A doubling construction for overlarge sets of Steiner triple systems

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ABSTRACT

Given an overlarge set of Steiner triple systems, each on v points, we construct an overlarge set of Steiner triple systems, each on $2v+1$ points. Overlarge sets with specified properties can be constructed in this way; in particular, we construct overlarge sets which cannot be derived from Steiner quadruple systems.

1. Introduction

A *t-design* based on a v-set, X, is a collection of k-subsets (blocks) chosen from X in such a way that each unordered t-subset of X occurs in precisely λ of the blocks. Such a design has parameters t-(v,k,λ). A 2-$(v,3,1)$ design is called a *Steiner triple system*, often denoted by $STS\,(v)$, and similarly a 3-$(v,4,1)$ design is called a *Steiner quadruple system*, $SQS\,(v)$. Two t-(v,k,λ) designs are said to be *disjoint* if and only if they have no block in common.

If the set of all the $\binom{v}{k}$ k-sets contained in X can be partitioned into mutually disjoint t-(v,k,λ) designs (all with the same parameters), then these designs are said to form a *large set*, denoted by $LS(t\text{-}(v,k,\lambda))$ or, in the case of Steiner triple systems, by $LS\,(STS\,(v))$. In particular, for v even, a one–factorization of K_v may be regarded as a $LS(1\text{-}(v,2,1))$; it is also often denoted by $OF(K_v)$. If a t-(v,k,λ) design has b blocks, then b must divide $\binom{v}{k}$ for a large set of these designs to exist. However, even where this condition is satisfied a large set may not exist; for example, there is no $LS\,(STS\,(7))$ [4].

Whether or not a large set exists, it may be possible to pack the designs neatly by enlarging the set of points on which they are based, sometimes by adjoining just one extra point. Thus, if the set of all the $\binom{v}{k}$ k-sets chosen from X can be partitioned into v mutually disjoint t-$(v-1,k,\lambda)$ designs, each missing a different point of X, then these designs are said to form an *overlarge set*, denoted by $OS(t\text{-}(v-1,k,\lambda))$ or, in the case of Steiner triple systems, by $OS\,(STS\,(v-1))$. We shall label the designs of an overlarge set by their missing elements. In particular, for v odd, a near–one–factorization of K_v may be regarded as an $OS(1\text{-}(v-1,2,1))$; it is also often denoted by $NOF(K_v)$. If a t-$(v-1,k,\lambda)$ design has b blocks, then b must divide $\binom{v}{k}$

for an overlarge set of these designs to exist. However, even where this condition is satisfied an overlarge set may not exist; for example, there is no $OS(5\text{-}(12,6,1))$ [7].

Given a $(t+1)\text{-}(v+1,k+1,1)$ design, $\mathcal{D}$, with $t = k-1$, we can form an $OS(t\text{-}(v,k,1))$ by choosing, for each $i = 1,...,v+1$, all the blocks of $\mathcal{D}$ containing i, and deleting i from each of them. These k-sets form design $\mathcal{D}_i$, and this overlarge set is said to be *derived* from $\mathcal{D}$. Note that, for different values of i, the designs $\mathcal{D}_i$ derived from $\mathcal{D}$ need not be isomorphic to each other. (This use of the term 'derived' is consistent with that of Rosa [6].)

This general construction shows that, for example, there is an $OS(STS(v))$ for every $v \equiv 1$ or 3 (modulo 6), since there is a $SQS(v+1)$ for every such v [1]. But this is not the only way in which such overlarge sets arise.

In this paper, we give a doubling construction for $OS(STS(v))$. That is, given an $OS(STS(v))$, we construct an $OS(STS(2v+1))$, and show that we have a wide choice of the substructures from which this overlarge set may be constructed. In particular, we can insist that it not be derived from any $SQS(2v+2)$.

