Nuclear Designs

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Abstract. A Nuclear Design $ND(v; k, \lambda)$ is a collection B of k-subsets of a v-set V, where $B = P \cap C$, where (V, P) is a maximum packing $(PD(v; k, \lambda))$ and (V, C) is a minimum covering $(CD(v; k, \lambda))$ with $|B|$ as large as possible. We construct $ND(v; 3, 1)$'s for all v and λ. Along the way we prove for every leave (excess) possible for $k = 3$, all v, λ, there is a maximum packing (minimum covering) achieving this leave (excess).

Introduction

If one wishes to construct tables of packings and coverings for v and λ, space can be saved by constructing the nuclear design, the packing supplement $P - N$ and the covering supplement $C - N$.

Formally let us define:

(a) The pair (V, P) is a v, k, λ *packing* design (a $PD(v; k, \lambda)$) iff $|V| = v$, P is a collection of k-subsets of V, so that every 2 subset is a subset of at most λ elements of P. $|P|$ as large as possible.

(b) The pair (V, C) is a v, k, λ covering design (a $CD(v; k, \lambda)$) iff $|V| = v$, C is a collection of k-subsets of V, so that every 2 subset is a subset of at least λ elements of C. $|C|$ as small as possible.

(c) The pair (V, N) is a v, k, λ nuclear design (an $ND(v; k, \lambda)$) iff $N = P \cap C$ and $|N|$ is as large as possible among all intersections of any maximum P and minimum C.

Indeed this opens up the entire question of the nuclear spectrum.

The *nuclear spectrum* spec $ND(v; k, \lambda)$, is the set of all integers n for which there exists a $PD(v; k, \lambda)$, (V, P) and a $CD(v; k, \lambda)$, (V, C) and $n = |P \cap C|$. This has been solved for $v \equiv 1, 3(6)$ $\lambda = 1$, $k = 3$ by Lindner and Rosa [**LR**], also the spectrum of intersections of two packings $v \equiv 0, 2 \bmod 6$ and $\lambda = 1$, has been solved by Hoffman and Lindner [**HL**].

Packings for $k = 3, 4$ and all λ, were dealt with in papers by Hanani, Brouwer, Assaf [**H1**][**B**][A2]. Coverings for $k = 3$ and all λ were dealt with also in [**FM**], [**H1**], for $k = 4$ and $\lambda = 1$ dealt with by Mills [**M1**][**M2**], and for $k = 4$ and all $\lambda > 1$ by Assaf [A1].

Two auxiliary concepts will be needed, that of the *leave* and the *excess* [MR][C].

The leave of a packing (V, P) is a graph (V, E) where $(xy) \in E$ with multiplicity m if $\{x, y\}$ is a subset of a $\lambda - m$ blocks of P. The excess of a covering (V, C) is a graph (V, E) where $(xy) \in E$ with multiplicity m if $\{x, y\}$ is a subset of $m - \lambda$ blocks of C.

The following necessary conditions hold for a leave of (V, P)

(a) $(v(v-1)/2) - |E| \equiv 0 \mod (k(k-1)/2)$
(b) for all $x \in V \deg(x) \equiv \lambda(v-1) \mod (k-1)$
(c) $|E|$ is minimal w.r.t. (a) and (b)
(d) for all $x \in V$, $\deg(x) \leq \lambda(v-1)$.

Similarly for an excess we have

(a) $v(v-1)\lambda/2 + |E| \equiv 0 \mod (k(k-1)/2)$
(b) for all $x \in V$, $\deg(x) \equiv \lambda(1-v) \mod (k-1)$
(c) $|E|$ is minimal w.r.t. (a) and (b)

We note that excesses and leaves are not necessarily unique as there may be many non isomorphic graphs on E edges with the degree of each vertex in the same congruence class mod $(k-1)$. For example the following are graphs with 4 edges and each degree even. (a) AB, BC, CD, DA, (b) AB, AB, BC, BC, (c) AB, CD, AB, CD and (d) AB, AB, AB, AB.

The following conjecture which is a natural generalization of the Wilson's [W] theorem would be of great help.

Conjecture. *There is a n_0 such that for all $n > n_0$, if (V, E) is a graph with $|V| = n$, and (V, E) satisfies the necessary conditions above to be a leave (excess), then there is a packing (covering) with a leave (excess) of (V, E).*

It is believed that this conjecture is not harder than Wilson's Theorem. Theorem 1 provides a proof for $k = 3, n_0 = 6$.

The construction techniques

In this section we briefly outline the construction techniques which can and do arise in the constructions of nuclear designs based on the leaves and excesses. The selections of which a particular leave to choose can be critical. Sometimes the packing in the literature which is given has a leave which is easily constructed but bad for the construction of nuclear designs e.g. $v \equiv 0 \mod 12, \lambda = 1, k = 4$, the leave given in [B] is a 2-factor of disjoint triangles where as what we would need is a 2-factor of disjoint quadrilaterals.

