Trains: An Invariant for One-Factorizations

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1. Introduction

We assume the standard ideas of graph theory. A *one-factor* in a graph G is a set of edges of G which together contain each vertex precisely once. A *one-factorization* is a set of disjoint one-factors whose union is the original graph.

In order to have a one-factor it is necessary that a graph have an even number of vertices, but this is not sufficient. There have been many papers written on the existence of one-factors; Tutte's famous paper [14] presents a necessary and sufficient condition. The problem of whether or not a given graph has a one-factorization is even more difficult. However, it is easy to see that the complete graph K_{2n} has a one-factorization for every positive integer n. For an excellent survey on one-factorizations of the complete graph, we refer the reader to [11].

The standard proof that K_{2n} has a one-factorization goes as follows. First, take the vertices of K_{2n} as $\{\infty, 0, 1, 2, \ldots, 2n-2\}$ where the elements are the integers modulo $2n-1$ except that ∞ is a new element satisfying the law "$\infty + x = \infty$". Then select a particular one-factor F_0 with edges:

$$(\infty, 0), (1, -1), (2, -2), \ldots, (n-1, n).$$

(This factor is easiest to understand geometrically, using the picture in Figure 1.) Finally, factor F_i is constructed from F_0 by the rule "add i to each vertex". F_i has edges

$$(\infty, i), (1+i, -1+i), (2+i, -2+i), \ldots, (n-1+i, n+i).$$

(Again the geometric picture is very simple: rotate the diagram of Figure 1 clockwise through i positions.) It is easy to check that $\{F_0, F_1, F_2, \ldots, F_{2n-2}\}$ constitute a one-factorization of K_{2n}. This particular factorization is called the *patterned factorization*, and denoted GK_{2n}.

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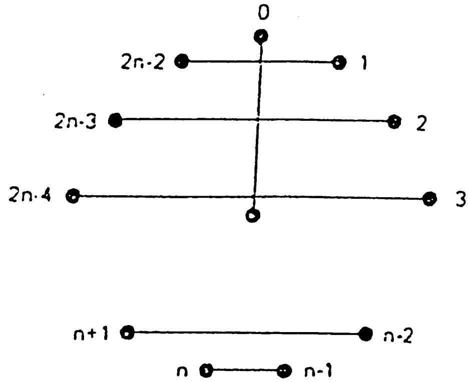

Figure 1

Another one-factorization of K_{2n} is frequently used. The definition varies, depending on whether n is even or odd. In either case the factorization is denoted GA_{2n} [2].

To construct GA_{2n}, one first partitions the vertex-set of K_{2n} into two sets of size n: for convenience, say they are $\{0,1,2,\ldots,n-1\}$ and $\{\overline{0},\overline{1},\ldots\overline{n-1}\}$. For $i = 0,1,\ldots,n-1$, let H_i denote the one-factor with edges $(0,\overline{i}),(1,\overline{1+i})$, $\ldots,(n-1,\overline{n-1+i})$ (where all arithmetic is performed modulo n). Then observe that the n one-factors $H_0, H_1, \ldots, H_{n-1}$, precisely cover all the edges joining the two sets of vertices.

Suppose n is even. Let $\{F_0, F_1, \ldots, F_{n-2}\}$ be the factors of GK_n, on the symbol set $\{0,\ldots,n-1\}$ and $\{F_0^+, F_1^+, \ldots, F_{n-2}^+\}$ be the same factors with each symbol x replaced by the symbol $\overline{x}$. (The symbols $n-1$ and $\overline{n-1}$ can by treated as the ∞ symbols.) Then

$$(F_0 \cup F_0^+),(F_1 \cup F_1^+),\ldots,(F_{n-2} \cup F_{n-2}^+), H_0, H_1, \ldots, H_{n-1}$$

constitute GA_{2n}. For odd n, GA_{2n} is defined similarly. (See Section 6.)

A one-factorization of K_{2n} is called *perfect* if the union of any pair of factors is always a Hamiltonian cycle in K_{2n}. We know that GK_{2n} is perfect when $2n-1$ is a prime, and GA_{2n} is perfect when n is prime. These are the only known infinite families of perfect one-factorizations, although other "sporadic" examples have been found (see [6]). A motivation for our discussion of this new invariant is that a commonly used invariant (using cycle structure) is totally ineffective in discerning nonismorphic perfect one-factorizations.

In this paper we will discuss an invariant of one-factorizations of K_n called the train. In Section 2 we describe this invariant and give some examples. In Section 3 we prove a theorem concerning the length of this invariant. Section 4 introduces another class of one-factorizations as motivation for the use of trains. In Sections 5 and 6 we explicitly compute the trains for GK_{2n} and GA_{2n} (n odd) respectively.

2. Trains

We are interested in describing one-factorizations of K_{2n}, and in distinguishing between nonisomorphic one-factorizations. Initially this was achieved by looking at cycle-structure, in the following sense. Each one-factorization consists of $2n-1$ factors. If these are paired, one constructs $(2n-1)(n-1)$ regular graphs of degree 2. In the case of K_8, one can distinguish between the isomorphism-classes of one-factorizations simply by counting how many of these graphs consist of two 4-cycles. In [17], two factors are called a *pair* if their union is not connected; three factors form a *division* if the union of all three is not connected. The pair and division structure is very useful in computing the automorphism groups of factorizations, and is also used in [17] to calculate automorphism groups of Room squares of side 7.

