On Small Faces in 4-critical Planar Graphs

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Abstract. It is shown that a 4-critical planar graph must contain a cycle of length 4 or 5 or a face of size k, where $6 \leq k \leq 11$.

A graph G is said to be k-critical if it has chromatic number k but every proper subgraph of G is $(k-1)$-colorable. It was proved by Grötzsch [5] that if G is a planar graph without triangles then G is 3-colorable. It follows that any 4-critical planar graph must have at least one triangular face. Grünbaum [6] proved that any such graph must have at least four triangular faces. Grünbaum's proof of this result contained some inaccuracies but a corrected proof was published by Aksionov [4].

It is natural to ask whether a 4-critical planar graph must contain a 4-cycle. The answer is no; the planar graph shown in Figure 1 is 4-critical and contains no 4-cycle. The graph was obtained via a construction described by Aksionov and Mel'nikov in [2] and [3]. Let G be a 4-critical graph and let $e = ab$ be an edge of G. Delete e and replace it by the graph H shown in Figure 2. The vertices a, b of H are to be identified vith the correspondingly labelled vertices of G. The graph in Figure 2 is 3-colorable. However, in every 3-coloring the vertices a and b are assigned different colors; it is a quasi-edge in the sense of Aksionov and Mel'nikov. The resulting graph is then easily seen to be 4-critical and is planar if G is planar. If we apply this operation to each of the three edges incident with a particular vertex of K_4 we get the graph in Figure 1. The graph in Figure 1 has six faces of size 5. There are three pairs of such faces; in each pair the faces share an edge. If we now apply the above construction to each of these edges we get a larger graph with the same property. This may be repeated as often as we please. Thus there exist arbitrarily large 4-critical planar graphs with no cycles of length 4 and having six faces of size 5. Furthermore, these graphs have no cycles of length 5, other than the boundaries of the pentagonal faces.

One may construct arbitrarily large 4-critical planar graphs with no cycles of length 5 and only four cycles of length 4. Let G be a 4-critical planar graph with exactly two 4-cycles and exactly two 5-cycles and suppose that these 5-cycles bound faces sharing an edge e. An example of such a graph is shown in Figure 3. Let $\mathcal{G}$ denote the collection of all such graphs. That $\mathcal{G}$ contains arbitrarily large graphs may be seen by noting that if one deletes the special edge e of any such graph and replaces it by a copy of H in the manner described earlier, one gets a larger member of $\mathcal{G}$. Recall the following (special case of a) construction of Hajós

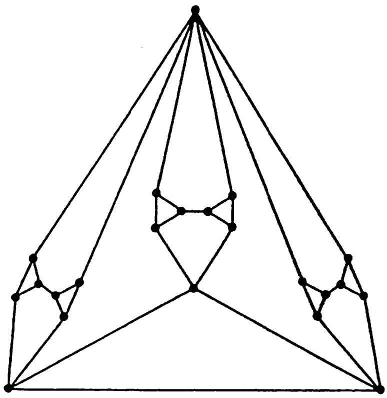

Figure 1

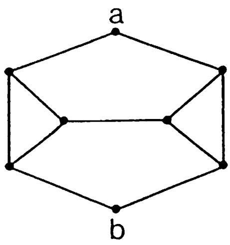

Figure 2

[7]: Let G_1 and G_2 be 4-critical planar graphs and let $e_1 = a_1 b_1$ and $e_2 = a_2 b_2$ be edges of G_1 and G_2. Delete e_1 and e_2, identify a_1 with a_2 and add the edge $b_1 b_2$. The resulting graph is then 4-critical and planar. Furthermore, if G_1 and G_2 are members of $\mathcal{G}$ and if e_1 and e_2 are chosen as the special edges which separate the faces of size 5, the resulting graph has no 5-cycles and has four cycles of length 4.

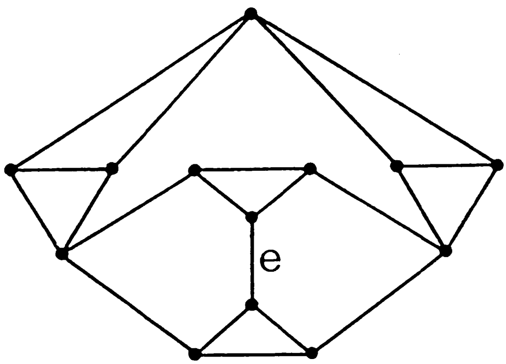

Figure 3

Steinberg (see [2], page 131 and [3], page 9) conjectured in 1975 that any 4-critical planar graph must contain a cycle of length 4 or 5. The object of this note is to prove that a counter-example to Steinberg's conjecture, if there is one, must contain a small non-triangular face.

Theorem. *Let G be a 4-critical planar graph. Then at least one of the following holds:*

(a) *G contains a cycle of length 4 or 5;*

(b) *every plane drawing of G contains a face of size k for some k satisfying* $6 \leq k \leq 11$.

