | 15 | 13 | 11 | 9 | 7 | 5 | 3 | 1 | 18 | 16 | 14 | 12 | 10 | 8 | 6 | 4 | 2 |
| $10^{-1}$ | 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| $11^{-1}$ | 7 | 7 | 14 | 2 | 9 | 16 | 4 | 11 | 18 | 6 | 13 | 1 | 8 | 15 | 3 | 10 | 17 | 5 | 12 |
| $12^{-1}$ | 8 | 8 | 16 | 5 | 13 | 2 | 10 | 18 | 7 | 15 | 4 | 12 | 1 | 9 | 17 | 6 | 14 | 3 | 11 |
| $13^{-1}$ | 3 | 3 | 6 | 9 | 12 | 15 | 18 | 2 | 5 | 8 | 11 | 14 | 17 | 1 | 4 | 7 | 10 | 13 | 16 |
| $14^{-1}$ | 15 | 15 | 11 | 7 | 3 | 18 | 14 | 10 | 6 | 2 | 17 | 13 | 9 | 5 | 1 | 16 | 12 | 8 | [illegible] |
| $15^{-1}$ | 14 | 14 | 9 | 4 | 18 | 13 | 8 | 3 | 17 | 12 | 7 | 2 | 16 | 11 | 6 | 1 | 15 | 10 | 5 |
| $16^{-1}$ | 6 | 6 | 12 | 18 | 5 | 11 | 17 | 4 | 10 | 16 | 3 | 9 | 15 | 2 | 8 | 14 | 1 | 7 | 13 |
| $17^{-1}$ | 9 | 9 | 18 | 8 | 17 | 7 | 16 | 6 | 15 | 5 | 14 | 4 | 13 | 3 | 12 | 2 | 11 | 1 | 10 |
| $18^{-1}$ | 8 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

An LT(17)

and $u \leq k - 1$. Thus $s + u \leq n$, so $s + u \neq 0$. We conclude that this triangle is Latin in the $a$ lines.

It is also instructive to view Latin triangles from a different perspective. Consider a triangular array of odd order $n = 2s - 1$. We may label each cell with its three line coordinates, as in section I. Then an LT($n$) may be represented as $n$ sets of $s$ triples from $\{1, 2, \ldots s\}$ where each set is the transversal of cells occupied by some fixed symbol from $\{1, 2, \ldots, n\}$. For example the LT(9) of figure 1 may be represented by the system below of triples from $\{1, 2, 3, 4, 5\}$.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 121 | 242 | 353 | 434 | 515 |
| 2 | 132 | 253 | 341* | 425 | 514* |
| 3 | 143 | 231 | 352 | 415 | 524 |
| 4 | 154 | 245 | 312 | 431 | 523 |
| 5 | 155 | 211 | 344 | 422 | 533 |
| 6 | 145 | 254 | 321 | 413 | 532 |
| 7 | 134 | 213 | 325 | 451 | 542 |
| 8 | 123 | 235 | 314* | 452 | 541* |
| 9 | 112 | 224 | 335 | 443 | 551 |

As an aside, it may be observed that the entries 2 and 8 could be interchanged in the four cells marked with * to produce a new triangle with the standard base (i.e., with first $a$-row 1 2 3...9). This type of interchange can also be effected with 4 and 6, in each of the last three cells listed for these entries.

The *row* coordinates of each cell of a triangle of order $n = 2s - 1$ always sum to $n+2$. Thus if the *line* coordinates $i, j, k$ are not identical to the row coordinates, then at most one of the corresponding row coordinates is larger than $s$. Using the fact that line $m$ consists of row $m$ paired with row $(n+2) - m$, it follows that exactly one of the following holds:

$$i + j + k = n + 2 \text{ or}$$

$$i - j + k \text{ or } j = i + k \text{ or } k = i + j.$$

The set of all triples from $\{1, 2, \ldots, s\}$ that satisfy the property above may be called triangular triples. An LT($n$) is a partition of the $(n^2 + n)/2$ triangular triples into $n$ sets of $s$ triples such that in each partition set, each of $1, 2, \ldots, s$ occurs exactly once in each coordinate position.

