On Simple Hamiltonian Cycles in a 2-Colored Complete Graph

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Abstract. Let the edges of the complete graph K_n be 2-colored. A *Simple Hamiltonian Cycle* is a Hamiltonian cycle in K_n that is either monochromatic or is a union of two monochromatic paths. The main result of this paper is that if n is an even integer greater than 4, then for every 2-coloring of the edges of K_n, there is a Simple Hamiltonian Cycle in K_n which is either monochromatic, or is a union of two monochromatic paths, where each path is of *even length.*

1. Introduction.

Let n be an integer, $n \geq 3$ and let the edges of the complete graph K_n be 2-colored. A *Simple Hamiltonian Cycle* (S.H.C.) is a Hamiltonian cycle in K_n that is either monochromatic or is a union of two monochromatic paths. The following theorem is due to A. Gyárfás.

Theorem 1. *If the edges of the complete graph K_n are 2-colored, then K_n contains a Simple Hamiltonian Cycle.*

An algorithmic proof of Theorem 1 appears in [2] along with some other related results concerning the covering of the vertices of K_n by monochromatic paths and cycles.

The object of this paper is to strengthen the conclusion of Theorem 1. The strengthened theorem, along with its proof, is presented in Section 2. The proof is by induction and required a search that was carried out by computer.

Next we mention some related results. In [3] R. Rado generalized a weaker version of Theorem 1 to infinite graphs, and in [4] H. Raynaud generalized Theorem 1 by proving a corresponding theorem for the complete symmetric directed graph. Recently in [1], P. Erdős, A. Gyárfás and L. Pyber proved that if the edges of K_n are r-colored, then the vertices of K_n can be covered by $P(r)$ disjoint monochromatic paths, where $P(r)$ is a polynomial in r and is independent of n.

2. Result and Proof.

In the sequel we shall use the phrase 'colored graph' to mean an edge colored graph. Suppose that r and s are positive integers such that $r + s = n$. An

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(r, s) - S.H.C is an S.H.C. consisting of a path of length r in the first color and a path of length s in the second color. An $(n, 0)$ - S.H.C.($(0, n)$ - S.H.C.) is a monochromatic Hamiltonian cycle in the first (second) color.

Theorem 2. *Let n be an integer, $n \geq 3$.*

(i) *If n is odd, then every 2-colored K_n contains an* S.H.C. *Moreover, for every two non-negative integers r and s such that $r + s = n$, there exists a 2-coloring of K_n such that the only* S.H.C.'s *in K_n are the* (r, s) - S.H.C.'s.

(ii) *If n is even and $n \geq 6$, then every 2-colored K_n contains an* (r, s) - S.H.C. *where both r and s are non-negative even integers and $r + s = n$. Moreover, for every two non-negative even integers r and s such that $r + s = n$, there exists a 2-coloring of K_n such that the only* S.H.C.'s *in K_n are the* (r, s) - S.H.C.'s.

(iii) *There exists a unique 2-coloring of K_4 for which K_4 does not contain a* $(0, 4)$ - S.H.C. *or a* $(4, 0)$ - S.H.C. *or a* $(2, 2)$ - S.H.C. *However it contains a* $(1, 3)$ - S.H.C. *and a* $(3, 1)$ - S.H.C. *All other 2-colorings of K_4 contain a* $(0, 4)$ - S.H.C., *or a* $(4, 0)$ - S.H.C. *or a* $(2, 2)$ - S.H.C.

Proof. The proof of (iii) is obtained by a direct verification. The partition of the edges of K_4 into 2 paths of length 3 each, is the mentioned unique 2-coloring. For the "moreover" parts of (i) and (ii), we exhibit a 2-coloring of K_n having the desired property. Let r and s be as in (i) or (ii). Without loss of generality we may assume that r is even. Color the disjoint union of $K_{r/2}$ and $K_{n-r/2}$ red and the complete bipartite graph $K_{r/2\,,\,n-r/2}$ blue. In the degenerate case $r = 0$ this coloring reduces to a red K_n. One can see that the only S.H.C.'s in this coloring are (r, s) - S.H.C's. The first part of (i) is already contained in Theorem 1 and is stated for the sake of completeness. Thus, it is left to prove the first part of (ii). The proof is by induction on n. The first case of $n = 6$ was checked manually. Using Microsoft Pascal and a backtracking program, an IBM-PC/AT was used to verify the following two claims.

Claim I. *Every 2-colored K_8 contains an (r, s) - S.H.C. where both r and s are non-negative even integers and $r + s = 8$.*

Claim II. *Consider any 2-coloring of K_{10}. Assume that for this 2-coloring there exists a 2-colored 8-cycle of the type shown either in* Fig. 1(a) *or* Fig. 1(b) *or* Fig. 1(c). *Then there exists an* (r, s) - S.H.C. *in K_{10} such that:*

1. *r and s are non-negative even integers and $r + s = 10$.*

2.(a) *The edge $\{1, 2\}$ was used in the* (r, s) - S.H.C. *if the 8-cycle is as shown in* Fig. 1(a).

