On Simmons' "Campaign Graphs"

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Abstract. In a recent paper, Gustavus J. Simmons introduced a new class of combinatorial-geometric objects he called "campaign graphs". A k-campaign graph is a collection of points and segments such that each segment contains precisely k of the points, and each point is the endpoint of precisely one segment. Among other results, Simmons proved the existence of infinitely many critical k-campaign graphs for $k \leq 4$. The main aim of this note is to show that Simmons result holds for $k = 5$ and 6 as well, thereby providing proofs, amplifications and a correction for statements of this author which Dr. Simmons was kind enough to include in a postscript to his paper.

For a positive integer k, a family **F** of points and straight-line segments in the plane is called by Simmons [6] a k-campaign graph provided each segment of **F** is incident with precisely k of the points of **F**, and each point of **F** is the endpoint of precisely one segment of **F**. Replacing the last condition by the requirement that every point of **F** be the endpoint of precisely m segments of **F**, where m is a fixed positive integer, we obtain the definition of a (k, m)-campaign graph or, as we prefer to say, a **(k, m) segment configuration**; to avoid repetitions of this long designation, we shall frequently shorten it to (k, m)-SC. Since the cases with $k = 1$ or 2 are rather obvious, we shall henceforth assume $k \geq 3$. Clearly, disjoint unions (as well as other kinds of combinations) of (k, m)-SCs are themselves (k, m)-SCs; thus it is reasonable to restrict attention to **critical** (k, m) segment configurations, that is, those which contain no proper subfamily which is a (k, m) segment configuration. The critical (k, m)-SCs are analogous to the well known (connected) configurations of points and lines of classical geometry; in fact, our results have to a large extent been made possible by previous research of such configurations (see the forthcoming account [3]).

In [6], Simmons proved (among other results) that there exist infinitely many critical $(k, 1)$ segment configurations for $k \leq 4$, and gave one example of a critical $(5, 1)$-SC. The first result we shall prove is:

Theorem 1. *For* $k = 5$ *or* 6 *there exist infinitely many critical* $(k, 1)$ *segment configurations.*

Remark. The statement of Theorem 1 regarding $k = 5$ was included in the Postscript of [6], but with no hint of how the family is constructed. Also included

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was this writer's drawing of a single example of a critical $(6,1)$-SC, to which we shall return below.

Proof of Theorem 1: We start by constructing, for each integer $n \geq 3$, a family $\mathbf{F}_n$ consisting of $36n$ points and $18n$ segments, and then proving that $\mathbf{F}_n$ is a $(5,1)$-SC. In Figure 1 we show the families $\mathbf{F}_n$ for $n = 3$ and 4; this should illustrate the idea of the construction. We must stress, however, that the drawings do not represent a proof of the existence of the segment configurations shown: the diagrams do not establish that the apparently straight line segments are in fact straight, or that they could be replaced by straight line segments with the required incidences. Indeed, it is well known that for configurations of points and lines there exist examples in which the lines appear to be straight but actually are not straight, and cannot be made straight without violating some of the incidence conditions even if the positions of the points of the configuration are also permitted to be changed. (The first such "non-stretchable" configuration was found by Schroeter [5] more than a century ago; for an account of this topic see [1], [2].) To show that our construction does not suffer from this shortcoming we shall assign precise locations to the points so as to enable us to establish that the collinearities claimed indeed take place.