The cycle structure is not so useful in K_{10}, because of the large number of nonisomorphic one-factorizations (396 of them). Gelling [8] counts the cycles through a vertex. In each of the 36 graphs obtained by taking a union of two factors, a given vertex lies in a 4-cycle, a 6-cycle or a 10-cycle. One can count how many times this occurs for each vertex. Such a count is more useful, but still far from ideal. In particular, cycle structure cannot possibly distinguish between nonisomorphic perfect factorizations.

Kotzig [10] introduced another way of distinquishing between one-factorizations. With each one-factoriza- tion $\mathcal{F}$ of K_{2n} associate $2n$ idempotent quasigroups $Q(\mathcal{F}, v)$, one for each vertex v of K_{2n}. The multiplication in $Q(\mathcal{F}, v)$ is defined as follows: $aa = a$, and if $a \neq b$ then $ab = c$ where (a, c) and (b, v) lie in the same one-factor of $\mathcal{F}$. Then $Q(\mathcal{F}, v)$ induces a set of cycles $(a_1, a_2, a_3, \ldots)$ by the iterative process $a_1a_2 = a_3,\ a_2a_3 = a_4, \ldots,\ a_ia_{i+1} = a_{i+2}$.

For a given v, the set of cycles partitions the edges of the K_{2n-1} obtained by deleting v. The lengths of these cycles form an invariant.

Anderson [1] used the Kotzig invariants to study one-factorizations arising from starters in cyclic groups (that is, one-factorizations derived by rotating a diagram in the same way that GK_{2n} is derived by rotating Figure 1). In particular, he applied them to some small perfect one-factorizations.

Another invariant was used by Gross [9]. This invariant was easy to compute, however was only defined for one-factorizations arising from starters in Abelian groups. It also sometimes distinquishes between perfect one-factorizations.

In this paper we will discuss an invariant of one-factorizations called trains. These were first introduced by White [18], and later used by Colbourn, Colbourn and Rosenbaum [4] and by Stinson [13], in discussing Steiner triple systems. Trains of one-factorizations were first discussed by Dinitz [5].

Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_{2n-1}\}$ is a one-factorization of K_{2n}. The *train* of $\mathcal{F}$ is a directed graph whose vertices are the $n(2n-1)^2$ triples $\{x, y, F\}$, where x and y are (an unordered pair of) vertices and F is a factor in $\mathcal{F}$ (i.e. $F = F_i$ for some i). There is exactly one edge leaving each vertex; the edge from $\{x, y, F\}$ goes to $\{z, t, G\}$ where:

$$(x, z) \text{ is an edge in } F;$$
$$(y, t) \text{ is an edge in } F;$$
$$(x, y) \text{ is an edge in } G.$$

We will sometimes think of this edge as a self-mapping ϕ on the vertices of the train where $\phi(x, y, F) = (z, t, G)$. It is obvious that isomorphic one-factorizations have isomorphic trains.

Following [13] we simplify trains by considering only the indegree sequence of the train. That is, with a one-factorization F we associate the sequence $t_0, t_1, t_2, \ldots$ where t_i equals the number of vertices in the train of F which have i edges directed into them. The sequence is normally written so as to terminate with the last nonzero element.

The sequence of indegrees allows us to separate many factorizations. In particular, Dinitz [5] uses it to prove that two perfect factorizations of K_{12} are nonisomorphic– the two sequences were $(330, 176, 165, 0, 55)$ and $(110, 506, 110)$– and to separate three nonisomorphic perfect factorizations of K_{20} and five of K_{24}.

In the Appendix we list the indegree sequence for the train of each one-factorization of K_{2n} for $2n = 8$ and 10. Note that these trains form a complete invariant for $2n = 8$ and almost a complete invariant when $2n = 10$. (Only the trains of one-factorizations numbered 16 and 26 have the same indegree sequence.)

3. The Maximum Length of a Train

One might think that the length of (the indegree sequence of) a train could be extremely long. There are $n(2n-1)^2$ vertices in the train of a one-factorization of K_{2n}, so an arbitrary digraph with all outdegrees 1 could have a vertex with indegree as large as $n(2n-1)^2$. However, the maximum indegree is much smaller.

Theorem 1. *The train of any one-factorization of K_{2n} has maximum indegree $2n-1$.*

Proof: Given a vertex $\{z, t, G\}$, assume there are edges directed into it from both $\{x, y, F\}$ and $\{u, v, F\}$. Then z is joined to one of $\{x, y\}$ and to one of $\{u, v\}$ by an edge of F, and so is t. So $\{x, y, F\} = \{u, v, F\}$ and thus there can be at most one edge into $\{z, t, G\}$ for each factor F in the one-factorization. Therefore, each vertex in the train has maximum indegree $2n-1$.

In the case of a perfect one-factorization we can say more.