2. The doubling construction

To state the construction neatly, we introduce a small amount of notation. Let Q be a set, $Q = \{q_1, \dots, q_n\}$. The complete graph on n vertices is usually denoted by K_n but we use the notation K_Q to indicate that the vertices are labelled with the elements of Q. (We may label the $n-1$ one–factors of a one–factorization of K_Q with any convenient symbols.) Similarly a Latin square of order n is said to be *based on* Q if its rows and columns are labelled from q_1 to q_n, and its symbols are those of Q. Such a square is denoted by L_Q and defines a quasigroup. L_Q is said to be *in standard form* if its initial column (column q_1) contains the elements of Q in order, that is, if $L_Q = \left[l_{ij}^Q\right]$ where $l_{iq_1}^Q = i$ for $i = q_1, q_2, \dots, q_n$.

Construction. Let $v \equiv 1$ or 3 (modulo 6) and let X and Y be two disjoint $(v+1)$-sets:

$$X = \{1, 2, \dots, v+1\}$$

and

$$Y = \{v+2, v+3, \dots, 2v+2\}.$$

Let

$$\mathcal{F} = \{F_{v+3}, F_{v+4}, \dots, F_{2v+2}\}$$

and

$$\mathcal{G} = \{G_2, G_3, \dots, G_{v+1}\}$$

be one–factorizations of K_X and K_Y, respectively. Let $L_X = \left[l_{ij}^X\right]$ and $L_Y = \left[l_{ij}^Y\right]$ be two Latin squares of order $v+1$, based on X and Y respectively, and both in standard form. Finally, let

$$\{\mathcal{A}_x : x \in X\}$$

and

$$\{\mathcal{B}_y : y \in Y\}$$

be two $OS(STS(v))$ based on X and Y, respectively. Let $S = X \cup Y$, so $|S| = 2v+2$. Define the following collections of triples on S.

(i) For $i = 1, 2, \ldots, v+1$,

$$\mathcal{C}_i = \left\{ \left\{ u, w, l_{ij}^X \right\} : [u, w] \in G_j \text{ and } j = 2, 3, \ldots, v+1 \right\}$$

and

$$\mathcal{D}_i = \mathcal{A}_i \cup \mathcal{C}_i .$$

Then $(S \setminus \{i\}, \mathcal{D}_i)$ is a $STS(2v+1)$ for $i = 1, 2, \ldots, v+1$.

(ii) For $i = v+2, v+3, \ldots, 2v+2$

$$\mathcal{C}_i = \left\{ \left\{ u, w, l_{ij}^Y \right\} : [u, w] \in F_j \text{ and } j = v+3, v+4, \ldots, 2v+2 \right\}$$

and

$$\mathcal{D}_i = \mathcal{B}_i \cup \mathcal{C}_i .$$

Then $(S \setminus \{i\}, \mathcal{D}_i)$ is a $STS(2v+1)$ for $i = v+2, v+3, \ldots, 2v+2$.

Then an $OS(STS(2v+1))$ based on S can be constructed by taking the following collection of $STS(2v+1)$:

$$(S \setminus \{i\}, \mathcal{D}_i) \qquad \text{for } i = 1, 2, \ldots, 2v+2 .$$

To show that this collection of $2v+2$ $STS(2v+1)$ forms an $OS(STS(2v+1))$ we show first that we have $\binom{2v+2}{3}$ triples and secondly that no triple is repeated.

To count the number of triples, note that in collection (i) of triples, we have the $\binom{v+1}{3}$ triples chosen from X and the $\frac{(v+1)}{2}v(v+1)$ triples constructed from $\mathcal{G}$ and L_X, and in collection (ii) we have the same number of triples again. That is, we have altogether

$$2\binom{v+1}{3} + v(v+1)^2 = \binom{2v+2}{3}$$

triples, as required. So provided no triple is repeated, all are used, and the designs are pairwise disjoint.

Now we check that no triple is repeated. We are given that all triples in $\mathcal{A}_i$ and $\mathcal{B}_i$ are distinct. This accounts for all triples in X or Y. Additional triples are either of the form $\{x, y_\alpha, y_\beta\}$, where $x \in X$ and $y_\alpha, y_\beta \in Y$ (in $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{v+1}$), or $\{x_\alpha, x_\beta, y\}$, where $x_\alpha, x_\beta \in X$ and $y \in X$ (in $\mathcal{D}_{v+2}, \mathcal{D}_{v+3}, \ldots, \mathcal{D}_{2v+2}$), so there is no repetition between these sets of triples. But within $\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{v+1}\}$ or $\{\mathcal{D}_{v+2}, \mathcal{D}_{v+3}, \ldots, \mathcal{D}_{2v+2},\}$ each edge of K_X is paired with each point of Y, and vice versa. So all of the triples are distinct.

