A CONSTRUCTION OF CORDIAL GRAPHS FROM SMALLER CORDIAL GRAPHS

S.M. Lee
Department of Mathematics and Computing Science
San Jose State University
San Jose, CA 95192

A. Liu[1]
Department of Mathematics
University of Alberta
Edmonton, ALTA, T6G 2G1

A binary labelling of a connected graph assigns 0 or 1 to each vertex of the graph, 0 to an edge joining two vertices having the same label, and 1 to an edge joining two vertices having opposite labels. For such a labelling, let $v(0)$, $v(1)$, $e(0)$ and $e(1)$ denote, respectively, the numbers of vertices labelled 0, vertices labelled 1, edges labelled 0 and edges labelled 1.

Cahit [1] defines a graph to be cordial if it has a binary labelling such that $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$. We list below some of the results proved in that paper.

(1) In any binary labelling of a Eulerian graph, $e(1)$ is even.
(2) A Eulerian graph is not cordial if it has a number of edges congruent to 2 (mod 4).
(3) The complete graph K_n is cordial if and only if $n \leq 3$.
(4) All complete bipartite graphs are cordial.
(5) All trees (see Figure 1(a) for a special case) are cordial.
(6) The cycle C_n (see Figure 1(b)) is cordial if and only if $n \not\equiv 2 \pmod 4$.
(7) The matching M_n (see Figure 1(c)) is cordial if and only if $n \not\equiv 2 \pmod 4$.
(8) All fans (see Figure 1(d)) are cordial.
(9) The wheel W_n (see Figure 1(e)) is cordial if and only if $n \not\equiv 3 \pmod 4$.

In this paper, we prove additional results via the following construction.

Theorem 1. *Let H be a graph with an even number of edges and a cordial labelling such that the vertices of H can be divided into ℓ parts $H_1, H_2, \ldots, H_\ell$, each consisting of an equal number of vertices labelled 0 and vertices labelled 1. Let G be any graph and $G_1, G_2, \ldots, G_\ell$, be any ℓ subsets of the vertices of G. Let (G, H) be the graph which is the disjoint union of G and H augmented by edges joining every vertex in G_i to every vertex in H_i, $1 \leq i \leq \ell$. Then G is cordial if and only if (G, H) is.*

Proof: The given cordial labelling of H and any cordial labelling of G induce a binary labelling of (G, H). Since $v(0) = v(1)$ for H and $|v(0) - v(1)| \leq 1$

[1] Supported by NSERC grant A5137.

for G, we have $|v(0)-v(1)| \leq 1$ for (G,H). Similarly, $e(0) = e(1)$ for H and $|e(0)-e(1)| \leq 1$ for G. Consider the augmenting edges between G_i and H_i, $1 \leq i \leq \ell$. Each vertex in G_i is incident with an even number of such edges. Since $v(0) = v(1)$ for H_i, half of these edges are labelled 0 and half labelled 1, regardless of the label on the vertex in G_i. Hence, $|e(0) - e(1)| \leq 1$ for (G,H). The converse can be proved in the same way. ∎

Note that apart from being cordial, there are no other restrictions placed on G. The subsets $G_1, G_2, \ldots, G_\ell$ are completely arbitrary. They do not have to be distinct or disjoint, and their union does not have to include every vertex of G.

In our applications of Theorem 1, we use the following graphs in the role of H. T_{2k} is the trivially cordial graph consisting of $2k$ isolated vertices. We take $H_1 = T_{2k}$. T_{4k} is the trivially cordial graph consisting of $4k$ isolated vertices. Here, each of H_1 and H_2 consists of k vertices labelled 0 and k labelled 1. M_{4k} is cordial by (7). Here, H_1 consists of all vertices on one side of the matching arranged so that exactly half of them are labelled 0. H_2 consists of all vertices on the other side of the matching. Other choices of H will also be encountered.