Theorem 2. *The train of a perfect one-factorization of K_{2n} has maximum indegree n, for $n > 2$.*

Proof: Suppose there is an edge $\{x, y, F\} \rightarrow \{z, t, G\}$ in the train of a perfect one-factorization of K_{2n}. Then xy is an edge of G, and either $\{xz, yt\}$ or $\{xt, yz\}$ belong to F — say the former case. If there were an edge from $\{x, y, F_1\}$ to $\{z, t, G\}$ where $F_1 \neq F$, then xt and yz must be edges of F_1. Then $F \cup F_1$ contains a 4-cycle, which is impossible when $n > 2$. So there is at most one edge into $\{z, t, G\}$ for each edge xy of G.

The maximum indegree $2n - 1$ can be realized for nearly all orders $2n$. To prove this we need a definition. Suppose $R = \{1, 2, \ldots, r\}$ and $S_1, S_2, \ldots, S_k$ are sets which form a partition of R. By an *incomplete Latin square* of side r with holes $S_1, S_2, \ldots, S_k$ we mean an $r \times r$ array with the following properties:

(i) if $\{i, j\} \subseteq S_k$ for some k, then the (i, j) cell is empty; otherwise it contains an element of R;

(ii) if $i \in S_k$, then both row i and column i contain every element of $R \backslash S_k$ precisely once.

Suppose $A = (a_{ij})$ is an incomplete Latin square of side $2n - 2$ with holes $\{1, 2\}, \{3, 4\}, \ldots \{2n-3, 2n-2\}$, and further suppose A is symmetric. Define

$$F_0 = \{\infty 0,\ 12,\ 34, \ldots, (2n-3)(2n-2)\}$$

and when $i > 0$

$$F_{2i-1} = \{\infty(2i-1),\ 0(2i)\} \cup \{xy : a_{xy} = 2i - 1\},$$
$$F_{2i} = \{\infty(2i),\ 0(2i-1)\} \cup \{xy : a_{xy} = 2i\}.$$

Then $\{F_0, F_1, \ldots F_{2n-2}\}$ is a one-factorization of K_{2n} and $\{\infty, 0, F_0\}$ has indegree $2n-1$ in the train. Fu [7] has proven the existence of a symmetric incomplete Latin square of side $2n-2$ with $n-1$ holes of size 2 whenever $2n-2 \neq 4$, so we have

Theorem 3. *There is a one-factorization of K_{2n} whose train has a vertex of indegree $2n - 1$ whenever $2n \neq 6$.*

The unique one-factorization of K_6 has no vertex of indegree 5 in its train, so the situation is completely determined.

Also note that in K_{10}, the trains of one-factorizations numbered 1 through 5 all have a vertex with indegree 9. (See Appendix)

4. Staircase Factorizations

A practical example of the use of trains occurred in the case of staircase factorizations. Following the appearance of [15], F. Bileski [3] outlined a new way to construct a one-factorization of K_{2n}. His technique is as follows. (See also [16]).

First, write down a diagram of cells, with $2n-1$ rows and columns. If this were a square array, only the cells on or above the back diagonal would be included.

Rows are labeled $2n, 2n-1, \ldots, 2$ and columns $1, 2, \ldots, 2n-1$. (This can be done with a single labeling, as shown in Figure 2: the label encompasses the column above it and the row to its left – there is, for example, no *row* labelled 1 because label 1 is below all rows.)

Now construct n paths of cells. Path 1 is vertical, pointing north, on the west side. Path 2 starts from the extreme east, moves one square west, then southwest as far as possible without meeting Path 1, then south for one cell.

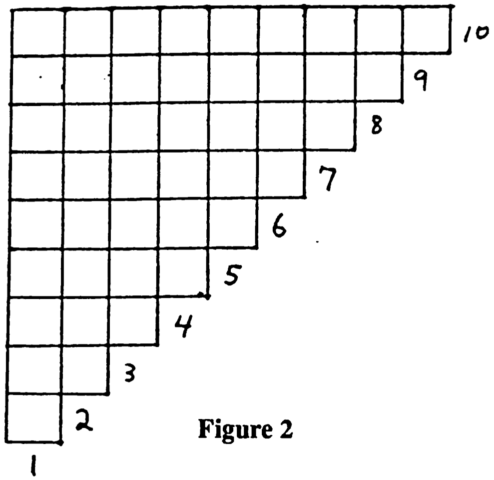

Figure 2

Every subsequent path meets the following description. Path i starts in the cell diagonally adjacent to the end of path $i-1$. Proceed northwest until it is impossible to proceed further (one would either cross Path 1 or escape from the diagram). It turns 45° and proceeds one step (it will either be possible to proceed north or to proceed west or north, but not both, so this instruction is unambiguous). Then turn a further 45° and go as far as possible. Turn a further 45° and move one cell. Turn a final 45° and go as far as possible.

These instructions sound complicated but are easy to follow. Path 1 is special. After that, odd-numbered paths go

NW as far as possible
N one step
NE as far as possible
E one step
SE as far as possible.

Even-numbered paths have the same description, except that the order of directions is NW, W, SW, S, SE. If we numbered the back-diagonal cells from 1(SW) to $2n-1$(NE), then if $i \geq 2$

for i odd, path i goes from cell i to cell $2n+1-i$
for i even, path i goes to cell i from cell $2n+1-i$.

