PAIRWISE BALANCED DESIGNS WITH BLOCK SIZES $5t+1$

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Abstract. In this paper we construct pairwise balanced designs (PBDs) having block sizes which are prime powers congruent to 1 modulo 5 together with 6. Such a PBD contains $n = 5r + 1$ points, for some positive integer r. We show that this condition is sufficient for $n \geq 1201$, with at most 74 possible exceptions below this value. As an application, we prove that there exists an almost resolvable BIB design with n points and block size five whenever $n \geq 991$, with at most 26 possible exceptions below this value.

1. Introduction.

A *pairwise balanced design* (or, PBD) is a pair $(X, \mathcal{A})$ such that X is a set of elements called *points*, and $\mathcal{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of points is contained in a unique block in $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, each of which is not less than 2, then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$.

Pairwise balanced designs are of fundamental importance in combinatorial design theory, being of interest in their own right, as well as having many applications in the construction of other types of designs. In general, one usually is interested in constructing (v, K)-PBDs for some specified set K. Denote $B(K) = \{v$: there exists a (v, K)-PBD$\}$. A set K is said to be PBD-*closed* if $B(K) = K$. For the concepts not defined in this paper, the reader is referred to [1].

In this paper, we investigate the set $B(P)$, where $P = \{6\} \cup P_{1,5}$, and $P_{1,5}$ is defined to be the set of prime powers congruent to 1 modulo 5. According to Wilson's theory of PBD-closed sets ([15]), there exists a constant N such that, for all $v \geq N$, $v \in B(P)$ if and only if v is congruent to 1 modulo 5. Unfortunately, this theory does not yield any reasonable upper bounds on N. However, we are able to give an upper bound on N: $N \leq 1201$. Further, there are at most 74 positive integers congruent to 1 modulo 5 for which an (n, P)-PBD does not exist. The possible exceptions are those shown in Table 1.

Table 1

$\underline{21}$, $\underline{26}$, $\underline{36}$, 46, 51, $\underline{56}$, 86, 116, 141, $\underline{146}$, 161, $\underline{166}$, $\underline{171}$, 196, 201, $\underline{206}$, $\underline{221}$, 226, $\underline{231}$, 236, 261, 266, $\underline{276}$, $\underline{286}$, 291, 296, $\underline{316}$, $\underline{321}$, 326, $\underline{336}$, 351, $\underline{356}$, 376, $\underline{386}$, 411, $\underline{416}$, $\underline{441}$, $\underline{446}$, 471, 476, 501, $\underline{536}$, $\underline{561}$, $\underline{566}$, 591, 596, $\underline{621}$, $\underline{626}$, $\underline{651}$, $\underline{686}$, $\underline{706}$, 711, $\underline{716}$, $\underline{741}$, $\underline{746}$, 766, $\underline{771}$, $\underline{776}$, $\underline{801}$, $\underline{806}$, $\underline{831}$, $\underline{866}$, $\underline{896}$, $\underline{926}$, $\underline{946}$, $\underline{956}$, 986, $\underline{1016}$, $\underline{1046}$, $\underline{1076}$, $\underline{1106}$, $\underline{1121}$, $\underline{1156}$, $\underline{1196}$.

For further reference, we name the set of these 74 possible exceptions Q.

In Section 6, we shall mention an application of this result to another type of designs. A $(v, k, k-1)$-BIBD is said to be *almost resolvable*, denoted by $AR(k, v)$, if its blocks can be partitioned into some families (*parallel classes*) such that each family forms a partition of $X \setminus \{x\}$ for some $x \in X$ (x is called a *singleton*). It is known ([7]) that an $AR(k, v)$ exists only if $v \equiv 1 \pmod{k}$. We shall show that $AR(5) \supset B(P)$. Further discussion shows that the integers underlined in Table 1 are also in $AR(5)$. Here $AR(5) = \{v\text{: an } AR(5, v) \text{ exists}\}$.

2. Recursive constructions for PBDs.

In this section, we describe several recursive constructions for PBDs with block sizes from $P = \{6\} \cup P_{1,5}$, where $P_{1,5} = \{n\text{: } n \geq 11 \text{ is a prime power such that } n \equiv 1 \pmod{5}\}$.