To simplify the typography, it is convenient to introduce the abbreviations

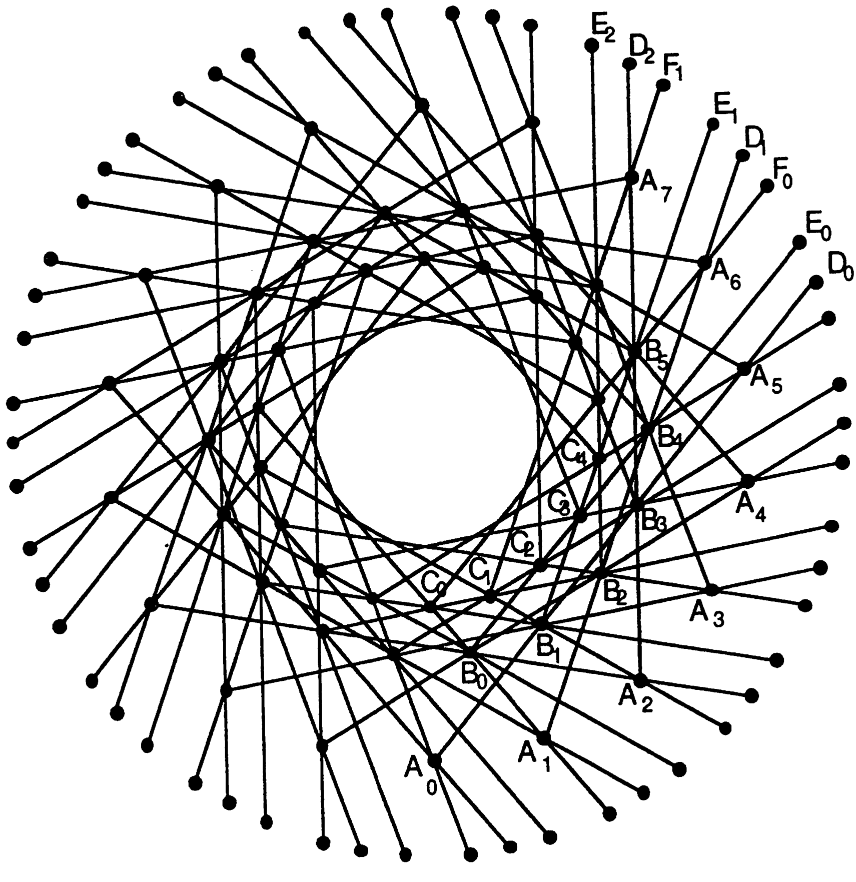
$$\begin{aligned} r_B &= \cos(n+2)\alpha/\cos(n-1)\alpha = \cos(2n+1)\alpha/\cos 2n\alpha, \\ r_C &= \cos(2n+1)\alpha/\cos(2n-1)\alpha \\ &= \cos(2n+1)\alpha \cdot \cos(n+1)\alpha/\cos 2n\alpha \cdot \cos(n-2)\alpha \end{aligned}$$

where $\alpha = \frac{\pi}{6n}$. The equality of the two trigonometric expressions for r_B can be established as follows (using $\cos 2n\alpha = \frac{1}{2}$, $\cos 3n\alpha = 0$, and the relations between sums and products of cosines):

$$\begin{aligned} 2\cos(n+2)\alpha \cdot \cos 2n\alpha \\ &= \cos(3n+2)\alpha + \cos(n-2)\alpha \\ &= \cos(3n+2)\alpha + 2\cos 2n\alpha \cdot \cos(n-2)\alpha \\ &= \cos(3n+2)\alpha + \cos(3n-2)\alpha + \cos(n+2)\alpha \\ &= 0 + \cos(n+2)\alpha \\ &= \cos 3n\alpha + \cos(n+2)\alpha \\ &= 2\cos(2n+1)\alpha \cdot \cos(n-1)\alpha. \end{aligned}$$

The validity of the expressions for r_C can be verified in a similar manner, and since cosine is strictly decreasing in the interval from 0 to π, it follows that $r_C < r_B < 1$.

Now we can define the "basic" points of $\mathbf{F}_n$ which we denote (as in Figure 1) by A_j, B_j and C_j; here and in the sequel, j ranges over the integers $0, 1, 2, \ldots, 6n-1$,



(a)

Figure 1. Examples of $(5, 1)$ segment configurations constructed in the proof of Theorem 1. (a) corresponds to $n = 3$, and has 108 points and 54 segments; (b) corresponds to $n = 4$ and has 144 points and 72 segments. To avoid clutter, only some of the points have been labelled.

and is to be understood $\mod 6n$. The remaining "non-basic" points D_j, E_j and F_j will be defined below. We choose

$$A_j = (\cos 2j\alpha, \sin 2j\alpha)$$
$$B_j = (r_B \cdot \cos(2j + 1)\alpha, r_B \cdot \sin(2j + 1)\alpha)$$
$$C_j = (r_C \cdot \cos 2j\alpha, r_C \cdot \sin 2j\alpha);$$

then we define the lines a_j, b_j and c_j as the straight lines containing the following