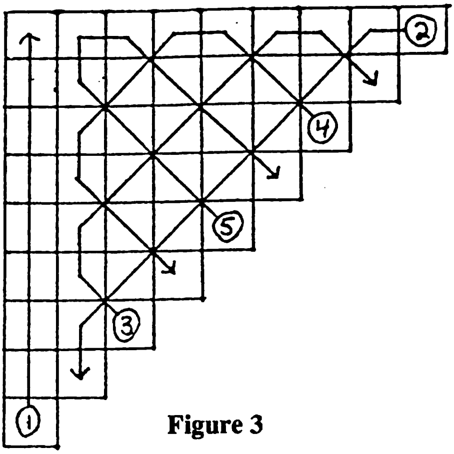

Figure 3

The labeling and the construction of paths is illustrated in Figure 3 for the case $2n = 10$. Paths are shown as lines joining the centers of the relevant cells.

To form a one-factorization, identify each cell of the staircase diagram with the unordered pair given by its coordinates. The first factor consists of the n pairs which are the first elements of the n paths, the second elements give the second factor, and so on.

It is naturally of interest to determine whether or not this factorization is different from the two main classes discussed earlier. Initially we could not decide this, so some experimental work was done. It was found that the trains of the staircased factorization and the patterned factorization are the same for all $2n \leq 36$. Using information from the calculation we constructed the following proof.

Theorem 4. *The staircase factorization is always isomorphic to the patterned factorization.*

Proof: For convenience we decreased the row and column numbers by 1. Then reverse the order of the odd labels, as shown in Figure 4. One can interpret the labels as $0, -0, 2, -2, \ldots (\mathrm{mod}\ 2n - 1)$ if preferred.

The starting point of a typical path (other than the first) is in row k, column $-k$. As one moves along the path, one gets cells and in general after $2h - 1$ steps comes cell $(2h - k, k + 2h)$; after $2h$ it is $(k - 2h, -k - 2h)$. So the sum of the two coordinates of the cell after $2h - 1$ steps is $4h$; the sum after $2h$ steps is $-4h$.

So the one-factor consisting of entry i from each path is the one-factor consisting of all the pairs with the same sum ($2i + 2$ if i is odd, $-2i$ if i is even). The contribution from path 1 is consistent ($i+1$, $-i$ respectively). These are the factors of GK_{2n}. So the staircased factorization is isomorphic to GK_{2n}.

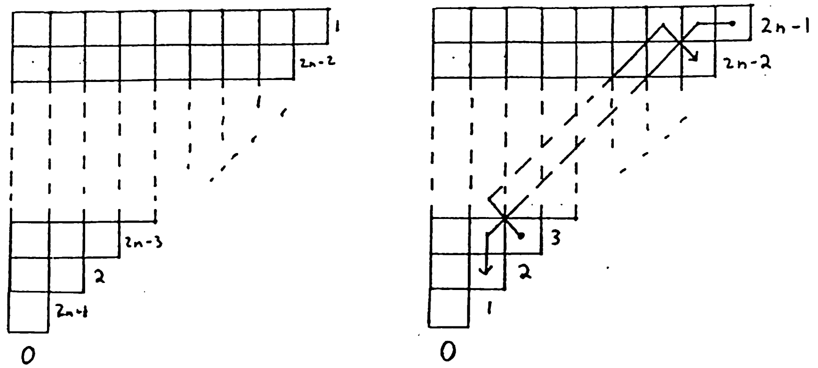

Figure 4

5. The Train of GK_{2n}

In this section we will determine the indegree sequence of the train of the patterned one-factorization GK_{2n} on the graph K_{2n}. This one-factorization was described in the introduction, but we will define it again below in an equivalent (but different looking) manner. The patterned one-factorization on K_{2n} may be defined by

$$GK_{2n} = \{F_1, F_2, \ldots, F_{2n-1}\}, \text{ where}$$
$$F_i = \{\infty, i\} \cup \{\{x, y\} : x + y = 2i\}.$$

(All arithmetic is modulo $2n-1$.) We will use the following notation: let $I = \{1, 2, \ldots, 2n-1\}$, $J = I \cup \{\infty\}$, and $I_x = I \backslash \{x\}$. Also let $S = \{(x, y, f) : x, y, f \in I, y > x\}$, $S' = \{(\infty, y, f) : y, f \in I\}$, and $V = S \cup S'$. We will let f denote the 1-factor F_f, where $f \in I$.

The train of GK_{2n} is a digraph whose vertices are the elements of V and whose outdegrees are all 1. The arcs can therefore be interpreted as the diagram of a mapping $V \to V$. The map is $\phi_1 \cup \phi_2 \cup \phi_3 \cup \phi_4$ where:

$$\phi_1 : (\infty, x, x) \to (\infty, x, x) \quad x \in I,$$
$$\phi_2 : (\infty, x, f) \to (f, 2f - x, x) \quad f \in I, x \in I_f,$$
$$\phi_3 : (x, y, x) \to \left(\infty, 2x - y, \frac{x+y}{2}\right) \quad x \in I, y \in I_x, \text{ and}$$
$$\phi_4 : (x, y, f) \to \left(2f - x, 2f - y, \frac{x+y}{2}\right) \quad f \in I, x \in I_f, y \in I_f.$$

Observe that ϕ_4 is a $1-1$ map from S onto itself. So the image of ϕ_4 on its domain is $S \backslash T$, where $T = \{\phi_4(x, y, f) : x \text{ or } y = f, 1 \leq x < y \leq 2n-1\}$.