Definition 2.1: A $TD(k, n)$-$TD(k, m)$ is a quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{A})$, which satisfies the following properties:

1. (1) X is a set of cardinality kn;
2. (2) $\mathcal{G} = \{G_i\text{: } 1 \leq i \leq k\}$ is a partition of X into k groups of size n;
3. (3) $\mathcal{H} = \{H_i\text{: } 1 \leq i \leq k\}$, where each $G_i \supset H_i$, and $|H_i| = m$, $1 \leq i \leq k$;
4. (4) $\mathcal{A}$ is a set of $n^2 - m^2$ blocks of size k, each of which intersects each group in a point;
5. (5) every pair of points $\{x, y\}$ from distinct groups, such that at least one of x and y is in $\cup_{1 \leq i \leq k} (G_i - H_i)$, occurs in a unique block of $\mathcal{A}$.

If $m = 0$ in a $TD(k, n)$-$TD(k, m)$, the design becomes a $TD(k, n)$. It is well-known that a $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. Denote by $N(n)$ the maximum number of MOLS of order n. For a list of lower bounds on $N(n)$, $n \leq 10,000$, we refer the reader to Brouwer [2] and Todorov [12].

Definition 2.2: An *incomplete* PBD (or, IPBD) is a triple $(X, Y, \mathcal{A})$, where X is a set of points, $X \supset Y$, and $\mathcal{A}$ is a set of blocks which satisfies the properties:

1. (1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$;
2. (2) any two points, not both in Y, occur in a unique block;
3. (3) any two points, both in Y, do not occur in any block.

We say that $(X, Y, \mathcal{A})$ is a (v, w, K)-IPBD if $|X| = v$, $|Y| = w$, and $|A| \in K$ for every $A \in \mathcal{A}$. Denote $IB_w(K) = \{v\text{: a } (v, w, K)\text{-IPBD exists}\}$.

The following construction is referred to as the *singular indirect product* (SIP) (see [**8, 10**]).

Theorem 2.3. *Suppose K is a set of positive integers and $u \in K$; suppose v, w and a are integers such that $0 \leq a \leq w \leq v$; and suppose that following designs exist:*

(1) *a $TD(u, v-a)$-$TD(u, w-a)$;*
(2) *a (v, w, K)-IPBD; and*
(3) *a $(u(w-a) + a, K)$-PBD.*

Then $u(v-a) + a \in IB_u(K) \cap IB_{u(w-a)+a}(K)$. Hence, in particular, $u(v-a) + a \in B(K)$.

If we let $w = a$ in the SIP, we obtain the *singular direct product* (SDP).

Theorem 2.4. *Suppose K is a set of positive integers and $u \in K$. Suppose v and w are non-negative integers such that $w \leq v$, there exists a $TD(u, v-w)$, a (v, w, K)-IPBD, and a (w, K)-PBD. Then $u(v-w) + w \in IB_u(K) \cap IB_v(K) \cap IB_w(K)$. Hence, in particular, $u(v-w) + w \in B(K)$.*

If we further specialize this construction by letting $w = a = 0$, we obtain the *Direct product* (DP).

Theorem 2.5. *Suppose K is a set of positive integers and $u, v \in K$. If there exists a $TD(u, v)$, then $uv \in IB_u(K) \cap IB_v(K)$. Hence, in particular, $uv \in B(K)$.*

Lemma 2.6. *If $n, m \in P$ and $n < m$, then $mn \in B(P)$.*

Proof: A $TD(n, m)$ exists since m is a prime power. Apply Theorem 2.5. ∎

Lemma 2.7. *If $n \in B(P)$ and $n \neq 11, 31$, then $6(n-1) + 1 \in B(P)$.*

Proof: Since $n \in B(P)$ and $n \neq 11, 31$, $n \equiv 1 \pmod 5$ and $N(n-1) \geq 4$. There exists a $TD(6, n-1)$. Apply Theorem 2.4 with $w = 1$. ∎

In order to apply SIP, we need incomplete transversal designs. We use constructions given in [**3, 14**] to produce them.