The easiest case is when a design exist then the maximum packing is the minimum covering and that is the nuclear design. The next case in order of difficulty is when we can arrange for $P \subset C$. This happens, for example when $v \equiv 5 \mod 6$, $k = 3, \lambda = 1$, but not every maximum packing has this property e.g. $v \equiv 0 \mod 12, k = 4, \lambda = 1$. The next case is for all $v \equiv 1 \mod (k-1)$ but $v(v-1)/2 \not\equiv 0$

mod $(k(k-1)/2)$. In this case the leave and excess are fixed graphs on E with $|E| < k(k-1)$ edges. The best approach is to find the smallest subset V' containing all vertices of non-zero degree of the leave, and forming a packing design on the minimal $V'' \subset V'$. Following this one finds a $ND(v''; k, \lambda)$. One can then use the theory of designs with holes **[RS]** to find $ND(v; k, \lambda)$.

There is a hidden difficulty in this and other reduction to small cases, i.e. does the purported nuclear design have $|V|$ as large as possible[1]?

The next most difficult cases are handled by using GDD's **[H1]**, the basic GDD constructions falls into two classes. The first is when the leave consists of many vertex disjoint isomorphic copies of the same graph. In this case we try to group some of the copies into subsets which can be the leave of specially constructed subdesigns and use GDD's to construct the larger design. The second is where except for a small (independent of V) fixed subgraph the rest is many isomorphic copies of the same fixed graph.

$(v; 3, \lambda)$-Nuclear Designs

The fist thing we do in this case is to decide what leaves and excesses can be used.

Theorem 1. *For $k = 3$ any $v, \lambda, v \geq 6$, the only graphs which can be leaves or excesses are as in Table 1 with the following abbreviations and all are realizable.*

Graphs of odd degrees

$1F$	A one factor on $6t$ vertices
$1FY$	A one factor on $6t-4$ vertices and a tree on 4 vertices with one vertex of degree 3

06	(a)	$1FH$	a one factor on $6t-6$ vertices and a graph AB, BC, BD, DF, DG
	(b)	$1F5$	a one factor on $6t-6$ vertices and a tree on 6 vertices with one vertex of degree 5
	(c)	$1FYY$	a one factor on $6t-8$ vertices and two trees each on 4 vertices with one vertex of degree 3
	(d)	$1F3$	a one factor on $6t-2$ vertices and a triple edge AB, AB, AB
	(e)	$1F-0$	a one factor on $6t-4$ vertices and a graph AB, BC, BC, CD

[1] The best results we have obtained for small values by taking a packing, judiciously throw away as few blocks as necessary and let the computer hill climb [S3] to a covering.

2			A double edge AB, AB ($\lambda \geq 2$).
$E4$	(a)	Q	A quadrilateral AB, BC, CD, DA
	(b)	4	A quadruple edge AB, AB, AB, AB
	(c)	2^2	2 double edges AB, AB, CD, CD
	(d)	∞	AB, AB, BC, BC

λ mod 6 \ v mod 6	0	1	2	3	4	5
0	0;0	0;0	0;0	0;0	0;0	0;0
1	$1F;1F$	0;0	$1F;1FY$	0;0	$1FY;1FY$	$E4;2$
2	0;0	0;0	$2;E4$	0;0	0;0	$2;E4$
3	$1F;1F$	0;0	06;06	0;0	$1FY;1FY$	0;0
4	0;0	0;0	$E4;2$	0;0	0;0	$E4;2$
5	$1F;1F$	0;0	$1FY;1FY$	0;0	$1FY;1F$	$2;E4$

Table (1)

Proof: The paper [FH], [H1] provide all the entries in the table below—when a leave (excess) is unique or unique for $\lambda = 1$ is *must* be achieved. Some of the entries can be obtained by simply adding the blocks of two packings (coverings) on the same set together. We can of course take the packings or coverings independently in order to achieve the desired leave (excess). We shall use the notation $(\lambda \equiv 1)+(\lambda \equiv 4)$ for example, to mean take the blocks of a design with $\lambda \equiv 1(6)$ and add them to the blocks of a design with $\lambda \equiv 4(6)$.

λ mod 6 \ v mod 6	0	1	2	3	4	5
0	0;0	0;0	0;0	0;0	0;0	0;0
1	$1F;1F$	0;0	$1F;1FY$	0;0	$1F;1FY$	$Q;2$
2	0;0	0;0	2;	0;0	0;0	2;
3	$1F;1F$	0;0		0;0	$1FY;1FY$	0;0
4	0;0	0;0	;2	0;0	0;0	;2
5	$1F;1F$	0;0		0;0	$1FY;1F$	2;

It is easy to see that the necessary conditions for the fifth column are

$$(0;0 \quad E4;2 \quad 2;E4 \quad 0;0 \quad E4;2 \quad 2;E4)^T,$$

and the necessary conditions on 2nd column are

$$(0;0 \quad 1F;1FY \quad 2;E4 \quad 06;06 \quad E4;2 \quad 1FY;1FY)^T.$$

The only non trivial calculation is $v \equiv 2(6)$, $\lambda \equiv 3(6)$ the necessary conditions force the leave (excess) to be a graph with $3t + 3$ edges and all degrees odd. It is easily seen that no degree can exceed 6 and the 5 graphs of 06 are the only possible ones.