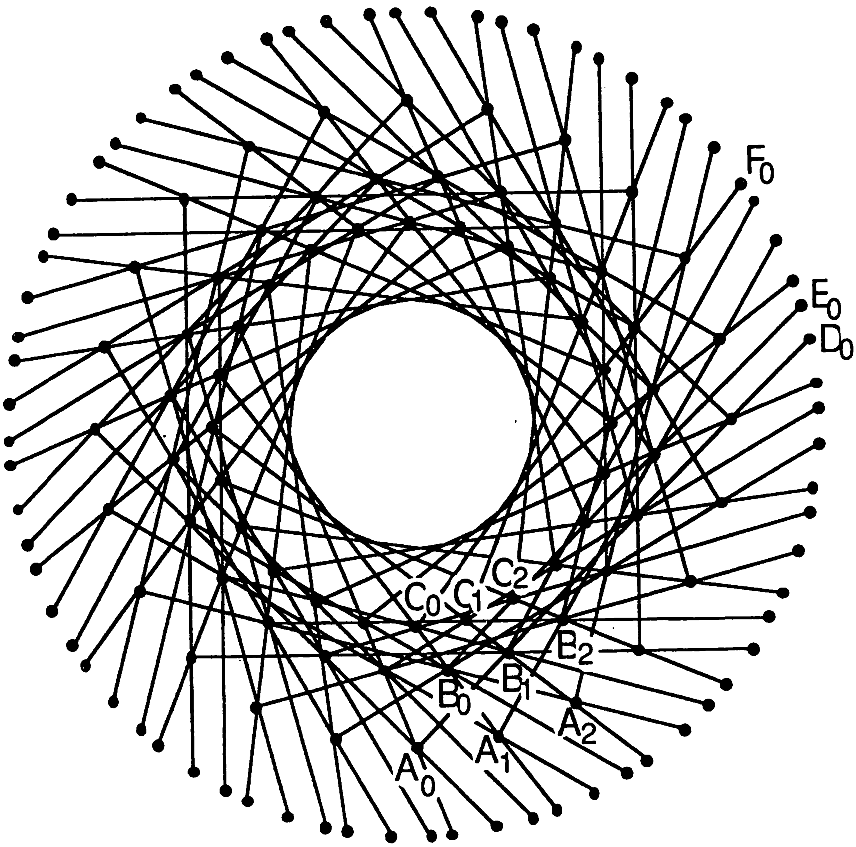


Figure 1 (b).

points:

$$a_j\colon A_j B_{j+1} B_{j+n} A_{j+n+2}\, D_j$$
$$b_j\colon B_j C_{j+2} C_{j+n} B_{j+n+1}\, E_j$$
$$c_j\colon C_j C_{j+2n-1}\, B_{j+2n-1}\, A_{j+2n} F_j.$$

In fact, the line c_j contains the points A_{j-1} and B_{j-1} as well. To show that the "basic" points are indeed collinear as claimed, it is simplest to proceed as follows. Due to the symmetry of the families of points and lines in question, the collinearity of $A_j B_{j+1} B_{j+n} A_{j+n+2}$ is equivalent to the assertion that the midpoint M_j of $A_j A_{j+n+2}$ is at the same distance from the center O of the figure as is the midpoint M_j' of $B_{j+1} B_{j+n}$. But this equality of distances is obvious, since

$OM_j = OA_j \cdot \cos(n+2)\alpha = \cos(n+2)\alpha$, while

$$OM'_j = r_B \cdot \cos(n-1)\alpha = \frac{\cos(n+2)\alpha}{\cos(n-1)\alpha} \cdot \cos(n-1)\alpha = OM_j.$$

Analogously, the collinearity of the points on b_j is equivalent to the coincidence of the midpoint of B_jB_{j+n+1} with the midpoint of $C_{j+2}C_{j+n}$, and the collinearity of the points on c_j is equivalent to the coincidence of the midpoints of $A_{j-1}A_{j+2n}$, $B_{j-1}B_{j+2n-1}$, and C_jC_{j+2n-1}; all these are immediate consequences of the trigonometric identities that appear in the definitions of r_B and r_C.

The "non-basic" points D_j, E_j and F_j are chosen arbitrarily on the appropriate lines, subject only to the conditions of being on the suitable halflines and not on any of the other lines. The $18n$ segments of $\mathbf{F}_n$ can now be defined as A_jD_j, B_jE_j and C_jF_j, and it follows that $\mathbf{F}_n$ is a $(5, 1)$ segment configuration, as claimed. Its criticality is obvious.