Theorem 2. *All generalized fans $F_{m,n}$ (see Figure 1(g)) are cordial.*

Proof: Take $H = T_{2k} = H_1$. If $m = 2k$, take $G = P_n = G_1$. If $m = 2k+1$, take $G = F_n$ and $G_1 = P_n$. Then $(G,H) = F_{m,n}$. Since P_n is cordial by (5) and F_n is cordial by (8), $F_{m,n}$ is cordial by Theorem 1. ∎

Theorem 3. *A bundle B_n (see Figure 1(f)) is cordial if and only if $n \not\equiv 2 \pmod 4$.*

Proof: If $n \equiv 2 \pmod 4$, then B_n is Eulerian. Since its number of edges is $3n \equiv 2 \pmod 4$, B_n is not cordial according to (2).

Now take $H = M_{4k}$ with H_1 and H_2 defined as before. If $n = 4k+i$, $i = 1,3,4$, take $G = B_i$. G_1 consists of one of the vertices of B_i not of degree 2, and G_2 consists of the other vertex not of degree 2. Then $(G,H) = B_n$. It is easy to verify that B_1, B_3 and B_4 are cordial. By Theorem 1, so is B_n. ∎

Theorem 4. *A generalized bundle $B_{m,n}$ (see Figure 1(i)) is cordial if and only if $m \equiv 0 \pmod 2$ or $n \not\equiv 2 \pmod 4$.*

Proof: Suppose $n \equiv 2 \pmod 4$. Take $H = T_{4k}$ with H_1 and H_2 defined as before. If $m = 2k$, take $G = M_n$. G_1 consists of vertices on one side of the matching and G_2 consists of the vertices on the other side. If $m = 2k+1$, take $G = B_n$. Let G_1 and G_2 be as in the case $m = 2k$. Note that the two vertices of B_n not of degree 2 are not in G_1 or G_2. Then $(G,H) = B_{m,n}$. Since M_n is cordial by (7) and B_n is cordial by Theorem 3, $B_{m,n}$ is cordial by Theorem 1.

Suppose $n = 4k+2$ and $m \equiv 0 \pmod 2$. Take $H = B_{m-2,4k}$, using the cordial labelling obtained above. Take $G = B_{2,2}$, which is shown to be cordial in

Figure 2. The same figure defines G_i and H_i, $1 \le i \le 4$, in the construction of $(G, H) = B_{m,n}$. By Theorem 1, it is a cordial graph.

Finally, if $n \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 2$, then $B_{m,n}$ is Eulerian. Since its number of edges is $2mn + n \equiv 2 \pmod 4$, $B_{m,n}$ is not cordial according to (2). ∎

Theorem 5. *If $m \equiv 1 \pmod 2$, a generalized wheel $W_{m,n}$ (see Figure 1(h)) is cordial if and only if $n \not\equiv 3 \pmod 4$. If $m \equiv 0 \pmod 2$, $W_{m,n}$ is cordial if and only if $n \not\equiv 2 \pmod 4$.*

Proof: Suppose $m = 2k+1$ and $n \not\equiv 3 \pmod 4$. Take $H = T_{2k} = H_1$, $G = W_n$ and $G_1 = C_n$. Then $(G, H) = W_{m,n}$. Since W_n is cordial by (9), $W_{m,n}$ is cordial by Theorem 1.

Suppose $m = 2k$ and $n \not\equiv 2 \pmod 4$. Take $H = T_{2k} = H_1$ and $G = C_n = G_1$. Then $(G, H) = W_{m,n}$. Since C_n is cordial by (6), $W_{m,n}$ is cordial by Theorem 1.

Suppose $W_{m,n}$ is cordial for some $m \equiv 1 \pmod 2$ and $n \equiv 3 \pmod 4$. Consider the one for which m is minimum, and any cordial labelling of it. Note that $m \ge 3$ since W_n is not cordial according to (9). Suppose two vertices not on C_n have opposite labels. Then $W_{m,n} = (G, H)$ where $H = H_1$ consists of these two vertices, $G = W_{m-2,n}$ and $G_1 = C_n$. Since $W_{m,n}$ is assumed to be cordial, so is $W_{m-2,n}$ by Theorem 1. This contradicts the minimality assumption.