This completes the proof that we have an $OS(STS(2v+1))$.

Example 1. Let $v = 3$, $X = \{1, 2, 3, 4\}$ and $Y = \{5, 6, 7, 8\}$. Let

$$\mathcal{F} = \{\{12, 34\}, \{13, 24\}, \{14, 23\}\}$$

and

$$\mathcal{G} = \{\{56, 78\}, \{57, 68\}, \{58, 67\}\}.$$

Then $\mathcal{F}$ and $\mathcal{G}$ are one–factorizations of K_X and K_Y, respectively. Let

$$L_X = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix} \quad \text{and } L_Y = \begin{matrix} 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 \\ 7 & 8 & 5 & 6 \\ 8 & 7 & 6 & 5 \end{matrix}.$$

Then L_X and L_Y are two Latin squares of order 4, based on X and Y respectively, and both in standard form. Finally, consider $\{\mathcal{A}_x : x \in X\}$ where

$$\mathcal{A}_1 = \{234\}, \mathcal{A}_2 = \{134\}, \mathcal{A}_3 = \{124\}, \mathcal{A}_4 = \{123\}$$

and $\left\{\mathcal{B}_y : y \in Y\right\}$ where

$$\mathcal{B}_5 = \{678\}, \mathcal{B}_6 = \{578\}, \mathcal{B}_7 = \{568\}, \mathcal{B}_8 = \{567\}.$$

Then $\{\mathcal{A}_x : x \in X\}$ and $\left\{\mathcal{B}_y : y \in Y\right\}$ are $OS(STS(3))$ based on X and Y, respectively. The $OS(STS(7))$ obtained using the above construction is shown in Table 1. It is isomorphic to the overlarge set E in [8], by the mapping (164532).

$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_8$
234	134	124	123	678	578	568	567
256	356	456	156	612	512	812	712
278	378	478	178	634	534	834	734
357	457	157	257	713	813	513	613
368	468	168	268	724	824	524	624
458	158	258	358	814	714	614	514
467	167	267	367	823	723	623	523

Table 1: An $OS(STS(7))$ constructed from two $OS(STS(3))$

3. Non–derived overlarge sets of Steiner triple systems

Note that there is no special relationship between the one–factorizations, $\mathcal{F}$ and $\mathcal{G}$, the Latin squares, L_X and L_Y, or the two $OS(STS(v))$ based on X and Y. This allows us great freedom in constructing overlarge sets; in particular, we now construct some which cannot be derived from Steiner quadruple systems. The case $v = 3$ is of course an exception: the only possible $OS(STS(3))$ is derived.

If the block bcd belongs to the design $\mathcal{D}_a$ in a derived $OS(STS(v))$, then the block $abcd$ belongs to the $SQS(v+1)$ from which it was derived. This means that the design $\mathcal{D}_b$ contains the block acd.

If we choose at least one of the Latin squares in the doubling construction to be back–circulant, then the overlarge set that we obtain cannot be derived. For if, say,

L_X is back–circulant, and if the first edge in the one–factor G_{v+2} is y_1y_2, then the design $\mathcal{D}_1$ contains the block $2y_1y_2$ but the block $1y_1y_2$ occurs in the design $\mathcal{D}_{v+1}$. In Example 1, we see that in particular, the block 256 belongs to $\mathcal{D}_1$ and the block 156 to $\mathcal{D}_4$.

This still leaves open the question of whether non–derived $OS(STS(v))$ exist in general, that is, including the cases which cannot be found from the doubling construction. Certainly there are 77 non–isomorphic $OS(STS(9))$ [9] and only one of these (# 77) is derived. Further, if S is the set $S = \{1, \ldots, 9, A, \ldots, E\}$, then the following $STS(13)$, based on the set $S \setminus \{A\}$, can be developed cyclically modulo 14 to give an $OS(STS(13))$, based on S, which is not derived.