In a similar way, for each integer $n \geq 4$ we construct a family $\mathbf{G}_n$. consisting of $42n$ points and $21n$ segments, which is a $(6, 1)$-SC. The construction is illustrated for $n = 5$ in Figure 2. For each of the values of n in question, we include in $\mathbf{G}_n$ the family $\mathbf{F}_n$ constructed above, but with a specialized choice of the points E_j and F_j: we choose E_j as the intersection of b_j with c_{j-n+3}, and F_j as the intersection of c_j with a_{j+n-3}; as before, D_j is free, except that it is now constrained to a smaller halfline of a_j. We also need to choose additional points, denoted G_j. The point G_j is the intersection point of the line determined by A_j and C_j (which also contains A_{j+3n} and C_{j+3n}) and the line b_{j-n-3}. The segments of $\mathbf{G}_n$ are now defined as A_jD_j, B_jE_j, C_jF_j and G_jG_{j+3n}, and it follows that $\mathbf{G}_n$ is a $(6, 1)$ segment configuration. ∎

The need for caution in questions of "stretchability" is well illustrated by Figure 3, which is the example of a $(6, 1)$ segment pattern I sent to Dr. Simmons, and which appears in the Postscript to [6]. To my acute embarrassment, despite the checks which I thought were complete, the apparent segments in this drawing do not all correspond to straight line segments. In fact, if the other incidences are kept, the segments are assumed straight and the whole family is assumed to have 18-fold rotational symmetry, then the triplets of segments which appear to meet at each of the "points" marked by the large dots actually determine small triangles! It therefore seems that this is a $(6, 1)$ configuration not of segments but of "pseudosegments". However, I cannot prove even this statement: although I consider this possibility most unlikely, it is conceivable that if one does not insist on the rotational symmetry of the figure, the additional freedom may lead to realizability by straight line segments. To resolve this question, one could parametrize all points and lines, and see whether the resulting systems of equations have solutions; such an approach was recently used to prove the realizability by straight lines (even in the rational plane) of all the combinatorial types of the configurations (11_3) and

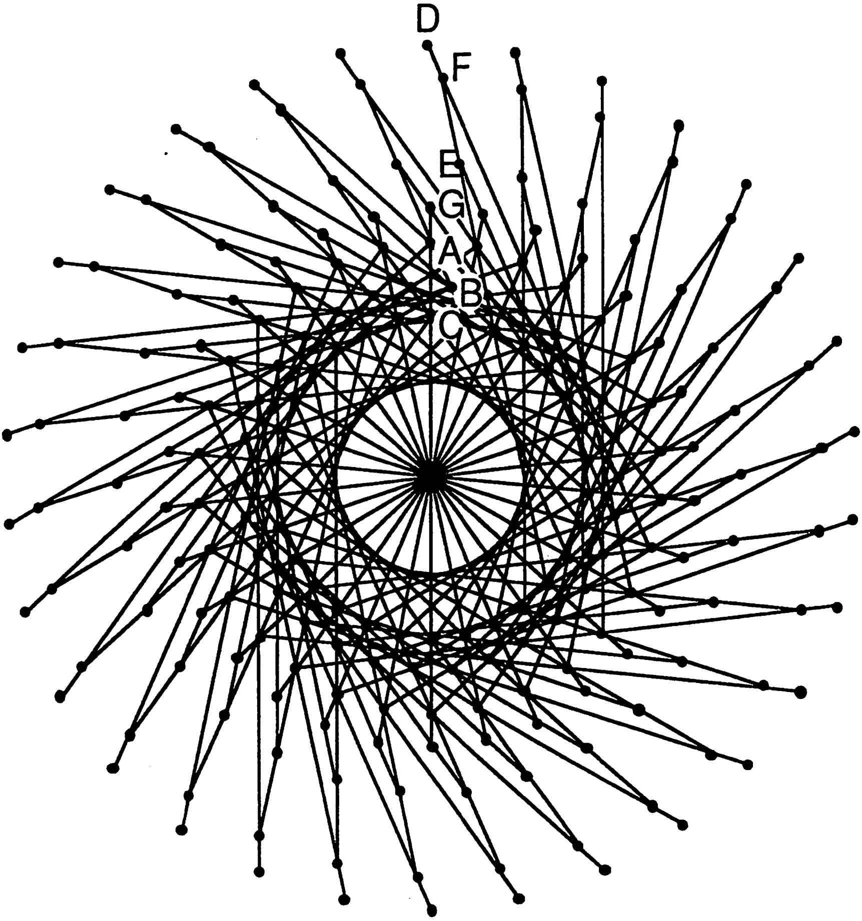


Figure 2. A critical $(6, 1)$ segment configuration constructed in the proof of Theorem 1; it corresponds to $n = 5$ and has 210 points and 105 segments.