Proof: Suppose that (a) doesn't hold for G and suppose that some plane drawing of G has no face of size k, $6 \leq k < t$. Let n,m and f denote the number of vertices, edges and faces of G, respectively, and let f_i denote the number of faces with i edges. Then

$$2m = \sum_{i \geq 3} i f_i = 3f_3 + \sum_{i \geq t} i f_i \geq 3f_3 + t(f - f_3),$$

from which we get

$$f \leq \frac{(t-3)f_3 + 2m}{t}$$

Since $3f_3 \leq m$ this gives

$$f \leq \frac{(t+3)m}{3t}.$$

From Euler's formula, $n + f - m = 2$, we get

$$n + \frac{3+t}{3t}m - m \geq 2$$

so that, since $t \geq 6$,

$$m \leq \frac{3t}{2t-3}n - \frac{6t}{2t-3} < 2n.$$

Define l by

$$m = 2n - l. \tag{1}$$

Let p denote the number of vertices of G of degree 3. Then

$$4n - 2l = 2m = \sum d(v) \geq 3p + 4(n-p)$$

from which we get

$$p \geq 2l. \tag{2}$$

Any vertex of degree 3 is incident with an edge e such that the faces on each side of e each has at least t edges; otherwise, there would be a 4-cycle. From (2), there are at least l such edges e. Split the set of edges of G into two sets as follows

$$E_1 = \{e : \text{both faces neighboring } e \text{ have at least } t \text{ edges}\}$$
$$E_2 = \{e : \text{one face neighboring } e \text{ is triangular}\}.$$

It is clear that

$$f_3 \le \frac{1}{3}|E_2|.$$

Let $\mathcal{F}$ denote the set of faces of G of size at least t. Set up a bipartite graph whose parts are $\mathcal{F}$ and $E_1 \cup E_2$ and in which $F \in \mathcal{F}$ is joined to $e \in E_1 \cup E_2$ if e is an edge of F. In this graph the degree of each element of E_1 is 2 and the degree of each element of E_2 is 1 so that the number of edges is $2|E_1| + |E_2|$. On the other hand, since the degree of each element of $\mathcal{F}$ is at least t, the number of edges is at least $t|\mathcal{F}|$. Thus

$$t|\mathcal{F}| \le 2|E_1| + |E_2|,$$

so that

$$f - f_3 = |\mathcal{F}| \le \frac{2|E_1| + |E_2|}{t}.$$

This gives

$$\begin{aligned} f &= f_3 + (f - f_3) \\ &\le \frac{1}{3}|E_2| + \frac{2|E_1| + |E_2|}{t} \\ &= \frac{6m + (t-3)|E_2|}{3t}, \quad \text{since } |E_1| = m - |E_2| \\ &\le \frac{6m + (t-3)(m-l)}{3t}, \quad \text{since } |E_2| \le m - l \\ &= \left(\frac{t+3}{3t}\right) m - \left(\frac{t-3}{3t}\right) l. \end{aligned}$$

By Euler's formula,

$$n + \left(\frac{t+3}{3t}\right) m - \left(\frac{t-3}{3t}\right) l - m \ge 2.$$

This implies, by (1), that

$$l \ge \frac{t-6}{t} n + 6 \tag{3}$$

It follows from (1) and (3) that

$$m = 2n - l \le 2n - \left(\frac{t-6}{t}\right) n - 6 = \left(\frac{t+6}{t}\right) n - 6.$$

If we choose $t = 12$ we get $m \le \frac{3}{2}n - 6$, contradicting the fact that $m \ge \frac{3}{2}n$ in a 4-critical graph. Thus $t \le 11$. ∎

Hajós [7] constructed 4-critical graphs with n vertices and m edges where

$$m = \begin{cases} \frac{5n}{3} & \text{if } n \equiv 0(\bmod 3) \\ \frac{5n-2}{3} & \text{if } n \equiv 1(\bmod 3) \\ \frac{5n+2}{3} & \text{if } n \equiv 2(\bmod 3) \end{cases}$$

and it has been conjectured that there do not exist 4-critical graphs of order n with fewer edges. If this conjecture is true we could take $t = 9$ at the last step in the proof and it would follow that every 4-critical planar graph has a cycle of length 4 or 5 or a face of size 6,7 or 8. We remark that the inequality $m \geq \frac{3n}{2}$ for 4-critical graphs has been improved by Gallai [4] to $m \geq \frac{20n}{13}$, but this stronger inequality does not allow us to choose a smaller value for t.

References

1. V.A. Aksionov, *On the extension of the 3-coloring of planar graphs, (in Russian)*, Discret. Analiz. **26** (1974), 3–19.
2. V.A. Aksionov and L.S. Mel'nikov, *Essay on the theme: The three color problem*, Combinatorics; Colloquia Mathematica Societatis János Bolyai **18** (1976), 23–34.
3. V.A. Aksionov and L.S. Mel'nikov, *Some counterexamples associated with the three color problem*,, Journal of Combinatorial Theory, Series **B 28** (1980), 1–9.
4. T. Gallai, *Kritisch Graphen I*, Publ. Math. Inst. Hung. Acad. Sci. **8** (1963), 165–192.
5. H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther Univ. Halle-Wittenberg, Math. Natur. Reihe. **8** (1959), 109–119.
6. B. Grünbaum, *Grötzsch's theorem on 3-coloring*, Michigan Math. J. **7** (1963), 303–310.
7. G. Hajós, *Über ein Konstruktion nicht n-färbbarer Graphen*, Wiss. Z. Martin-Luther Univ. Halle-Wittenberg, Math. Natur. Reihe **10** (1961), 116–117.