123 346 458 4DE 16E 67D 178 6BC 8CD 256 689 1BD 9BE
57B 5CE 28E 24B 35D 479 14C 37E 39C 38B 29D 159 27C

In the overlarge set, this will be the design $\mathcal{D}_A$ and it contains the block 6BC. Hence if the overlarge set were derived, the design $\mathcal{D}_B$ would contain the block 6AC. But in fact, since 35D belongs to $\mathcal{D}_A$, and since the overlarge set is cyclic, 6AC belongs to $\mathcal{D}_8$. (The 70 non–isomorphic cyclic $OS(STS(13))$ are classified in [10].)

To show that non–derived $OS(STS(v))$ exist for all $v \equiv 1$ or 3 (modulo 6) we now apply the following result, found by Hartman [2] and subsequently proved more simply by Lenz [3].

Theorem 1. *A Steiner quadruple system of order $v + 1$ which is an extension of a Steiner quadruple system of order 8 exists for all $v \equiv 1$ or 3 (modulo 6) and $v \neq 1, 3, 9$ or 13.*

The $OS(STS(v))$ derived from such a $SQS(v+1)$ has eight $STS(v)$, say, $\mathcal{C}_i$ for $i = 1, \ldots, 8$, each containing a $STS(7)$ $\mathcal{E}_i$ on $\{1, \ldots, 8\} \setminus \{i\}$ These eight subdesigns $\mathcal{E}_i$ for $i = 1, \ldots, 8$ together form a $OS(STS(7))$. We may now substitute for each $\mathcal{E}_i$ the corresponding design $\mathcal{D}_i$ of the $OS(STS(7))$ given in Table 1. Since that overlarge set is not derived, neither is the resulting $OS(STS(v))$, completing the proof of

Theorem 2. *For every $v \equiv 1$ or 3 (modulo 6), there exists an $OS(STS(v))$ which is not derived from a $SQS(v+1)$.*

4. The smallest examples of the doubling construction

The smallest case of the doubling construction starts from an $OS(STS(3))$ and gives an $OS(STS(7))$. There are precisely eleven non–isomorphic $OS(STS(7))$ [8] and we show that seven of them can be obtained by doubling. First we recall from [8] the function $f : \binom{8}{4} \to \binom{8}{4}$ given by $f(\{x_1, x_2, x_3, x_4\}) = \{y_1, y_2, y_3, y_4\}$ where, for $i = 1, 2, 3, 4$, y_i is defined by $\{x_1, x_2, x_3, x_4\} \setminus \{x_i\} \in \mathcal{D}_{y_i}$. Therefore in a derived $OS(STS(7))$, every quadruple of the corresponding $SQS(8)$ is fixed by f, and in an $OS(STS(7))$ from the doubling construction, the sets X and Y are fixed by f. Since the overlarge sets B, F, G and K have no blocks fixed by f, they cannot be constructed by doubling. The other seven overlarge sets can be, as shown in Table 2.

Since overlarge set A is derived from a $SQS(8)$, each of its blocks is fixed by f, and only one possible choice of X and Y is shown. For the other overlarge sets, all possible choices of X and Y are given, that is, three each for D and E and one each for the others.

$OS(STS(7))$	X / Y	$\mathcal{F}$ / $\mathcal{G}$	L_X / L_Y	$\{\mathcal{A}_x : x \in X\}$ / $\{\mathcal{B}_y : y \in Y\}$
A	{1, 2, 3, 8}	{12 38} {13 28} {18 23}	1 8 3 2 2 3 8 1 3 2 1 8 8 1 2 3	{238} {138} {128} {123}
	{4, 5, 6, 7}	{45 67} {46 57} {47 56}	4 7 6 5 5 6 7 4 6 5 4 7 7 4 5 6	{567} {467} {457} {456}
C	{1, 2, 5, 6}	{12 56} {15 26} {16 25}	1 2 6 5 2 1 5 6 5 6 2 1 6 5 1 2	{256} {156} {126} {125}
	{3, 4, 7, 8}	{34 78} {37 48} {38 47}	3 8 7 4 4 7 8 3 7 4 3 8 8 3 4 7	{478} {378} {348} {347}
D	{1, 2, 5, 6}	{12 56} {15 26} {16 25}	1 2 5 6 2 5 6 1 5 6 1 2 6 1 2 5	{256} {156} {126} {125}
	{3, 4, 7, 8}	{34 78} {37 48} {38 47}	3 4 7 8 4 7 8 3 7 8 3 4 8 3 4 7	{478} {378} {348} {347}

Table 2: The $OS(STS(7))$ obtained by using the doubling construction. Labels are those of [8].