Sufficiency of the 5th column

Packing $\lambda \equiv 4(6)$ For Q take $(\lambda \equiv 1) + (\lambda \equiv 3)$; For $4, 2^2, \infty$ take two copies of $(\lambda \equiv 2)$.

$\lambda \equiv 1(6), \lambda > 1$, The rest of $E4$ is obtained by $(\lambda \equiv 3) + (\lambda \equiv 4)$.

Covering $\lambda \equiv 2(6)$, Q and ∞ can be obtained by adding 2 blocks to a packing. For $4, 2^2$ we take $2 \times (\lambda \equiv 4) + (\lambda \equiv 3)$ for $\lambda > 2$. For $\lambda = 2$ we take two copies of a cover for $\lambda = 1$.

$\lambda \equiv 5(6)$, we take $(\lambda \equiv 3) + (\lambda \equiv 2)$.

Sufficiency of the 2nd column

Packing: $\lambda \equiv 3(6)$. The graphs $1F - 0-$, and $1F3$ can be obtained by $(\lambda \equiv 1) + (\lambda \equiv 2)$. To obtain the remaining three graphs we note that 3 one factors on $v-8$ points can be united to form a configuration which consists of disjoint copies G where G is a hexagon $abcdef$ and edges ae, bd, fc. Removing two triangles from G leaves the 1 factor fc, ab, ed. Let us take 3 copies of a packing with $\lambda = 1$ and ensure that these $v - 8$ have a leave in the form of this configuration. We add also to the packing the triangles afe, bdc for each copy. Let two of the leaves have union which consists of the 4 cycles $ABCD$, and $A'B'C'D'$ on the remaining points.

To obtain $1FYY$, let the third factor be $AD, BC, A'B', D'C'$ and add $B'C'D'$, BCD to the packing.

To obtain $1FH$, let the third factor be $AA', DD', CB, C'B'$ and add $A'B'C'$, ABC to the packing.

To obtain $1F5$, we start with $1FH$ (obtained from previous step) leave but ensure that $D'CB$ is a block. We then remove $D'CB$ and add DCB. (See Diagram 1).

For $\lambda \equiv 4(6)$

(a) $\infty, 2^2, 4$ can all be obtained by doubling $(\lambda \equiv 2)$.

(b) Q can be obtained by $(\lambda \equiv 1) + (\lambda \equiv 3)$.

For $\lambda \equiv 5(6)$, take $(\lambda \equiv 2)$ and $1F - 0-$, to get a graph $AB\ BC\ BC\ CD\ BD\ BD$ + a one $1F$ on $6t - 2$ points, add BCD to the packing to get a $1FY$.

Covering: $\lambda \equiv 2(6)$, Q, ∞ can be obtained by adding two blocks to a packing. For $\lambda > 2$; $4, 2^2$ can be obtained by doubling a $(\lambda \equiv 4)$. This leaves the case $\lambda = 2$, and excesses $2^2, 4$.

Excess 2^2

In this case we use a $(\lambda \equiv 1)$ nuclear design and a $(\lambda \equiv 1)$ packing. It will be seen that a $(\lambda \equiv 1)$ nuclear design has a leave which consists for

$v = 6t + 2$ of t copies of the graph on $abcdef$ whose edges are ab, bc, ac, ad, be. Let us look at $t - 1$ of these. Take the 1 factor from the packing to have edges ad, be, cf. Thus adding the triangles abd, bec, acf gives an empty excess on $v - 8$ points. For the remaining copy of the hexogan and the triangle abc to the cover. We make sure that on the 8 remaining points the union of the leaves is two four cycles $ABCD$, $EFGH$ we add ABC, ADC, EFG, FHG to get an excess of 2^2.

Excess 4

We handle this case as follows, first we will exhibit a packing for $v = 8$. We then note that by Stern and Lenz [SL] K_{6t}, $K > 1$ can be decomposed into $t - 1$ orbits of triangles and 5 one factors. This is also an easy exercise using Rosa's Skolem sequence technique. Thus $2K_{6t}$ can be decomposed into $2t - 3$ orbits of triples and 16 one factors. We can thus find a covering on $6t + 8$ for $t > 1$ whose excess is 4. $v = 14$, we do separately.