Lemma 2.8. *Suppose there exist: a $TD(k, m)$, a $TD(k, m+1)$, a $TD(k+1, t)$, and $0 \leq u \leq t$. Then there exists a $TD(k, mt+u)$-$TD(k, u)$.*

Lemma 2.9. *Suppose there exist: a $TD(k, m)$, a $TD(k, m+1)$, a $TD(k, m+2)$, a $TD(k+2, t)$, a $TD(k, u)$, and $0 \leq u, v \leq t$. Then there exists a $TD(k, mt + u + v)$-$TD(k, v)$.*

Lemma 2.10. *Suppose there exist: a $TD(k, m)$, a $TD(k, m+1)$, a $TD(k, m+2)$, a $TD(k+u+1, t)$, and a $TD(k+1, m+u)$. Then there exists a $TD(k, mt + u + v)$-$TD(k, v)$, where $0 \leq v \leq t-1$.*

Corollary 2.11. *Suppose there exist: a* $TD(8,t)$, *a* $TD(6,k)$, *where* $0 \leq a,k \leq t$. *Then there exists a* $TD(6,7t+k+a)$ *-*$TD(6,a)$.

Corollary 2.12. *Suppose there exist: a* $TD(7+w,t)$, *a* $TD(6,m)$, *a* $TD(6,m+1)$, *a* $TD(6,m+2)$, *and a* $TD(6,m+w)$. *Then there exists a* $TD(6,mt+w+a)$ *-* $TD(6,a)$ *for* $0 \leq a \leq t$.

A *group-divisible design* (or, GDD), is a triple $(X,\mathcal{G},\mathcal{A})$, which satisfies the following properties:

(1) $\mathcal{G}$ is a partition of X into subsets called *groups*,
(2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,
(3) every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\{|G|: G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe group-type: a group-type $1^i.2^j.3^k\ldots$ denotes i occurrences of 1, j occurrences of 2, etc. As with PBDs, we will say that a GDD is a (K,λ)-GDD if $|A| \in K$ for every $A \in \mathcal{A}$. If the group-type is T, we further write a (K,λ)-GDD as (K,λ)-GDD(T). We shall only deal with the case $\lambda = 1$ until Section 6. In this case, we omit λ in the above notations.

We often construct PBDs from GDDs by filling in the groups as follows.

Theorem 2.13. *Suppose there exists a* P*-GDD on* v *points such that for some* G_0 *there exists a* $(|G_0|+w,P)$*-PBD, and for any other group* G *there exists a* $(|G|+w,w,P)$*-IPBD. Then there exists a* $(v+w,P)$*-PBD. Further,* $v+w \in IB_{|G_0|+w}(P)$.

Proof: An analogue to SDP construction. ∎

If $w = 0$ or 1, we have

Theorem 2.14. *Suppose there exists a* P*-GDD on* v *points such that for each group* G, $|G|+\varepsilon \in B(P)$, *where* $\varepsilon = 0$ *or 1. Then* $v+\varepsilon \in B(P)$.

Corollary 2.15. *If* $v \in B(6)$, $N(u) \geq 4$, *and* $u+\varepsilon \in P$, *where* $\varepsilon = 0$ *or 1, then* $vu+\varepsilon \in B(P)$.

Proof: Since $v \in B(6)$, there is a $\{6\}$-GDD(1^v). Give weight u to each point and use a $TD(6,u)$ as input design, which exists since $N(u) \geq 4$. We have a P-GDD(u^v). Apply Theorem 2.14. ∎

Using Wilson's Fundamental Constructions for GDDs ([15]) and Theorem 2.13, we have the following lemmas, which are essentially Theorem 2.17 and Theorem 2.20 in [16], where R_5^* should be replaced by P, and the proofs are still valid.

Lemma 2.16. *Suppose* $N(t) \geq 12$, $0 \leq s \leq t$. *Then,* $B(P) \supset \{5t+1, 15s+1\}$ *implies* $65t+15s+1 \in B(P)$. *Further, if* $5t+a \in IB_a(P)$ *and* $15s+a \in B(P)$, *then* $65t+15s+a \in B(P)$.