Two easy alternative descriptions of T are

$$T = \{(f, 2f - y, \frac{f+y}{2}) : f \in I, y \in I_f\}$$
$$= \{(3f - 2g, f, g) : g \in I, f \in I_g\}.$$

In what follows we use the words "image of ϕ_i" to mean the collection of all $\phi_i(x, y, f)$ which are defined, with multiple elements counted multiply. We obtain the following images for the mappings ϕ_1, ϕ_2 and ϕ_3.

$$\text{Image of } \phi_1 = P = \{(\infty, x, x) : x \in I\}.$$
$$\text{Image of } \phi_2 = Q = \{(2f - x, f, x) : f \in I, x \in I_f\}$$
$$= \{(2f - x, f, x) : x \in I, f \in I_x\}.$$
$$\text{Image of } \phi_3 = R = \{\left(\infty, 2x - y, \frac{x+y}{2}\right) : x \in I, y \in I_x\}$$
$$= \{(\infty, 4g - 3y, g) : g \in I, y \in I_g\}.$$

The train has image $Z = P + Q + R + S - T$. If we let $P_x = \{$elements of P with last entry $x\}$. Then clearly $Z_x = P_x + Q_x + R_x + S_x - T_x$.

If $3 \not| 2n - 1$, then for x fixed $\{4x - 3y : y \neq x\} = I \backslash \{x\}$. So $P_x + R_x = \{(\infty, y, x) : y \in I\}$ and thus every member of S' appears exactly once in $P + R$. If $2n - 1 = 3t$,then $(\infty, 4x - 3y, x) = (\infty, 4x - 3z, x)$ whenever $3y - 3z$ (i.e. $y = z \pm t$). In that case $P_x \cup R_x$ contains t different elements three times each. So $P + R$ contains $t(2n - 1) = \frac{1}{3}(2n - 1)^2$ elements, each of frequency 3.

Next consider Q_x. Does it contain repetitions? Now $\phi_2(\infty, x, f) = \phi_2(\infty, x, g)$ implies that either $(2f - x, f, x) = (2g - x, g, x)$ (which implies $g = f$) or $(2f - x, f, x) = (g, 2g - x, x)$ for some $f \neq g$. This means that $g = 2f - x$ and $f = 2g - x$, so $f = 4f - 3x$. This occurs if and only if $3f = 3x$. Since $f \neq x$, the necessary and sufficient condition is that 3 divides $2n - 1$. If $2n - 1 = 3t$, then $f = x + t$ or $x + 2t$. So to summarize, we have that if $3 \not| 2n - 1$, then Q_x contains $2n - 2$ distinct elements. But if $3 | 2n - 1$, say $2n - 1 = 3t$, then Q_x contains one element twice ($(x + t, x + 2t, x)$) and $2n - 4$ elements once each.

Clearly T_x contains no repetitions. But can Q_x and T_x have common elements? There are two possibilities, either $(2f - x, f, x) = (3f - 2x, f, x)$ or $(2f - x, f, x) = (g, 3g - 2x, x)$. The first of these is clearly impossible. For the second one we get $g = 2f - x$ and $f = 3g - 2x$ so $f = 6f - 5x$. Equality is impossible when $(5, 2n - 1) = 1$; if $2n - 1 = 5u$, then the entries of Q_x with $f = x + iu, i = 1, 2, 3, 4$, all occur in T_x. So there is a 4-element overlap: $(x + 2iu, x + iu, x)$ for $i = 1, 2, 3, 4$. We can see then that $Q_x \backslash T_x$ has $2n - 6$ elements.

Can this overlap be among the repeats in the case $3 | 2n - 1$ (i.e. $2n - 1 = 15v$)? Then $u = 3v$ and $t = 5v$. The overlap is $(x + 3iv, x + 6iv, x)$: $i = 1, 2, 3, 4$, while the repeats are $(x + 5v, x + 10v, x)$. So they are distinct.

Now we can write out the indegree sequence of the train of GK_{2n}. There are four cases:

(i) $(2n-1, 15) = 1$.
Every element of V has indegree 1, except for the elements of T (indegree 0) and those of R (indegree 2). Since $|T| = |R| = (2n-1)(2n-2)$, the sequence is

$$(2n-1)(2n-2),\ (2n-1)(2n^2-5n+4),\ (2n-1)(2n-2).$$

(ii) $(2n-1, 15) = 3$.
S' contains $\frac{1}{3}(2n-1)^2$ elements of indegree 3 and $\frac{2}{3}(2n-1)^2$ elements of indegree 0. Also Q contains $2n-1$ elements twice and $(2n-1)(2n-4)$ elements once each. As none of these are elements of T, they give rise to vertices of indegree 3 and 2 respectively. So the sequence is

$$\begin{aligned}&\Big(\frac{2}{3}(2n-1)^2 + (2n-1)(2n-2), *, (2n-1)(2n-4), \frac{1}{3}(2n-1)^2 \\ &\quad + 2n-1\Big) \\ &= \Big(\frac{2}{3}(2n-1)(5n-4), (2n-1)(2n^2-7n+6), (2n-1)(2n-4), \\ &\quad \frac{2}{3}(2n-1)(n+1)\Big)\end{aligned}$$

