On Balanced Graphs

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Abstract. A graph G is defined to be balanced if its average degree is at least as large as the average degree of any of its subgraphs. We obtain a characterization of all balanced graphs with minimum degree one. We prove that maximal Q graphs are strictly balanced for several hereditary properties Q. We also prove that a graph G is balanced if and only if its subdivision graph $S(G)$ is balanced.

Introduction

The graphs in this paper are finite, undirected and without loops and multiple lines. Throughout the paper $p = p(G)$ stands for the number of vertices and $q = q(G)$ stands for the number of lines of a (p, q) graph G. The values p and q are called the order and size of G, respectively. Terms not defined here are used in the sense of Harary [2].

The notion of a balanced graph originated in the work of Erdös and Renyi [1] on Random Graphs.

For a (p, q) graph G, we define the average degree $d(G)$ and the maximum average degree $m(G)$ of G as follows:

$$d(G) = 2q/p; \quad m(G) = \max_{H \subseteq G} d(H).$$

We observe that if G is a connected graph then $d(G) < 2$ if and only if G is a tree and $d(G) = 2$ if and only if G is unicyclic.

A graph G is said to be balanced if $d(H) \leq d(G)$ for every subgraph H of G and is strictly balanced if $d(H) < d(G)$ for every proper subgraph H of G. Clearly G is balanced if and only if $m(G) = d(G)$ and is strictly balanced if and only if $H \subseteq G$ and $d(H) = m(G)$ imply that $H = G$.

One can easily verify that trees, cycles, complete graphs and complete bipartite graphs are strictly balanced; whereas, a unicyclic graph which is not a cycle is balanced but not strictly balanced.

Since the number of lines of a subgraph of a graph G attains its maximum if and only if it is an induced subgraph of G, a graph G is balanced if and only if $d(H) \leq d(G)$ for every induced subgraph H of G.

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ARS COMBINATORIA 32(1991), pp. 221-223

2. Balanced Graphs

Theorem 1. *A graph G is balanced if and only if for every component H of $G, d(H) = d(G)$ and H is balanced.*

Proof: The Theorem is obvious if G is connected. We prove the Theorem for G with exactly two components. The result then follows by induction.

Let G be a (p, q) graph with exactly two components G_1 and G_2 of order and size (p_1, q_1) and (p_2, q_2) respectively so that $p = p_1 + p_2$ and $q = q_1 + q_2$. Suppose G is balanced. If $d(G_1) < d(G_2)$ then $d(G) < d(G_2)$ which is a contradiction. Hence $d(G_1) = d(G_2) = d(G)$ and both G_1 and G_2 are balanced.

Conversely suppose G_1 and G_2 are balanced and $d(G_1) = d(G_2) = d(G)$. Then for any subgraph H of $G, d(H \cap G_1) \leq d(G_1) = d(G)$ and $d(H \cap G_2) \leq d(G_2) = d(G)$ so that $d(H) \leq d(G)$ and hence G is balanced. ∎

We define a deficit function f for a graph G as follows: For any (p_o, q_o) subgraph H of $G, f(H) = d(G)p_o - 2q_o$.

Clearly $f(G) = 0$ and G is balanced if and only if $f(H) \geq 0$ for every subgraph H of G. For any two subgraphs H_1 and H_2 of $G, f(H_1 \cup H_2) = f(H_1) + f(H_2) - f(H_1 \cap H_2)$. Thus if H_1 and H_2 have no vertex in common and $f(H_1 \cup H_2) < 0$, then either $f(H_1) < 0$ or $f(H_2) < 0$. Hence we have

Theorem 2. *A graph G is balanced if and only if $d(H) \leq d(G)$ for every connected induced subgraph H of G.*

Theorem 3. *Let G be a connected graph such that $\delta(G) = 1$. Then G is balanced if and only if G is either a tree or a unicyclic graph.*

Proof: Suppose G is a connected balanced graph with $\delta = 1$. If G is neither a tree nor a unicyclic graph, then $p(G) < q(G)$. Now if u is a vertex of degree one in G, then $d(G - u) > d(G)$ which is a contradiction. The converse is obvious. ∎

The following Theorem generalizes Theorem 3.

Theorem 4. *Let G be a graph with k components and let $\delta(G) = 1$. Then G is balanced if and only if each component of G is either a tree of order p/k or a unicyclic graph.*

Proof: Suppose G is balanced. Let H be a component of G containing a vertex of degree one. By Theorem 3, H is a tree or a unicylic graph. If H is a tree, then by Theorem 1, each component of G must be a tree of order p/k and if H is unicyclic, each component of G must be unicyclic. The converse immediately follows from Theorem 1. ∎

Remark

Let G be a connected balanced graph with $\delta > 1$ and let u be a vertex of degree δ in G. Then $d(G - u) \leq d(G)$, which implies that $q \leq \delta p$.

For a given property Q, call a graph G a maximal Q graph if no line can be added without loosing the property Q.

Theorem 5. *Let Q be a hereditary property of graphs. Let f be a positive strictly monotonic increasing linear function defined on $[x, \infty]$ such that every maximal Q graph G with p vertices has $f(p)$ lines. Then every maximal Q graph is strictly balanced.*

Proof: Let G be a maximal Q graph with p vertices and $f(p)$ lines.

Let H be a proper induced subgraph of G with p_o vertices and q_o lines. Since H has $Q, q_o \leq f(p_o)$. Now $f(p)/p$ represents the slope of the line joining the point $(0,0)$ to the point $(p, f(p))$ in the Euclidean plane. Since f is a positive strictly increasing linear function, $p_o < p$ implies $f(p_o)/p_o < f(p)/p$ and hence $d(H) < d(G)$. ∎

Corollary.

i) Every maximal planar graph is balanced.

ii) Every maximal outerplanar graph is balanced.

We denote by $S(G)$, the graph obtained from G by subdividing each line of G exactly once. $S(G)$ is called the subdivision graph of G.

Theorem 6. *Let G be a connected graph. Then $S(G)$ is balanced if and only if G is balanced.*

Proof: Suppose $S(G)$ is balanced. Then for any connected induced subgraph H of $G, d(S(H)) \leq d(S(G))$ which implies $d(H) \leq d(G)$. Conversely suppose G is balanced. Let H_1 be a connected induced subgraph of $S(G)$. We claim that $d(H_1) \leq d(S(G))$. This is trivial if H_1 is a tree. Hence we assume that H_1 is not a tree.

If there exists a subgraph H of G such that $S(H) = H_1$, then $d(H) \leq d(G)$ from which it follows that $d(S(H)) = d(H_1) \leq d(S(G))$. Otherwise there exists a maximal subgraph H_2 of H_1 such that $S(H) = H_2$ for some subgraph H of G. Since H_1 is not a tree, $d(H_1) \leq d(H_2)$. Also $d(H_2) = d(S(H)) \leq d(S(G))$ and hence $d(H_1) \leq d(S(G))$. ∎

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