Lemma 2.17. *Suppose* $N(t) \geq 15, 0 \leq u \leq t, 0 \leq w \leq t$. *Then,* $B(P) \supset \{5t+1, 15u+1, 5w+1\}$ *implies* $75t + 15u + 5w + 1 \in B(P)$. *Further, if* $5t + a \in IB_a(P)$, *at least one of* $15u + a$ *and* $5w + a$ *belongs to* $IB_a(P)$, *and the other is in* $B(P)$, *then* $75t + 15u + 5w + a \in B(P)$.

3. The case $v \equiv 1, 6 \pmod{15}$.

As an easy corollary of the known results on $B(6)$ (see [**9, 17**]) we have

Lemma 3.1. *If* $v \equiv 1, 6 \pmod{15}$ *is a positive integer and* v *is not in Table 2, then* $v \in B(6)$ *and* $v \in B(P)$.

Table 2

$\underline{16}$, 21, 36, 46, 51, $\underline{61}$, $\underline{81}$, 141, 166, 171, 196, 201, 226, 231, $\underline{246}$, $\underline{256}$, 261, 276, 286, 291, 316, 321, 336, $\underline{346}$, 351, 376, $\underline{406}$, 411, $\underline{436}$, 441, $\underline{466}$, 471, $\underline{486}$, $\underline{496}$, 501, $\underline{526}$, 561, 591, $\underline{616}$, 621, $\underline{646}$, 651, $\underline{676}$, 706, 711, $\underline{736}$, 741, 766, 771, $\underline{796}$, 801, 831, $\underline{886}$, $\underline{891}$, $\underline{916}$, 946, $\underline{1071}$, $\underline{1096}$, $\underline{1101}$, $\underline{1131}$, $\underline{1141}$, 1156, $\underline{1161}$, $\underline{1176}$, $\underline{1186}$, $\underline{1191}$, $\underline{1221}$, $\underline{1246}$, $\underline{1251}$, $\underline{1276}$, $\underline{1396}$, $\underline{1401}$, $\underline{1456}$, $\underline{1461}$, $\underline{1486}$, $\underline{1491}$, $\underline{1516}$, $\underline{1521}$, $\underline{1546}$, $\underline{1611}$, $\underline{1641}$, $\underline{1671}$, $\underline{1816}$, $\underline{1821}$, $\underline{1851}$, $\underline{1881}$, $\underline{1971}$, $\underline{2031}$, $\underline{2241}$, $\underline{2601}$, $\underline{3201}$, $\underline{3471}$, $\underline{3501}$, $\underline{4191}$, $\underline{4221}$, $\underline{5391}$, $\underline{5901}$.

We shall show, in this section, that the integers underlined in Table 2 are also in $B(P)$.

Theorem 3.2. *Suppose* $v \equiv 1, 6 \pmod{15}$ *is a positive integer. If* v *is not in Table 2, or* v *is an underlined number in Table 2, then* $v \in B(P)$.

Proof: We need only discuss those numbers underlined in Table 2. The first three and 256 are prime powers belonging to P. Lemma 2.6 takes care of $246, 486, 496, 891$. Corollary 2.15 takes care of $466 = 31.15 + 1$, $1221 = 111.11$ and $4191 = 381.11$. Lemma 2.7 takes care of 1141. From a $(325, 5, 1)$-RBIBD (see [**16**]) we can obtain a $(406, \{6, 81\})$-PBD by adding infinite points to its 81 parallel classes. Therefore, $406 \in B(P)$. Lemma 2.16 and Lemma 2.17 also take care of some numbers shown in Table 3 and Table 4. The remaining numbers are all done by SIP, as shown in Table 5. The required incomplete TDs are all constructed by Lemma 2.8 and Lemma 2.11. ∎

Table 3 Applications of Lemma 2.16

$t = 16$, $a = 1$	1071	1101	1131	1161	1176	1191	1251
$t = 17$, $a = 6$	1186						
$t = 25$, $a = 1$	1641	1881					