(iii) $(2n-1, 15) = 5$.
We know that $Q \cap T$ has $4(2n-1)$ elements. On comparing with case (i), this means that the sequence is

$$\begin{aligned}&((2n-1)(2n-2) - 4(2n-1), *, (2n-1)(2n-2) - 4(2n-1)) \\ &= ((2n-1)(2n-6), (2n-1)(2n^2-5n+12), (2n-1)(2n-6))\end{aligned}$$

(iv) $(2n-1, 15) = 15$.
Similarly to the above, we modify the solution (ii), obtaining

$$\begin{aligned}&\Big(\frac{2}{3}(2n-1)(5n-4) - 4(2n-1), *, (2n-1)(2n-4) - 4(2n-1), \\ &\quad \frac{2}{3}(2n-1)(n+1)\Big) \\ &= \Big(\frac{10}{3}(2n-1)(n-2), (2n-1)(2n^2-7n+14), (2n-1)(2n-8), \\ &\quad \frac{2}{3}(2n-1)(n+2)\Big).\end{aligned}$$

As examples of the above trains we get:
The train of GK_4 is 12,0,0,6.
The train of GK_6 is 0,75.
The train of GK_8 is 12,0,0,6.
The train of GK_{10} is 0,75.
The train of GK_{12} is 12,0,0,6.
The train of GK_{14} is 0,75.
The train of GK_{16} is 12,0,0,6.
The train of GK_{22} is 0,75.
The train of GK_{36} is 0,75.

6. The Train of GA_{2n}, n Odd and $n \geq 5$

In this section we will determine the train for the 1-factorization GA_{2n} of K_{2n}, with n odd, $n \geq 5$. As discussed in the introduction, the vertices of K_{2n} are $0, 1, \ldots, n-1, \overline{0}, \overline{1}, \ldots, \overline{n-1}$

When n is odd, we construct GK_{n+1} on symbols $\{\infty, 0, 1, \ldots, n-1\}$ and call the factors $F_0, F_1, \ldots, F_{n-1}$. Also construct GK_{n+1} on the symbols $\{\infty, \overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ and call the factors $F_0^+, F_1^+, \ldots, F_{n-1}^+$. Now $F_i \cup F_i^+$ is not a one-factor, because ∞ appears twice. But, if we delete edges (∞, i) and $(\infty, \overline{i})$ and add $(i, \overline{i})$ we obtain a one-factor; call it K_i. In general, we will denote this new one-factor K_f simply by f where $f \in \{0, 1, \ldots, n-1\}$.

We define a further set of one-factors by $g_i = \{(x, \overline{x+i}) | x \in Z_n\}$ for $1 \leq i \leq n-1$. Then $K_0, K_1, \ldots, K_{n-1}, g_1, g_2, \ldots, g_{n-1}$ is a one-factorization of K_{2n} called GA_{2n}.

From the definition above it is easy to compute the images of all vertices in the train. (Note again that each vertex in the train is a triple containing two vertices in K_{2n} and a one-factor in GA_{2n}). The following is a list of all possible cases of mappings depending on the structure of the triple in the domain. By convention we assume $\{x, y, f\} \in \{0, 1, \ldots, n-1\}, x \neq y, f \neq x, f \neq y$.

$$\phi_1 : (x, y, f) \to \left(2f - x, 2f - y, \frac{1}{2}(x + y)\right) \quad (x < y)$$

$$\phi_2 : (x, y, x) \to \left(2x - y, \overline{x}, \frac{1}{2}(x + y)\right)$$

$$\phi_3 : (\overline{x}, \overline{y}, f) \to \left(\overline{2f - x}, \overline{2f - y}, \frac{1}{2}(x + y)\right) \quad (x < y)$$

$$\phi_4 : (\overline{x}, \overline{y}, x) \to \left(x, \overline{2x - y}, \frac{1}{2}(x + y)\right)$$

$$\phi_5 : (x, y, g_i) \rightarrow \left(\overline{x+i}, \overline{y+i}\frac{1}{2}(x+y)\right) \quad (x < y)$$

$$\phi_6 : (\overline{x}, \overline{y}, g_i) \rightarrow \left(x - i, y - i, \frac{1}{2}(x+y)\right) \quad (x < y)$$

$$\phi_7 : (x, \overline{x}, f) \rightarrow (2f - x, \overline{2f - x}, x)$$

$$\phi_8 : (x, \overline{x}, x) \rightarrow (x, \overline{x}, x)$$

$$\phi_9 : (x, \overline{x}, g_i) \rightarrow (x - i, \overline{x+i}, x)$$

$$\phi_{10} : (x, \overline{y}, f) \rightarrow (2f - x, \overline{2f - y}, g_{y-x})$$

$$\phi_{11} : (x, \overline{y}, x) \rightarrow (\overline{x}, 2f - y\frac{1}{2}(x+y))$$

$$\phi_{12} : (x, \overline{y}, y) \rightarrow (2f - x, \overline{2x - y}, g_{y-x})$$

$$\phi_{13} : (x, \overline{y}, g_i) \rightarrow (y - i, \overline{x+i}, g_{y-x})$$