$v = 8$

$V = \{0,1,2,3,4,5\} \cup \{A,B\}$

$B =$ (a) $iAB, i = 0 \ldots 5$

(b) $A03, B03, A14, B14, A25, B25$

(c) $0 + i, 1 + i, 2 + i, i \in Z_5$

(d) $024, 135$

$v = 14$

$V = \{0,1,2,3,4,5\} \cup \{0',1',2',3',4',5'\} \cup \{A,B\}$

a) Let F_1 through F_{10} be a one factorization of $2K_6$ on $\{0',1',2',3',4',5'\}$. We form triples by attacking these one factors to the points 0, 1, 2, 3, 4, 5, A, B, A, B, respectively.

Add the triples

b) $ABi, i = 0 \ldots 5$.

c) $[i, i+1, i+2] i = 0 \ldots 5$, and $[0,2,4]$, $[1,3,5]$.

d) $[A, i, i+3]$, $[B, i, i+3]$, $i = 0,1,2$.

$\lambda \equiv 3(6)$, $1FY$ from $\lambda \equiv 1$, and Q from $\lambda \equiv 2$ with the removal of two triangles gives a $1FY$, $1FH$, $1FF$. [We must be careful in coverings to make sure that if we wish to remove a triangle from an excess that the cover actually contains that triangle. It is easy to ensure here]. A $1FY$ and ∞ will give the $1F$ removing two triangles will give the $1F3$ and $1F - 0 -$.

$\lambda \equiv 5(6)$. Take $(\lambda \equiv 3)$ cover with excess $1FH$ and $(\lambda \equiv 2)$ cover with excess ∞, and remove two triangles, to obtain a $1FY$.

This completes the proof.

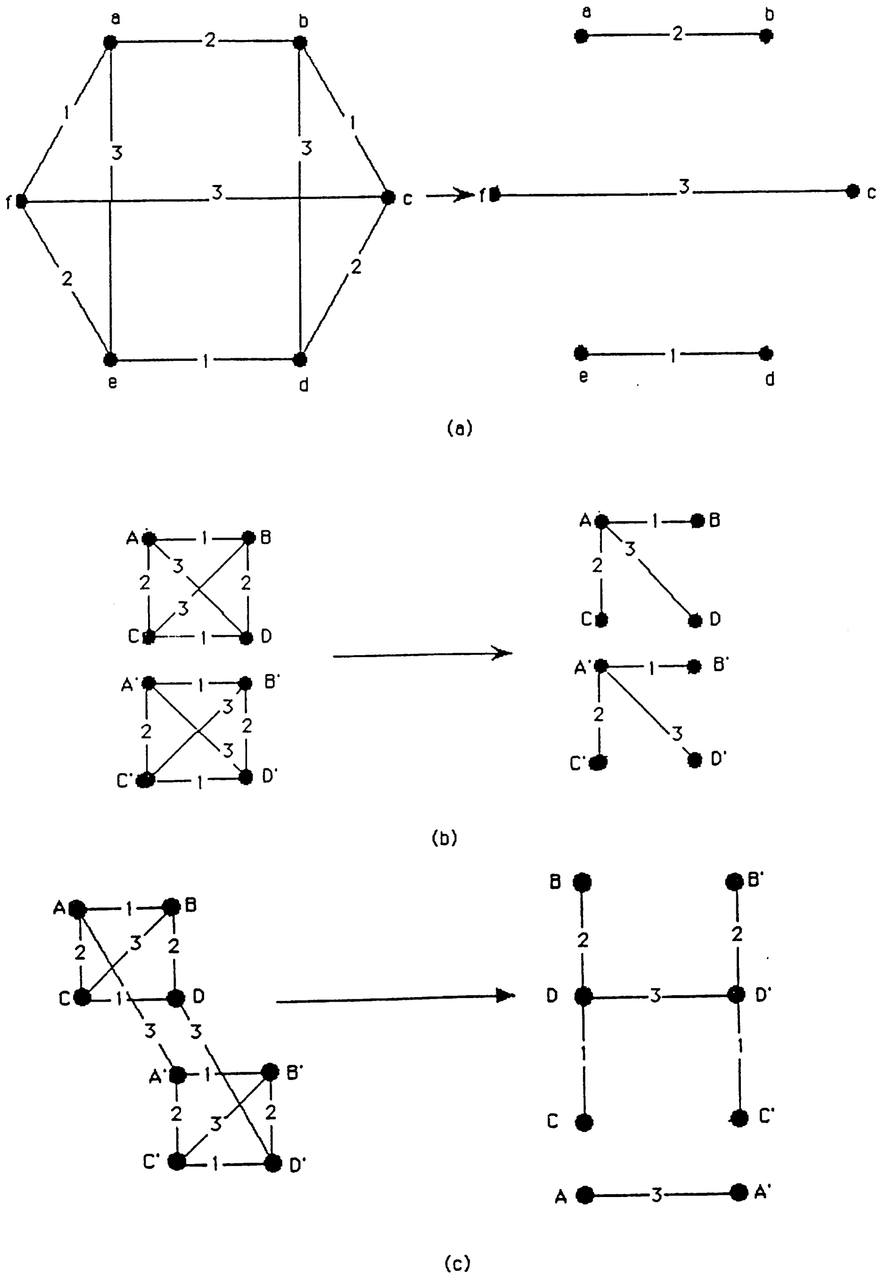
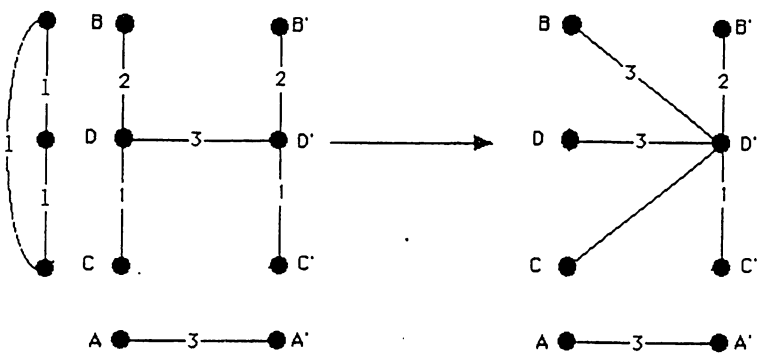