For each $n$ such that $n + 2$ is prime we can produce an LT($n$) as a system of triangular triples by permuting the line coordinates. We begin with the triples having first coordinate 1, i.e. the cells of the first $a$-row of a triangle. In the example shown here with $n = 9$ these cells are presented in the first column.

| | | | | |
|---|---|---|---|---|
| 121 | 242 | 353 | 434 | 515 |
| 132 | 254 | 325 | 413 | 541 |
| 143 | 235 | 312 | 451 | 524 |
| 154 | 213 | 341 | 425 | 532 |
| 155 | 211 | 344 | 422 | 533 |
| 145 | 231 | 314 | 452 | 532 |
| 134 | 253 | 321 | 415 | 542 |
| 123 | 245 | 352 | 431 | 514 |
| 112 | 224 | 335 | 443 | 551 |

The triples in each column are formed by replacing each coordinate in the triples of column 1 by the corresponding entry in the appropriate column of the 5 by 5 table shown. For example, to form the third column of the triple system, we use the third column of the square to produce 353 from 121.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 4 | 5 | 3 | 1 |
| 3 | 5 | 2 | 1 | 4 |
| 4 | 3 | 1 | 5 | 2 |
| 5 | 1 | 4 | 2 | 3 |

In general, for $n = 2s - 1$, the triples with first coordinate 1 are: 121, 132, 143,...,1 $s(s-1)$, $1ss$, $1(s-1)s$,...,123,112. For each $i$, $1 \leq i \leq s$, let $f_i$ be the permutation of $\{1, 2, \ldots, s\}$ defined by

$$f_i(x) = \begin{cases} ix(\mod n+2) & \text{if } ix \leq s \\ -ix(\mod n+2) & \text{if } ix > s \end{cases}$$

and define the action of $f_i$ on triples coordinatewise. It is straightforward to show for each triple xyz with first coordinate 1 that $f_i(xyz)$ is a triangular triple, that $\{f_i(xyz) : 1 \leq i \leq s\}$ is a transversal, and that no triple appears in 2 different transversals. The Latin triangles constructed in this way can be shown to be the same triangles contructed algebraically earlier in this section.

## IV. Comments

We call an LT($n$) symmetric about a median if there exists a pairing of entries such that whenever a symbol $i$ occurs as an entry in the triangle, its mate $i\prime$ occurs in the corresponding position on the other side of the median. For example, the LT(8) of Figure 2 is symmetric about its vertical median, with pairs 1 and 8, 2 and 7, 3 and 6, and 4 and 5. The LT(9) of Figure 1 has three axes of symmetry. All triangles presented in [1] and [2] are symmetric about at least one median. Figure 5 illustrates an LT(8) with no axis of symmetry.

```
              9
             5 4
            4 8 5
           6 5 4 3
          7 9 8 1 2
         2 3 1 9 6 7
        3 1 2 8 7 9 6
       1 2 3 4 5 6 7 8
```

Figure 5

The fact that line coordinates do not represent unique positions in triangles of even order $2s$ may be exploited to produce many different Latin triangles from a given triangle. For example, the three vertices may be permuted. The other ambiguous coordinate triples, on the outside rows of the triangle, are $1xx$, $x1x$, and $xx1$ where $2 \leq x \leq s$. Interchanging the two entries in the cells in $a$-row 1 represented by $1xx$ af an LT($2s$) yields a Latin triangle, as does interchanging the entries in the cells represented by $x1x$ or $xx1$. If we wish to keep a fixed horizontal base, we may choose to interchange any of the $2(s-1)$ pairs of cells $x1x$ or $xx1$. (Interchanging both these pairs of entries of a triangle symmetric about the vertical diagonal preserves symmetry.)

Continuing with the idea of producing different Latin triangles with the same horizontal base as a given LT($2s$), consider the three transversals that contain the three vertices. We may relabel all entries of one of these transversals using any one of the three symbols appearing in these transversals and relabel the other two transversals with the remaining symbols. Each of the 6 possible relabelings produces a Latin triangle. Now by permuting just the vertex entries as necessary we can restore the base of the triangle. Similarly for each of the $s-1$ pairs of transversals with entries along the base of the triangle in cells $1xx$ for $2 \leq x \leq s$, we can relabel all the entries in a pair of transversals and then restore the base by permuting the entries in the two $1xx$ cells. If $s \geq 4$ then the triangle has nontrivial interior so that relabeling transversals is distinct from interchanging symbols on the outside edges. Thus if there exists an LT($2s$) with $2s \geq 8$, there exist at least $2^{2(s-1)} \cdot 6 \cdot 2^{s-1} = 3 \cdot 2^{3s-2}$ Latin triangles of order $2s$ having the same horizontal base.

For $n = 2s = 8$, a computer search produced exactly $3072 = 3 \cdot 2^{10}$ LT($n$)'s with horizontal base $1, 2, 3, \ldots, 8$. Thus every LT(8) may be obtained from any other by a sequence of interchanges as described above.