It follows that all vertices of $W_{m,n}$ not on C_n have the same label. Note that $m + n$ is even. Hence, there are m more vertices on C_n with label 1 than those with label 0, so that among the edges not on C_n, there are m^2 more with label 1 than those with label 0. It follows that there are exactly $\frac{1}{2}(n - m^2)$ edges with label 1 on C_n. However, $\frac{1}{2}(n - m^2) \equiv 1 \pmod 2$ and C_n is Eulerian. This contradicts (1). The case $m \equiv 0 \pmod 2$ and $n \equiv 2 \pmod 4$ can be dealt with in the same way. ∎

The following result includes both (3) and (4) as special cases.

Theorem 6. *A complete k-partite graph is cordial if and only if the number of parts with an odd number of vertices is at most 3.*

Proof: A complete k-partite graph with no odd parts is clearly cordial, as we can assign the label 0 to exactly half of the vertices in each part. If it has 1, 2 or 3 odd parts, let G be a complete graph consisting of a single vertex in each odd part and H be the graph induced by the remaining vertices. Each G_i consists of one vertex in G, and the corresponding H_i consists of all vertices of H not in the same part of the vertex in G_i. Then the complete k-partite graph is equal to (G,H). By (3), K_n is cordial if $n \le 3$. Hence, our graph is cordial by Theorem 1.

Suppose there is a cordial complete k-partite graph with at least 4 odd parts. Consider one with the smallest number of vertices, and any cordial labelling of it. According to (3), K_n is not cordial if $n \le 4$. Hence, the graph has at least one part with at least two vertices. Suppose two vertices in this part have opposite labels. Let $H = H_1$ consist of these two vertices, G be the graph induced by the remaining vertices and G_1 consist of all vertices not in the same part as H. Then (G, H) is the original graph, which is assumed to be cordial. By Theorem 1, G is also cordial. However, H is also a complete k-partite graph with at least 4 odd parts, contradicting the minimality assumption.

It follows that all vertices in each part of the original graph must have the same label. Hence, $e(1) = v(0)v(1)$ and $e(0) \leq \frac{1}{2}v(0)\,(v(0)-1) + \frac{1}{2}v(1)\,(v(1)-1)$. Since $(v(0)-v(1))^2 \leq 1$, we have $1 \geq e(1)-e(0) \geq \frac{1}{2}(v(0)+v(1) -(v(0)-v(1))^2) \geq \frac{1}{2}(v(0)+v(1)-1)$. It follows that $3 \geq v(0) = v(1)$. However, with at most 3 vertices, the graph cannot have at least 4 odd parts. This completes the proof of the Theorem. ∎

Cahit's consideration of cordial graphs is motivated by the study of graceful graphs. This is a subject with a vast literature, and we will not discuss it here. A very enjoyable and informative account is given in [2].

Cahit regards cordial graphs as a weaker version of graceful graphs, although there are cordial graphs which are not graceful. While it is not known whether all trees are graceful, (5) gives the affirmative answer that they are all cordial. Theorem 4 completely solves the problem of which generalized bundles are cordial. In [3], where these graphs are called mirror-sums, only a partial answer to the problem of which of them are graceful is obtained.

Acknowledgement.

The authors thank the referee for helpful suggestions which improve the presentation of the paper, in particular the formulation of Theorem 1. The first author thanks NSERC for support during a visit to Edmonton in perusing this research.

References

1. I. Cahit, *Cordial graphs: a weaker version of graceful and harmonious graphs*, Ars Combinatoria **23** (1987), 201–207.
2. Martin Gardner, *Golomb's graceful graphs*, in "Wheels, Life and Other Mathematical Amusements", W.H. Freeman, 1983, pp. 152–165.
3. S.M. Lee and A. Liu, *A construction of k-graceful graphs from complete bipartite graphs*, Bull. Math. SEA. **12** (1988), 23–30.