(12_3) of eleven or of twelve lines and points, each incident with three of the others (see [7]). However, the size of the problem in the present case seems to make it intractable by the methods available so far.

Concerning (k, m)-SCs with $m \geq 2$ we have the following general result, which is analogous to well known facts about configurations of points and lines.

Theorem 2. *If a critical* $(k, 1)$ *segment configuration exists, then there exist critical* (k, m) *segment configurations for every* $m \geq 2$.

Proof: The proof is quite straightforward, using a "direct product" approach. Let $\mathbf{F}$ be a (k, m)-SC and $\mathbf{F}^*$ a (k, m^*)-SC, with points $\{P_\nu \mid \nu \in N\}$ and $\{P_\nu^* \mid \nu \in N^*\}$ and segments $\{S_\mu \mid \mu \in M\}$ and $\{S_\mu^* \mid \mu \in M^*\}$ respectively. We take copies of $\mathbf{F}$ and $\mathbf{F}^*$ such that no line determined by points of one is parallel to a

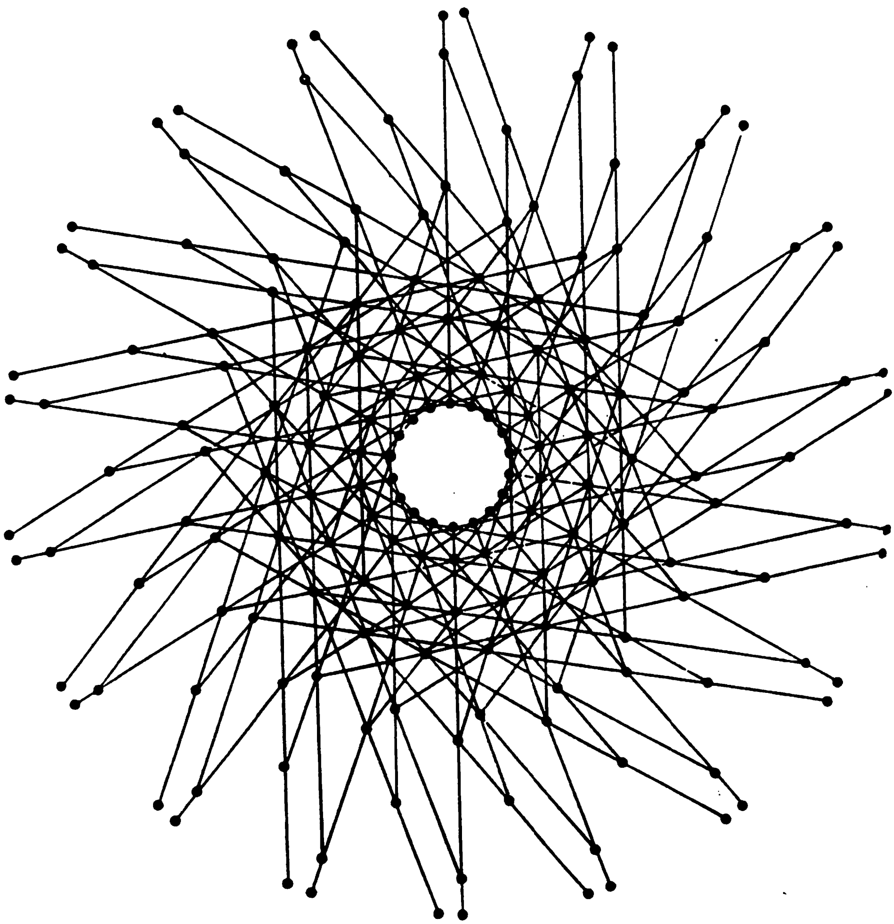


Figure 3. A "fake" $(6,1)$ segment configuration, in which the segments which appear to be straight cannot all be straight if the symmetry and incidences are to be preserved.