Let P_i = image of Q_i. There are six types of vertices in the train and we partition the P_i accordingly:

Type 1 vertices are of the form (x, y, f) and arise from the sets P_1 and P_6;

Type 2 vertices are of the form $(x, \overline{y}, f)$ and arise from the sets P_2, P_4, P_7, P_8, P_9;

Type 3 vertices are of the form $(\overline{x}, \overline{y}, f)$ and arise from the sets P_3 and P_5;

Type 4 vertices are of the form $(x, \overline{y}, g_i)$ and arise from the sets P_{10} and P_{13};

Type 5 vertices are of the form $(\overline{x}, \overline{y}, g_i)$ and arise from the set P_{11};

Type 6 vertices are of the form (x, y, g_i) and arise from the set P_{12}.

To compute the indegrees of all **Type 1** vertices, we explicitly compute P_1 and P_6. We get

$$P_1 = \left\{(a, b, h) | a \neq h, h \neq \frac{3a - b}{2}, h \neq \frac{3b - a}{2}\right\}$$

$$P_6 = \left\{(a, b, h) | a \neq b \text{ and } h \neq \frac{a+b}{2}\right\}.$$

So there are three types of Type 1 vertices.

$$1):\left\{(a,b,h)|h\neq\frac{3a-b}{2},h\neq\frac{3b-a}{2},h\neq\frac{a+b}{2}\right\}.$$

These will have indegree 2.

$$2):\left\{(a,b,h)|h=\frac{3a-b}{2},\text{ or } h=\frac{3b-a}{2},\text{ and } h=\frac{a+b}{2}\right\}.$$

These will have indegree 1.

$$3):\left\{(a,b,h)|h\neq\frac{3a-b}{2}\text{and } h\neq\frac{3b-a}{2},\text{ and } h=\frac{a+b}{2}\right\}.$$

These will have indegree 1.

The indegrees were deduced by noting that both ϕ_1 and ϕ_6 are $1-1$. Also, since it cannot be that $\frac{1}{2}(3a-b)=\frac{1}{2}(a+b)$, there is no fourth case.

Now by counting, we find $\binom{n}{2}(n-3)$ vertices with indegree 2 and $\binom{n}{2}\cdot 2+\binom{n}{2}\cdot 1$ vertices with indegree 1.

For **Type 2** we again note that $\phi_2,\phi_4,\phi_7,\phi_8$ and ϕ_9 are all $1-1$. Now computing the P_i we see that

$$\begin{aligned}
P_2 &= \left\{\{a,\bar{b},\frac{3b-a}{2}\}|a\neq b\right\},\\
P_4 &= \left\{\{a,\bar{b},\frac{3a-b}{2}\}|a\neq b\right\},\\
P_7 &= \{\{a,\bar{a},h\}|a\neq h\},\\
P_8 &= \{\{a,\bar{a},a\}\}, \quad \text{and}\\
P_9 &= \left\{\{a,\bar{b},\frac{a+b}{2}\}|a\neq b\right\}.
\end{aligned}$$

If $\frac{1}{2}(3b-a)=\frac{1}{2}(3a-b)$, then $a=b$, a contradiction. So $P_2\cap P_4=\phi$. Also, if $\frac{1}{2}(3b-a)=\frac{1}{2}(a+b)$, then again, $a=b$. So, $P_2\cap P_9=\phi$ and $P_4\cap P_9=\phi$. Clearly P_7 and P_8 are disjoint from each other and from P_2,P_4, and P_9.

So these are all distinct sets and thus every element in $P_2\cup P_4\cup P_7\cup P_8\cup P_9$ has indegree 1. Now $|P_2|+|P_4|+|P_7|+|P_8|+|P_9|=n(n-1)+n(n-1)+n(n-1)+n+n(n-1)=4n^2-3n$ more vertices with indegree 1.

Type 3 is analogous to Type 1. We get $\binom{n}{2}(n-3)$ additional vertices with indegree 2 and $\binom{n}{2}\cdot 3$ additional vertices with indegree 1.

to do **Type 4** we must first note that ϕ_{10} is *not* $1-1$. Since $\phi_{10}(x,\overline{y},f)=2f-x,2f-y,g_{y-x})$, then for any k, $0\leq k\leq n-1$,

$$\phi_{10}\left(x+k,\overline{y+k},f+\frac{k}{2}\right)=(2f-x)\overline{2f-y},g_{y-x}).$$

So ϕ_{10} appears to be n to 1. However, if $x + k = f + \frac{k}{2}$ or if $y + k = f + \frac{k}{2}$, then we are in cases corresponding to mappings ϕ_{11} or ϕ_{12}, respectively. Thus, if $k = 2(f - x)$ or $k = 2(f - y)$, then we are not in this case and so these images must be different. Except for these two possibilities, all $(x + k, \overline{y + k}, f + \frac{k}{2})$ have the same image under ϕ_{10}. Therefore ϕ_{10} is $n - 2$ to 1.