Diagram 1:
The number on the edge indicates which $\lambda = 1$ packing it comes from.



(d)

Diagram 1 (continued):
The number on the edge indicates which $\lambda = 1$ packing it comes from.

We note that the entries in the table where the graph is non-spanning i.e. $v \equiv 2 \mod 6$, $\lambda = 2,4$ and $v \equiv 5 \mod 6$, $\lambda = 1,2,4,5$ are all nuclear packings provided the leave is $Q, 2$ or ∞.

For the next part we will need as a basic construction tool a GDD with t-groups of size 6 and blocks of size 3, these exist for all λ and $t \geq 3$ **[H1]**. We will also need a GDD with all groups of size 12 and exactly one group of size 18, this can be obtained by multiplication of a GDD $[3; 1,4]$ **[H1]** and GDD $[3; 1,4,3^*]$ **[HR]** by 4.

Definition: The defect graph for a set of blocks of size 3 on v points with λ replications has $\{x,y\}$ an edge with label $-k$ if $\{x,y\}$ is in $\lambda - k$ blocks, and with label $+k$ if $\{x,y\}$ is in $\lambda + k$ blocks. [i.e. leaves have negative labels and excesses positive labels.]

Lemma 1. *Let P be the number of blocks in a $PD(v;3,\lambda)$ and N be the number of blocks in an $ND(v;3,\lambda)$ then $P - N$*

(a) $= 0$ *v odd*
(b) $\geq \lfloor \frac{v}{6} \rfloor$ *v even, except $v \equiv 2(6), \lambda \equiv 3(6)$*
(c) $\geq \lfloor \frac{v}{6} \rfloor$ $v \equiv 2(6), \lambda \equiv 3(6)$.

Proof: We think of proceeding from a packing to a covering by adding as few blocks as possible to the packing, and removing blocks from the system if there is a triangle in the defect all of whose labels are 1.

The following table summarizes the results. The total defect is the sum of the labels on all edges.

	Total Defect from packing	Minimum number of triangles to remove the negative edges	A: Total excess from adding these triangles	B: Needed excess for covers	Bound on $P-N=\lceil\frac{A-B}{3}\rceil$
$6t$	$-3t$	$3t$	$6t$	$3t$	t
$6t+2, \lambda\not\equiv 3(6)$	$-(3t+1)$	$3t+1$	$6t+2$	$3t+2$	t
$6t+2, \lambda\equiv 3$ leave $1F3$	$-(3t+4)$	$3(t-1)+7$	$6t+8$	$3t+4$	$t+1$
other leaves	$-(3t+4)$	$3(t-1)+5$	$6t+2$	$3t+4$	$t-1$
$6t+4$	$-(3t+3)$	$3t+2$	$6t+3$	$3t+3$	t

We have that the nuclear design is the packing design except when the leave is $1F$, $1FY$ or 06.

Definition: A t-nucleus will be a packing on $6t$ vertices whose leave is t disjoint copies of the graph whose vertices are a, b, c, $\ell(a)$, $\ell(b)$, $\ell(c)$ and whose edges are ab ac bc $a\ell(a)$ $b\ell(b)$ $c\ell(c)$.