line determined by points of the other. Then a $(k, m+m^*)$ segment configuration $\mathbf{G} = \mathbf{F} \otimes \mathbf{F}^*$ is obtained by taking the points $\{P_\nu + P_\nu^* \mid \nu \in N, \nu \in N^*\}$ and the segments $\{P_\nu + S_\mu^* \mid \nu \in N, \mu \in M^*\}$ and $\{P_\nu^* + S_\mu \mid \nu \in N^*, \mu \in M\}$. It is clear that the resulting configuration $\mathbf{G}$ is critical. ∎

The construction is illustrated in Figure 4, in which both $\mathbf{F}$ and $\mathbf{F}^*$ are the smallest $(3,1)$-SC, consisting of 6 points. The construction leads to very large configurations even for moderate values of k and m. Moreover, graphical representations of the resulting configurations are so cluttered that they provide no enlightment. Hence there may be some interest in an alternative construction for the special case of $m = 2$ and $k = 3$ or 4, which leads to smaller and more symmetric configurations.

For the construction of infinitely many critical (3,2) segment configurations we proceed as we did in the proof of Theorem 1. We first specify, for each odd integer $n \geq 3$, a family of $12n$ points and $12n$ segments, and then show that it is a critical $(3,2)$ segment configuration $\mathbf{H}_n$. In Figure 5 we illustrate the construction for $n = 3$. Putting again $\alpha = \frac{\pi}{6n}$, we define

$$r_B = \cos 2n\alpha / \cos(n-2)\alpha = \cos(2n+2)\alpha / \cos(n+4)\alpha;$$

the equality of the two trigonometric expressions can be verified as in the earlier situations. Then we define, for $j = 0, 1, 2, \ldots, 6n-1$, the points

$$A_j = (\cos 2j\alpha, \sin 2j\alpha)$$
$$B_j = (r_B \cdot \cos(2j+1)\alpha, r_B \cdot \sin(2j+1)\alpha).$$

Using the trigonometric relation which appears in the definition of r_B and the symmetry of this set of points, it is simple to verify that for each j the following quadruplets of points are collinear:

$$A_j B_{j+(n-3)/2}\, B_{j+(3n+5)/2}\, A_{j+2n+2}$$
$$A_j B_{j+(n+1)/2}\, B_{j+(3n-1)/2}\, A_{j+2n}.$$

Consequently, the $12n$ points A_j and B_j and the $12n$ segments $A_j B_{j+(3n+5)/2}$ and $B_{j+(n+1)/2} A_{j+2n}$ form a $(3,2)$ segment configuration.

We still have to show that the configurations $\mathbf{H}_n$ are critical. A simple way to do this is as follows. It is clear that the segments of every $(k,2)$ segment configuration form a collection of circuits. However, in the case of an $\mathbf{H}_n$, using the fact that n is odd it is easy to check that the segments form a single circuit, hence $\mathbf{H}_n$ is critical, as claimed. (All the other assertions concerning $\mathbf{H}_n$ are valid for even n as well.) ∎

The smallest $(3,2)$ segment configuration obtainable by this method is the one in Figure 5; it has 36 points. A smaller example of a $(3,2)$ segment configuration, with only 21 points, is shown in Figure 6. It is derived from a configuration (21_4) of points and lines discussed in [4]. It is not known whether smaller examples exist, with or without the added assumption that the segments form a single circuit.

It seems that a somewhat analogous construction of $(4,2)$ segment configurations is possible. We have not investigated this question systematically, but show in Figure 7 an example with 72 points; this is the smallest number of points in a configuration of this type we have been able to find.

Simmons [6] characterized the $(3,1)$-SCs, and pointed out the great proliferation of the $(4,1)$-SCs. The central problem concerning segment configurations is whether for every k there exist $(k,1)$-SCs.