$OS(STS(7))$	X / Y	$\mathcal{F}$ / $\mathcal{G}$	L_X / L_Y	$\{\mathcal{A}_x : x \in X\}$ / $\{\mathcal{B}_y : y \in Y\}$
D	{1, 3, 5, 7}	{13 57} {15 37} {17 35}	1 3 5 7 3 1 7 5 5 7 1 3 7 5 3 1	{357} {157} {137} {135}
	{2, 4, 6, 8}	{24 68} {26 48} {28 46}	2 8 6 4 4 6 8 2 6 4 2 8 8 2 4 6	{468} {268} {248} {246}
D	{1, 4, 5, 8}	{14 58} {15 48} {18 45}	1 4 5 8 4 5 8 1 5 8 1 4 8 1 4 5	{458} {158} {148} {145}
	{2, 3, 6, 7}	{23 67} {26 37} {27 36}	2 7 6 3 3 2 7 6 6 3 2 7 7 6 3 2	{367} {267} {237} {236}
E	{1, 2, 5, 6}	{12 56} {15 26} {16 25}	1 2 5 6 2 5 6 1 5 6 1 2 6 1 2 5	{256} {156} {126} {125}
	{3, 4, 7, 8}	{34 78} {37 48} {38 47}	3 8 7 4 4 7 8 3 7 4 3 8 8 3 4 7	{478} {378} {348} {347}
E	{1, 3, 5, 7}	{13 57} {15 37} {17 35}	1 3 5 7 3 5 7 1 5 7 1 3 7 1 3 5	{357} {157} {137} {135}
	{2, 4, 6, 8}	{24 68} {26 48} {28 46}	2 8 6 4 4 6 8 2 6 4 2 8 8 2 4 6	{468} {268} {248} {246}

Table 2: (cont'd) The $OS(STS(7))$ obtained by using the doubling construction. Labels are those of [8].

$OS(STS(7))$	X	$\mathcal{F}$	L_X	$\{\mathcal{A}_x : x \in X\}$
	Y	$\mathcal{G}$	L_Y	$\{\mathcal{B}_y : y \in Y\}$
E	{1, 4, 5, 8}	{14 58} {15 48} {18 45}	1 4 5 8 4 5 8 1 5 8 1 4 8 1 4 5	{458} {158} {148} {145}
	{2, 3, 6, 7}	{23 67} {26 37} {27 36}	2 7 6 3 3 6 7 2 6 3 2 7 7 2 3 6	{367} {267} {237} {236}
H	{1, 3, 5, 7}	{13 57} {15 37} {17 35}	1 3 5 7 3 1 7 5 5 7 3 1 7 5 1 3	{357} {157} {137} {135}
	{2, 4, 6, 8}	{24 68} {26 48} {28 46}	2 8 6 4 4 6 8 2 6 4 2 8 8 2 4 6	{468} {268} {248} {246}
I	{1, 3, 5, 7}	{13 57} {15 37} {17 35}	1 3 5 7 3 1 7 5 5 7 3 1 7 5 1 3	{357} {157} {137} {135}
	{2, 4, 6, 8}	{24 68} {26 48} {28 46}	2 8 6 4 4 6 2 8 6 4 8 2 8 2 4 6	{468} {268} {248} {246}
J	{1, 4, 5, 8}	{14 58} {15 48} {18 45}	1 4 5 8 4 5 8 1 5 8 1 4 8 1 4 5	{458} {158} {148} {145}
	{2, 3, 6, 7}	{23 67} {26 37} {27 36}	2 7 6 3 3 6 2 7 6 3 7 2 7 2 3 6	{367} {267} {237} {236}

Table 2: (cont'd) The $OS(STS(7))$ obtained by using the doubling construction. Labels are those of [8].

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