Lemma 2. *Let (V,B) be a packing whose leave is $1F$, $1FY$, $1FH$, $1FYY$, $1F5$. Then if (V,B) is a packing containing blocks which when added to the leave form a t-nucleus, on $6t+u$ points $u=0,2,4$ ($u=8$ if $v\equiv 2(6)$, $\lambda\equiv 3(6)$), then (V,B)-\{ triangles of the t-nucleus is a nuclear design\}.*

Proof: Let (V,B') be the packing - \{the triangles of the t-nucleus\}. By Lemma 1 this will be a nuclear design, it can be completed to a covering. We add blocks for $(a,b,\ell(a))$ $(b,c,\ell(b)(c,d,\ell(a))$ to the t-nucleus to get a cover whose excess is $(a,\ell(c))$ $(b,\ell(a))$ $(c,\ell(b))$. (See Diagram 2).

For $v\equiv 2(6)$ $\lambda\not\equiv 3(6)$, Let AB be the points not in the nucleus and a any point in the nucleus add the block (A,B,a) to complete the $1FY$ excess.

For $v\equiv 4(6)$, Let $ABCD$ be the points not in the nucleus and leave AB, CD, BD add blocks $\{ABC\}$, $\{ABD\}$ to get excess BA, CA, AD.

For $v\equiv 2(6)$, $\lambda\equiv 3(6)$, Let $ABCD$, $EFGH$ be the points not in the nucleus. The following table gives the desired excess, always a $1FYY$.

Leave Type	Leave	Blocks Added	Excess
$1FYY$	AB CB DB EF GF FH	ABC ACD EFG EFH	AB AC AD EF EG EH
$1FH$	AB BC BE DE EF GH	ABC DEF CBE DGH	AC BC CE DH DG DF
$1F5$	AB AC AD AF AF GH	ABC EAD CGH AFE	EF EA ED CH CB CG

Lemma 3. *There is a packing for $v=6t+2$, $\lambda\equiv 3(6)$ which has a leave of $1FYY$, $1FH$, $1F5$, and has a $(t-1)$-nucleus. Thus the packing minus the nucleus is a nuclear design.*

Proof: We note in the construction from the ($\lambda\equiv 1$) packing that one of the triangles added to the packing from each six point graphs is precisely the one we want to remove for a nucleus. (See Diagram 1).