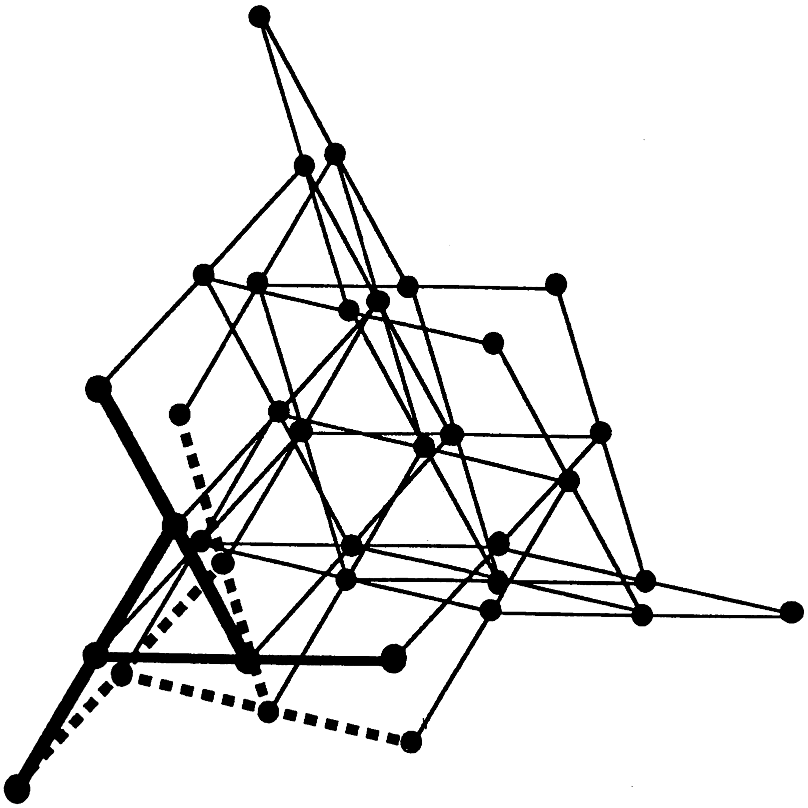


Figure 4. A $(3, 2)$ segment configuration which illustrates the construction used in the proof of Theorem 2. In this example the two "factors" are isomorphic; one copy is emphasized by heavy lines.

Conjecture. For every k there exist infinitely many critical $(k, 1)$ segment configurations.

The results of Simmons [6] and of the present paper establish the validity of this conjecture for $k \leq 6$, but give no clues for larger values of k. So far, I only found one "fake" $(7, 1)$-SC, with "pseudosegments" which apparently cannot be made straight. On the other hand, even in cases with $k = 5$ or 6, in which we know that there are infinitely many different $(k, 1)$-SCs, it is not known whether the symmetries exhibited in the constructions we made are essential. One way (which is also of independent interest) of investigating this would be to try to find $(5, 1)$ or $(6, 1)$ segment configurations in the rational plane, that is, under the

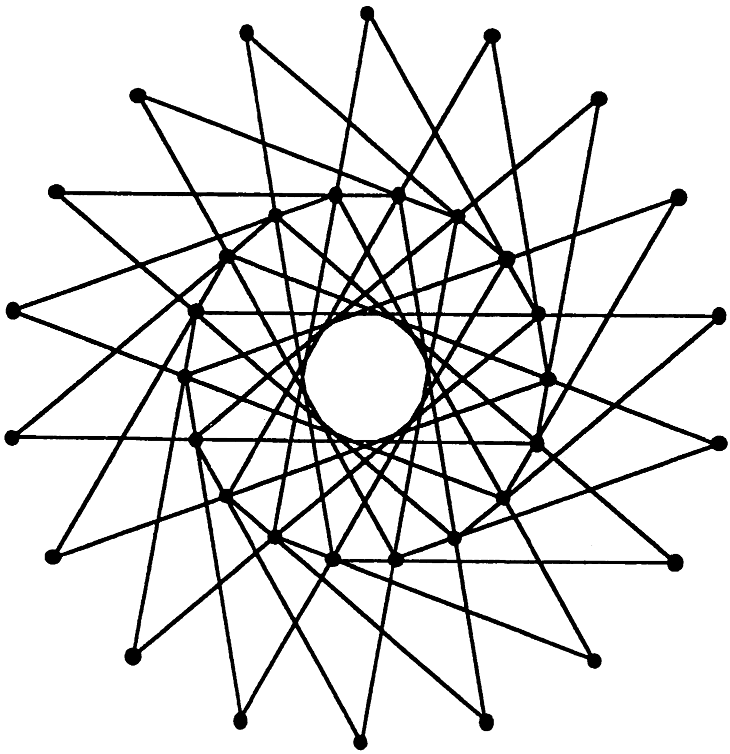


Figure 5. A (3, 2) segment configuration obtained by the construction explained in the text.

added restriction that all the points have rational coordinates.

Another open question is whether $(k, 1)$-SCs exist for all k if one allows the use of "pseudosegments" (that is, curvilinear arcs) but requires them to behave like straight line segments in as far as pairwise intersections are concerned. More specifically, no two should have more than one point in common, and if the common point is relatively interior to both, they must cross each other. The (6, 1)-SC in Figure 3 satisfies these conditions, as does the (7, 1)-SC with pseudolines mentioned above.