Now, to finish Type 4, from the above discussion we have that every vertex in $P_{10} = \{(a, \overline{b}, g_i) | a \neq b, i = a - b\}$ has indegree at least $n - 2$. Note that if $P_{13} = \{(a, \overline{b}, g_i) | a \neq b, i \neq a - b\}$, then necessarily $i \neq a - b$ and so $P_{13} \cap P_{10} = \phi$. Thus in P_{10} there are $n(n-1)$ vertices, each with indegree $n - 2$. Since ϕ_{13} is $1 - 1$, there are also $n(n-1)(n-1)$ vertices in P_{13} which each have indegree 1.

The **Type 5** vertices are just the image under ϕ_{11}. Since $P_{11} = \{\{\overline{a}, \overline{b}, g_{a-b}\} | a \,/\!= b\}$, then there are $n(n-1)$ additional vertices of degree 1.

Type 6 is analogous to Type 5, so there are $n(n-1)$ more vertices of degree 1.

The total is

- $\binom{n}{2}3 + (4n^2 - 3n) + \binom{n}{2}3 + n(n-1)(n-1) + n(n-1) + n(n-1)$
  $= n(n^2 + 7n - 7)$ vertices of indegree 1
- $\binom{n}{2}(n-3) + \binom{n}{2}(n-3) = n(n-1)(n-3)$ vertices of indegree 2
- $n(n-1)$ vertices of indegree $n - 2$.

By subtracting from the total number of vertices, which is $\binom{2n}{2}(n-1)$, we find that there are $2n(n-1)(n-3)$ vertices of indegree 0.

So the vector for GA_{2n} is

$$2n(n-1)(n-3), n(n^2 + 2n - 7), n(n-1)(n-3), \overbrace{0, 0, \ldots, 0}^{n-5}, n(n-1).$$

As examples of the above trains we get:

The trains of GA_{10} is 80, 265, 40, 20.

The trains of GA_{14} is 336, 637, 168, 0, 0, 42.

The trains of GA_{18} is 864, 1233, 432, 0, 0, 0, 0, 72.

The trains of GA_{22} is 1760, 2101, 880, 0, 0, 0, 0, 0, 0, 110.

The trains of GA_{26} is 3120, 3289, 1560, 0, 0, 0, 0, 0, 0, 0, 0, 156.

The trains of GA_{30} is 5040, 4845, 2520, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 210.

From the results of this and the previus section we can see that for n odd, $n \geq 5$ that GA_{2n} is not isomorphic to GK_{2n} (since their trains are different). In fact it is interesting to note that while the train of GK_{2n} has length at most 4, the train of GA_{2n} has length $n - 1$.

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References

1. Anderson, B. A., *Some perfect one-factorizations*, Congr. Numer..
2. Anderson, B. A., *Symmetry groups of some perfect one-factorizations of complete graphs*, Discr. Math..
3. Bileski, F., *Letters to W. Pages (4/2/87) and W. D. Wallis (6/21/87).*
4. Colbourn. M. J., Colbourn, C. J., and Rosenbaum, W. L., *Trains: an invariant for Steiner triple systems*, Ars Comb. **13** (1982), 149–162.
5. Dinitz, J. H., *Some Perfect Room Squares*, J. Combin. Math. Combin. Comput. **2** (1987), 29–36.
6. Dinitz, J. H. and Stinson, D. R., *Some perfect one-factorizations from starters in finite fields*, J. Graph Theory **13, No. 4**, 405–415.
7. Fu, C. M., "The intersection problem for pentagon systems", Ph.D. Thesis, Auburn University, 1987.
8. Gelling, E. N., "On 1-factorizations of the compete graphs and the relationship to round robin schedules", M.A. Thesis, Univ. of Victoria, 1973.
9. Gross, K. B., *Equivalence of Room designs, II*, J. Combin. Th. **A 17** (1974), 299–316.
10. Kotzig, A., *Groupoids and partitions of complete graphs*, in "Combinatorial Structures and Their Applications", New York, 1970, pp. 215–221.
11. Mendelsohn, E. and Rosa, A., *One-factorizations of the compete graph*, J. Graph Theory Vol **9** (1985), 43–65.
12. Petrenyuk, L. P. and Petrenyuk, A. Y., *Intersection of perfect one-factorizations of complete graphs*, Cybernetics **16** (1980), 6–9.
13. Stinson, D. R, *A comparison of two invariant for Steiner triple systems: fragments and trains*, Ars Comb. **16** (1983), 69–76.
14. Tutte, W. T., *The factorization of linear graphs*, J. London Math. Soc. **22** (1947), 107–111.
15. Wallis, W. D., *One-factorization of graphs–tournament applications*, College Math J. **18** (1987), 118–123.
16. Wallis, W. D., *Staircase Factorizations*, Gazette Austral. Math. Soc. (to appear).
17. Wallis, W. D., Street, A. P. and Wallis, J. S., "Combinatorics: Roon squares, sum-free sets, Hadamard matrics", Springer, Heidelberg, 1972.
18. White. M. S., *Triple systems as transformations and their paths along triads*, Trans. Amer. Math. Soc. **14** (1913), 6–13.