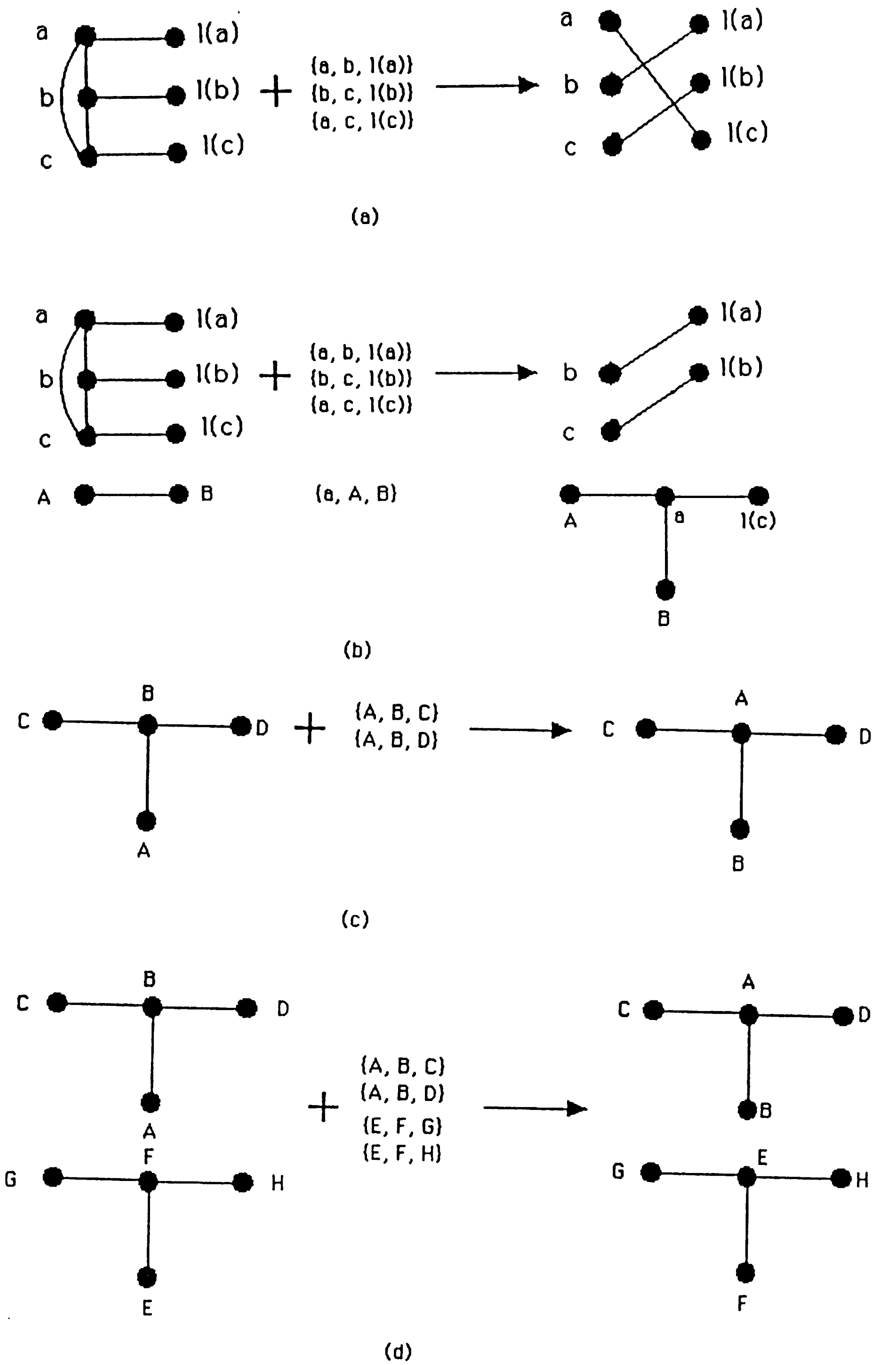
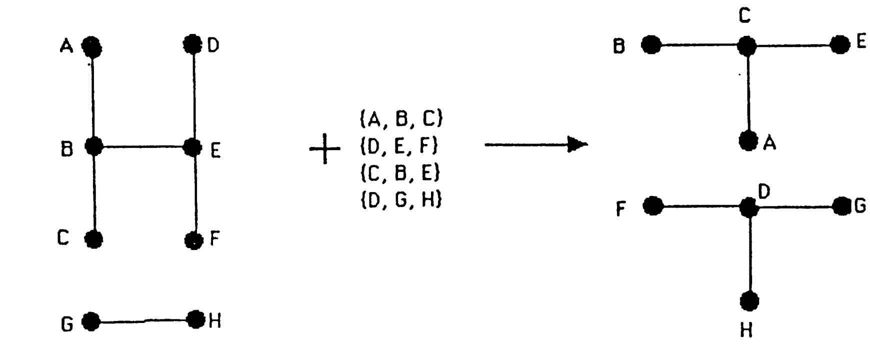
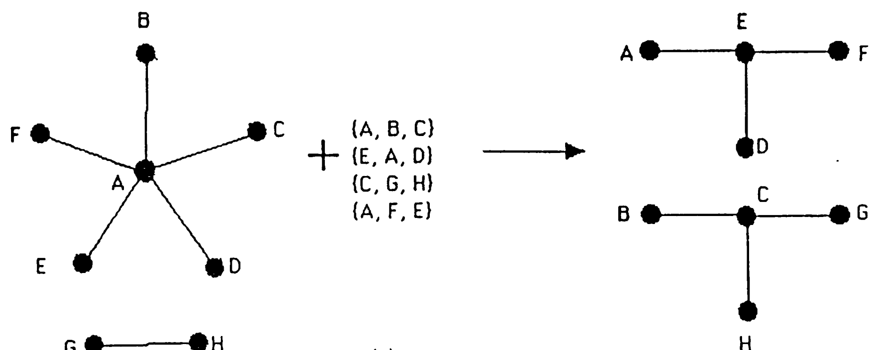

a
b
c
l(a)
l(b)
l(c)
{a, b, l(a)}
{b, c, l(b)}
{a, c, l(c)}
(a)
A
B
{a, A, B}
(b)
C
B
D
A
{A, B, C}
{A, B, D}
(c)
{E, F, G}
{E, F, H}
G
F
H
E
(d)

Diagram 2



(e)



(f)

Diagram 2 (continued)

Lemma 4. *The following designs exist.*

(a) $ND(6;3,1)$; $ND(8;3,1)$
(b) $ND(16;3,1)$; $ND(22;3,1)$ *with a subsystem of order* 4 *whose leave consists of a* 3*-star which comes from the subsystem and a nucleus on* 12*; resp.* 18 *points.*
(c) $ND(10;3,3)$, *with a subsystem of order* 4 *whose leave consists of a* 3 *star which comes from the subsystem and a nucleus on* 6 *points. (Such an* $ND(10;3,1)$ *does not exist).*

Proof:

(a) Throw away any block from a packing.
(b) $v \equiv 16, 18$. We start with a subsystem of order 4, on $\underline{1}, \underline{2}, \underline{3}, \underline{4}$.

We need a decomposition of K_{12} and K_{18} into some triples and 5 one factors, F_0, F_1, F_2, F_3, F_4, so that F_0 is part of a nucleus. We now add blocks of the form $\{\underline{i}xy, xy \in F_i, 1 \le i \le 4\}$. (Ideally we would like to decompose K_6 but it is too small to contain a nucleus).