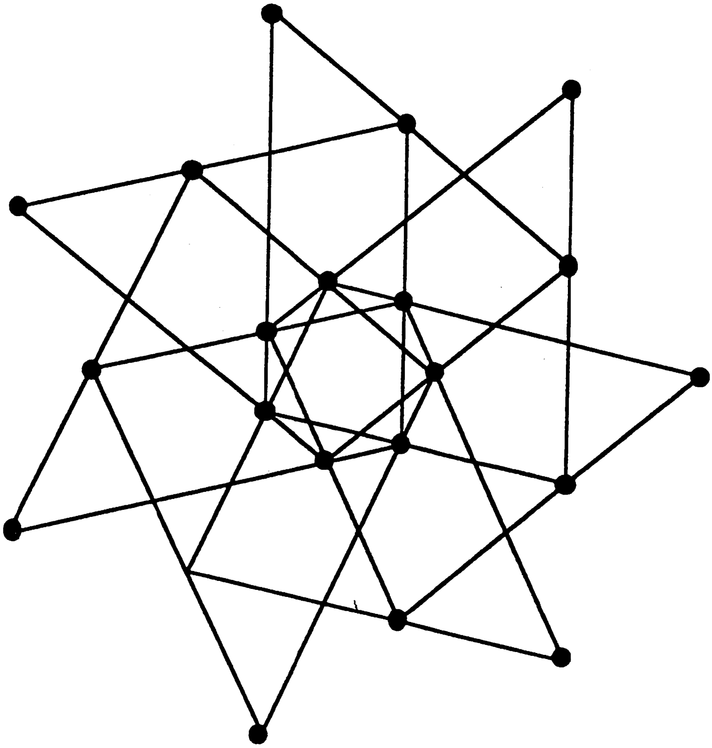


Figure 6. The smallest $(3, 2)$ segment configuration known.

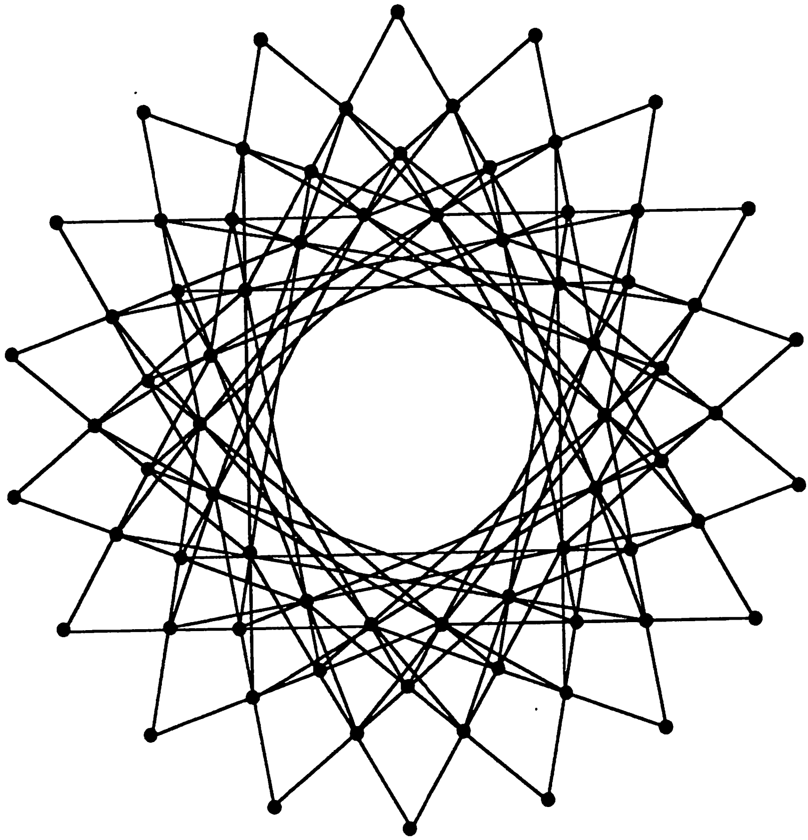


Figure 7. A (4, 2) segment configuration with 72 points and 72 lines; no smaller (4, 2)-SC is known.

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