(b) *The edge $\{5, 6\}$ was used in the* (r, s) - S.H.C. *if the 8-cycle is as shown in* Fig. 1(b).

(c) *The edges* $\{3,4\}$ *and* $\{6,7\}$ *were used in the* (r,s) - S.H.C. *if the 8-cycle is as shown in* Fig. 1(c).

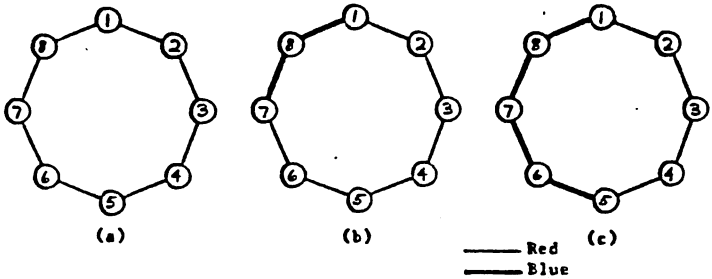

Fig. 1

A traditional demonstration of the proof of the claims is possible, but requires a long computer print-out. We assume the claims above to be true and complete the demonstration of the proof of (ii) of Theorem 2 by induction. Consider a 2-coloring of K_n where n is even and $n \geq 10$. Disregarding two vertices, say x and y, of K_n and applying the induction hypothesis, we get an (r,s) - S.H.C. in K_{n-2} such that r and s are both even and $r+s=n-2$. We shall consider three cases (a), (b) and (c) corresponding to cases 2(a), 2(b) and 2(c) of Claim II respectively.

(a) Assume that the S.H.C. in K_{n-2} is monochromatic, say red. Let $v_1, v_2, \ldots, v_8, \ldots, v_{n-2}, v_1$ be the red S.H.C. Consider the 2-colored K_{10} induced by the vertices $v_1, v_2, \ldots, v_8, x, y$, recoloring, if necessary, the edge $\{v_1, v_8\}$ red. This 2-colored K_{10} has a red 8-cycle as shown in Fig. 1(a). By replacing the red edge $\{v_1, v_8\}$ by the red path $v_8, v_9, \ldots, v_{n-2}, v_1$, we obtain an (r,s) - S.H.C. in K_n where r and s are both even.

(b) Assume that K_{n-2} contains an $(n-4,2)$ - S.H.C. such that $\{v_i, v_{i+1}\}$ for $i=1,\ldots,n-4$ are red and $\{v_{n-3}, v_{n-2}\}$ and $\{v_{n-2}, v_1\}$ are blue. Consider the 2-colored K_{10} induced by the vertices $v_1, v_2, v_3, v_4, v_5, v_{n-4}, v_{n-3}, v_{n-2}, x, y$, recoloring, if necessary, the edge $\{v_5, v_{n-4}\}$ red. This 2-colored K_{10} has an 8-cycle as shown in Fig. 1(b). By Claim II, 2(b), there is an (r', s') - S.H.C. in K_{10} such that both r' and s' are even and the edge $\{v_5, v_{n-4}\}$ is used in the S.H.C. By replacing the red edge $\{v_5, v_{n-4}\}$ by the red path $v_5, v_6, \ldots, v_{n-4}$ we obtain an (r,s) - S.H.C. in K_n where both r and s are even.

(c) Assume that K_{n-2} contains an $(n-2-s, s)$ - S.H.C. where $s \geq 4$. Let $v_1, v_2, \ldots, v_{n-2}, v_1$ be the $(n-2-s, s)$ - S.H.C. such that $\{v_i, v_{i+1}\}$ for $i=1,\ldots,n-2-s$ are red while $\{v_i, v_{i+1}\}$ for $i=n-1-s,\ldots,n-2$ and $\{v_{n-2}, v_1\}$ are blue. Consider the 2-colored K_{10} induced by the vertices $v_1, v_2, v_3, v_{n-s-2}, v_{n-s-1}, v_{n-s}, v_{n-3}, v_{n-2}, x, y$, recoloring, if necessary, the edge $\{v_3, v_{n-s-2}\}$ red and the edge $\{v_{n-s}, v_{n-3}\}$ blue. This 2-colored K_{10} has an 8-cycle as shown in Fig. 1(c). By Claim II, 2(c), there is an (r', s') - S.H.C. in K_{10} such that both r' and s' are even and the edges $\{v_3, v_{n-2-s}\}$ and $\{v_{n-s}, v_{n-3}\}$ are used in the S.H.C. By replacing the red edge $\{v_3, v_{n-2-s}\}$ by the red path $v_3, v_4, \ldots, v_{n-2-s}$, and replacing the blue edge $\{v_{n-s}, v_{n-3}\}$ by the blue path $v_{n-s}, \ldots, v_{n-3}$ we obtain an (r, s) - S.H.C. in K_n where both r and s are even. This completes the proof of (ii), and, hence, the proof of Theorem 2 is complete.

References

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