Another method of producing an LT($n$) from a given Latin triangle of the same order is a form of scalar multiplication. Let $0 < k < n+2$ with $(k, n+2) = 1$. Let $*$ denote multiplication modulo $n+2$. If $(a, b, c)$ is a triple of row coordinates of a triangle of order $n$, then exactly one of $(k*a, k*b, k*c)$ or $(-k*a, -k*b, -k*c)$ is a row triple. Define $k*(a, b, c)$ to be this row triple. For example, with $n = 9$, $4*(2,3,6) = (8,1,2)$ and $5*(2,3,6) = (1,7,3)$. It is straightforward to show that if $\tau$ is a transversal then $k*\tau = \{k*(a,b,c) : (a,b,c) \in \tau\}$ is a transversal. Furthermore if $T$ is a Latin triangle then $k*T$ is a set of transversals which is a Latin triangle. Scalar multiplication may be applied to triangles of even or odd order, but $k*T$ is not necessarily different from $T$.

An exhaustive computer search produced 51 Latin triangles of order 9. Each has at least one axis of symmetry and six have three lines of symmetry. One of these (Figure 1) is invariant under scalar multiplication. The other five Latin triangles with three lines of symmetry are all related by scalar multiplication, as each can be obtained as a scalar multiple of any other. The 45 LT(9)'s with exactly one axis of symmetry reduce to 15 equivalence classes containing a triangle and its two rotations.

A computer search for LT(11)'s yielded one with no axis of symmetry and in fact no median having constant entries. This is the least odd order for which either property is possible.

In [2], for odd $n$, two LT($n$)'s are defined to be orthogonal if upon superposition, each unordered pair of symbols occurs in some cell. For $n$ even, the definition requires that each unordered pair of distinct symbols occurs in some cell. This definition suffers from the fact that although

2<br>3 1

and

1<br>3 2

are related by a permutation of symbols,

1<br>2 3

is orthogonal to the first but not the second.

We propose a definition of orthogonal Latin triangles that is preserved under permutations and does not distinguish between even and odd cases: two Latin triangles are orthogonal if superposition produces no repeated ordered pairs. Clearly triangles orthogonal under the definition in [2] are also orthogonal in this sense.

Since Latin triangles of orders 3 and 5 must have this form:

| | | |
|---|---|---|
| | | c |
| | | b d |
| b | | e c a |
| c a | | d a e b |
| a b c | and | a b c d e |

the three medians prevent the existence of orthogonal LT(3)'s and LT(5)'s. The property of having three constant medians is shared by LT(9)'s, so there are no orthogonal LT(9)'s. The three LT(7)'s below are mutually orthogonal using the proposed definition, but not the previous one.

| | | |
|---|---|---|
| 4 | 3 | 5 |
| 5 3 | 2 4 | 4 6 |
| 3 4 5 | 6 5 1 | 7 3 2 |
| 2 1 7 6 | 7 3 4 2 | 6 4 5 1 |
| 7 6 4 2 1 | 5 1 7 3 6 | 2 5 1 7 3 |
| 6 5 1 7 3 2 | 4 6 2 1 7 5 | 3 1 7 6 2 4 |
| 1 2 3 4 5 6 7 | 1 2 3 4 5 6 7 | 1 2 3 4 5 6 7 |

The smallest order $n$ for which existence of an LT($n$) is not determined by [1] and [2] is $n = 14$. We have constructed an LT(14), shown in section I. This triangle is generated by the following three transversals in line coordinates.

| $\tau_1$ | $\tau_2$ | $\tau_3$ |
|---|---|---|
| 111 | 234 | 267 |
| 232 | 457 | 346 |
| 353 | 542 | 673 |
| 474 | 625 | 735 |
| 565 | 313 | 122 |
| 646 | 166 | 414 |
| 727 | 771 | 551 |

Recall that there are six cells whose line coordinates are a permutation of $1xx$, for $x \neq 1$. The three permutations of coordinates of triples in $\tau_1$ together with the 6 permutations of $\tau_2$ and $\tau_3$ constitute the required 15 transversals. Similarly, we have constructed an LT(20), LT(26), and LT(32).

## References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2. F. Y. Halberstam, R. B. Richter, *A Note On Latin Triangles*, Combinatorial Mathematics: Proceedings of the Third International Conference, Annals of New York Academy of Sciences (1989), 213-215.
3. Steven S. Perry and David McGough, personal communication, July, 1990.