Table 4 Applications of Lemma 2.17

$t = 16,\ a = 1$	1246 1276 1401 1461
$t = 19,\ a = 1$	1481 1491 1516 1521 1546 1611 1671
$t = 25,\ a = 1$	1971 2031 2241
$t = 31,\ a = 1$	2601
$t = 37,\ a = 1$	3201 3471 3501
$t = 17,\ a = 6$	1396 1456
$t = 23,\ a = 6$	1816 1821 1851
$t = 53,\ a = 6$	4221
$t = 71,\ a = 6$	5391 5901

Table 5 Applications of SIP

(v equation)	w	(PBD with flat)	incomplete TD	$u(w-a)+a$
$346 = 6(66-11)+16$	11	$66 = 6.11$	$TD(5,56) - TD(6,1)$	16
$436 = 6(72-2)+16$	6	76	$72 = 7.9+7+2$	16
$526 = 6(87-2)+16$	6	91	$87 = 7.11+8+2$	16
$616 = 6(102-2)+16$	6	106	$102 = 7.13+9+2$	16
$646 = 6(107-2)+16$	6	111	$107 = 15.7+2$	11
$676 = 11(61-1)+16$	6	66	$TD(6,61) - TD(6,1)$	11
$736 = 6(122-2)+16$	6	126	$122 = 7.16+8+2$	16
$796 = 6(132-2)+16$	6	136	$132 = 7.17+11+2$	16
$886 = 6(147-2)+16$	6	151	$147 = 8.12+2$	16
$916 = 6(152-2)+16$	6	156	$152 = 7.19+17+2$	16
$1096 = 6(182-2)+16$	6	186	$182 = 7.23+19+2$	16

4. A preliminary bound.

In this section, we shall show that $v \in B(P)$ whenever $v \geq 3401$ and $v \equiv 1 \pmod 5$. We first use Lemma 2.17 to prove

Theorem 4.1. *Suppose there exists a series of positive integers* $\{t_i\}_{i=1,2,\ldots}$, *such that for each* i

(1) $N(t_i) \geq 15$,
(2) $5t_i + 1 \in B(P)$,
(3) $t_i \leq t_{i+1} \leq (6t_i - 78)/5$.

Then $v \in B(P)$ *whenever* $v \geq 75t_1 + 1181$ *and* $v \equiv 11 \pmod{15}$.

Proof: For each i, applying Lemma 2.17 with $t = t_i$, $78 \leq u \leq t_i$ and $w = 2$, we have $75t_i + 15u + 1 \in B(P)$ since $B(P) \supset \{15u+1, 11\}$ from Theorem 3.2. This gives an interval $[75t_i + 1181, 90t_i + 11]$ such that $v \in B(P)$ whenever $v \equiv 11 \pmod{15}$ and v is in the interval. By condition (3), each interval is overlapped with the next interval. The conclusion then follows. ∎

Lemma 4.2. $v \in B(P)$ *whenever* $v \equiv 11 \pmod{15}$ *and* $v \geq 11606$.

Proof: Take a series $\{t_i\}_{i=1,2,\ldots}$ as follows:

$$139, 151, 163, 169, 181, 199, 223, 241, 271, 307, 349, 397, 457, 523,$$
$$607, 709, 829, 967, 1129, 1327, 1567, 1861, 2203, 2617, 3121, 3727, \ldots$$

$$t_{j+1} = t_j + 6 \;\text{ if } j \geq 26.$$

We check the conditions in Theorem 4.1. The first 26 terms are all prime powers congruent to 1 modulo 6 and the other terms are all odd and greater than 3603. From [3] we have $N(t_i) \geq 15$. Since $t_i \equiv 1 \pmod 6$ and $5t_i + 1 \equiv 6 \pmod{15}$, we have from Theorem 3.2 that $5t_i + 1 \in B(P)$ if $5t_i + 1 > 1156$, namely, $t_i > 231$. For the first seven terms it is easily verified that $5t_i + 1 \in B(P)$. The condition (3) can also be checked easily. We then apply Theorem 4.1 to obtain a bound $v \geq 75.139 + 1181 = 11606$. ∎

Lemma 4.3. $v \in B(P)$ *whenever* $v \equiv 11 \pmod{15}$ *and* $3401 \leq v \leq 11606$.