K_{12}: Blocks $[0+i\ 1+i\ 5+i], i = 0, \ldots 11 \bmod 12$. The five factors are $F_0 = [i, i+6]\ i = 0 \ldots 5$, F_1 through F_4 from the decomposition of the distances 2 and 3 mod 12 into 1 factors [SL]. The nucleus is $F_0 \cup \{[0\ 1\ 5], [2\ 3\ 4]\}$.

K_{18}: Blocks $[O+i\ 1+i\ 5+i], [0+i\ 6+i\ 8+i] \bmod 18, i = 0, \ldots, 17$. The five factors are $F_0 = [i, i+9]\ i = 0 \ldots 9$, F_1 through F_4 from the decomposition of the distances 3 and 7 mod 12 into 1 factors.

The nucleus is $F_0 \cup \{[0\ 1\ 5], [2\ 3\ 7], [4\ 5\ 9]\}$.

(c) $v = 10$ As in (b) but we need a decomposition of $3K_6$ into triples and 13 one-factors, so that one of the one factors is part of a nucleus.

$3K_6$: Blocks $[0\ 2\ 4], [1\ 3\ 5]$. $F_0 = [i, i+3] \bmod 6, F_1, F_2$ from decomposing distance 1 mod 6. F_2 through F_{12} from decomposing $2K_6$ into one factors.

The nucleus is $F_0 \cup [0\ 2\ 4]$.

Theorem 1. *For all $v, \lambda, v \geq 6$, an $ND(v; 3, \lambda)$ exist having the maximal number of blocks that arithmetic and degree conditions allow. i.e.*

(a) *For v odd the packing is a nuclear design.*
(b) *For v even (except $v \equiv 2, \lambda \equiv 3$), the nuclear design has $P - \lfloor \frac{v}{6} \rfloor$ blocks.*
(c) *For $\lambda \equiv 3(6), v \equiv 2(6)$ the nuclear design has $P - \lfloor \frac{v}{6} \rfloor + 1$ blocks.*

Proof: By construction we need only consider $\lambda = 1$, v even.

If $v \equiv 0(6), v \geq 18$, There is a $GDD[3; 1, 6, 6t]$ [H1]. Build a $ND(6; 3, 1)$ on each group.

If $v \equiv 2(6)$, $v \geq 20$ There is a $GDD[3; 1, 6, 6t]$ [H1]. For each group g build a ND on the set $g \cup \{AB\}$.

If $v \equiv 4(6)$ then

(i) $v = 12t + 4, v \geq 36$, Use a $GDD[3; 1, 12, 12t]$ [H1] using the ND on 16 points with a subsystem of order 4.
(ii) $v = 12t + 10, v \geq 54$, use a $GDD[3; 1, 12t + 6, 18^*]$ [HR] put a ND on 16 points with a subsystem of size 4 on $\{A, B, C, D\} \cup g$ for groups of size 12 and a ND on 22 points with a subsystem of size $\{A, B, C, D\} \cup g$ on the unique group of size 18.

The small designs for $v < 12$ are trivial, the missing cases are thus $v = 12, 14, 28, 34, 46$.

$v = 12$, Take the $STS(014), (068) \bmod 13$, remove 0 and the blocks 197, 458.

The remaining 4 cases will be done by taking $V = \{A, B, C, D\} \times Z_{6t}$, and a subsystem of size 4 on $\{A, B, C, D\}$ and decomposing Z_{6t} into two blocks, a nucleus and 4 1-factors, the one factors are always obtained by decomposing two distances.

$v = 14$

BLOCKS $[0\ 1\ 5] \bmod 12 - A; A = (015)(348)$

ONE FACTORS: Decompose distances 2, 3 [SL]

NUCLEUS: A $\cup$ distance 6 mod 12

$v = 28$

BLOCKS [0 3 10] [0 2 8] mod 24; [0 1 5] mod 24 $- A$; A = (0 1 5) (3 4 8) (6 7 11) (9 10 14) (12 13 17).

ONE FACTORS: distances 9, 11 mod 24 **[SL]**

NUCLEUS: A ∪ distance 12 mod 24

$v = 34$

BLOCKS: [0 10 22] [0 2 9] [0 11 14] mod 30; [0 1 5] mod 30 $- A$; A = (0 1 5) (3 4 9) (6 7 11); (9 10 14), (12 13 17) (15 16 20).

ONE FACTORS: distance 6,13 mod 30 **[SL]**

NUCLEUS: A ∪ 15 mod 30

$v = 46$

BLOCKS: [0 12 26] [0 10 17] [0 6 19] [0 3 11] [0 18 20] mod 42; [0 1 5] mod 42 $- A$; A = (0 1 5) (3 4 8) (6 7 11) (9 10 14) (12 13 17) (15 16 20) (18 19 23) (21 22 26).

ONE FACTORS: distance 9,15 mod 42.

NUCLEUS: A ∪ distance 21 mod 42.

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