Proof: We again use Lemma 2.17 with parameters shown in Table 6. $N(t) \geq 15$ is obvious since t is a prime power. For any v under consideration, v is in one of the intervals in Table 6. We write $v = 75t + 15u + 176$, where $176 = 11.16 \in B(P)$. When $a = 1$, $5t + a \in B(P)$. If $15u + 1 \in B(P)$, we have $v \in B(P)$ by Lemma 2.17. If $15u + 1 \notin B(P)$, it is readily checked by Theorem 3.2 that $15(u-1) + 1 \in B(P)$. We then write $v = 75t + 15(u-1) + 191$ where $191 \in B(P)$, and obtain $v \in B(P)$ also. When $a = 6$, $5t + 6 \in IB_6(P)$. If $15u+6 \in IB_6(P)$, we have $v \in B(P)$ by Lemma 2.17. If $15u+6 \notin IB_6(P)$, it is readily checked by Theorem 3.2 that $15(u-1)+6 \in IB_6(P)$ for $24 \leq u \leq t$. Therefore, $v = 75t + 15(u-1) + 191 \in B(P)$. This completes the proof. ∎

Table 6 Applications of Lemma 2.17

t	a	5t + a	u	v
127	1	$636 \in B(6)$	0–127	9701–11606
109	1	$546 \in B(6)$	0–109	8351–9986
103	1	$516 \in B(6)$	0–103	7901–9446
89	6	$451 \in B(6)$	24–89	7211–8186
79	1	$396 \in B(6)$	0–79	6101–7286
73	1	$366 \in B(6)$	0–73	5651–6746
61	1	$306 \in B(6)$	0–61	4751–5666
53	6	$271 \in B(6)$	24–53	4511–4946
49	1	$246 = 41.6 \in IB_6(P)$	0–49	3851–4586
43	1	$216 \in B(6)$	0–43	3401–4046

5. The spectrum.

It remains to discuss the values $v \equiv 11 \pmod{15}$ and $v < 3401$.

Lemma 5.1. *If* $v \equiv 11 \pmod{15}$ *and* $1211 \leq v < 3401$, *then* $v \in B(P)$.

Proof: Apply Lemma 2.6 with $m = 151, n = 11$ to obtain $1661 \in B(P)$. Apply Lemma 2.17 with $(t, a) = (16,1), (19,1), (25,1), (27,1), (31,1), (37,1)$ and $(23,6)$. We can take suitable u (Theorem 3.2) and w to cover the interval, noticing that $11, 41, 71, 101, 131, 176, 191 \in B(P)$. These leave $v = 1781$, for which we use SIP construction by writing $1781 = 6(296 - 1) + 5$. Since $301 \in B(P)$, we obtain $1781 \in B(6, 11)$, hence $1781 \in B(P)$. ▮

Lemma 5.2. $B(P) \supset \{176, 341, 371, 506, 551, 581, 611, 656, 671, 731, 791, 836, 851, 1001, 1136, 1166\}$.

Proof: We apply direct product to obtain $v \in B(P)$ where $176 = 11.16, 341 = 11.31, 656 = 16.41, 671 = 11.61, 1136 = 16.71$. Corollary 2.15 works for the values: $836 = 76.11$, $1001 = 91.11$, $1166 = 106.11$. A $(405, 5, 1)$-RBIBD exists (see [4]), from which we can add infinite points to parallel classes to obtain a $(506, \{6, 101\})$-PBD. Therefore, $506 \in B(P)$. The SIP will work for the other five values, where $371 = 6(61 - 1) + 11$, $551 = 6(91 - 1) + 11$, $611 = 6(101 - 1) + 11$, $731 = 6(121 - 1) + 11$ and $791 = 6(131 - 1) + 11$, and the required $66, 96, 106, 126, 136 \in B(6)$ all come from Table 2. We apply SIP also for the remaining values to get $581 \in B(6, 71)$ and $851 \in B(6, 101)$. From a $(85, 5, 1)$-RBIBD ([16]) we have $106 \in IB_{21}(6)$. Write $581 = 6(95 - 10) + 71$. The required $TD(6, 95) - TD(6, 10)$ comes from Lemma 2.11 with $95 = 7.11 + 8 + 10$. Similarly, a $(125, 5, 1)$-RBIBD yields a $(156, 31, \{6\})$-IPBD. Write $851 = 6(139 - 14) + 101$. The required $TD(6, 139) - TD(6, 14)$ comes from Lemma 2.12 with $m = 7, t = 17, w = 6, a = 14$ and $139 = 7.17 + 6 + 14$. ▮

Lemma 5.3. *For* $v \equiv 11 \pmod{15}$, *there are at most 37 values of* v *shown in Table 1, for which a* (v, P)*-PBD does not exist.*

Proof: There are all together 80 values of v such that $v \equiv 11 \pmod{15}$ and $11 \leq v \leq 1211$. Among them there are 27 prime integers. The other 16 values are contained in $B(P)$ from Lemma 5.2. This leaves 37 values below 1211. Combine Lemma 4.2, Lemma 4.3, and Lemma 5.1, the conclusion then follows. ▮

Theorem 5.4. *For any positive integer* $v \equiv 1 \pmod{5}$ *there exists a* (v, P)*-PBD, where* $P = \{6\} \cup P_{1,5}$ *and* $P_{1,5} = \{n: n \geq 11$ *is a prime power such that* $n \equiv 1 \pmod{5}\}$, *with 74 possible exceptions shown in Table 1.*

Proof: Combining Theorem 3.2 and Lemma 5.3. ▮

6. Application to almost resolvable BIB designs.

As mentioned in Section 1 we shall show here that $AR(5)$ contains all positive integers $v \equiv 1 \pmod 5$ except posibly when v is an integer not underlined in Table 1.

Let $(X, \mathcal{G}, \mathcal{A})$ be a (K, λ)-GDD(T). For $G \in \mathcal{G}$, let P_G be a subset of $\mathcal{A}$ such that the blocks in P_G form a partition of $X \setminus G$. P_G is called a *parallel class with hole* G. The GDD is called a (K, λ)*-frame* with type T if $\mathcal{A}$ can be partitioned into disjoint parallel classes with holes. The idea of a frame was first introduced by Hanani [6] and the concept was popularized by Stinson [11]. This notion is useful in the construction of resolvable and almost resolvable designs. In this section we only deal with $(\{5\}, 4)$-frames and we omit the parameters for simplicity of notation.

The following Lemmas are slight modifications of the constructions in [11].

Lemma 6.1. *Suppose there is a frame of type* m^n. *If* $N(t) \geq 4$, *then there exists a frame of type* $(mt)^n$.

Lemma 6.2. *Suppose there is a frame of type* $\{t_1, t_2, \ldots, t_n\}$, *and* $\varepsilon \geq 0$. *For* $1 \leq i \leq n$, *suppose there is a frame of type* $T_i \cup \{\varepsilon\}$, *where* $\sum_{t \in T_i} t = t_i$. *Then there is a frame of type* $\{\varepsilon\} \cup (\cup_{1 \leq i \leq n} T_i)$.

Lemma 6.3. *Let* $(X, \mathcal{G}, \mathcal{A})$ *be a GDD, and let* $w\colon X \to Z^+ \cup \{0\}$ *(we say that* w *is a weighting). For each* $A \in \mathcal{A}$, *suppose there is a frame of type* $\{w(x) : x \in A\}$. *Then there is a frame of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

We also need

Lemma 6.4. *Let* $v = p^r$ *be any prime power and* $k > 2$ *such that* k *is a divisor of* $v - 1$, *then there exists an* $AR(k, v)$.

Proof: Let $X = GF(v)$. For any λ, u, $z \in GF(v)$ satisfying $u \neq z$, $\lambda \neq 1$ and $\lambda^k = 1$, form a block $B_z = \{u, \lambda u + (1 - \lambda) z, \lambda^2 u + (1 - \lambda^2) z, \ldots, \lambda^{k-1} u + (1 - \lambda^{k-1}) z\}$. For any element u' in this block, the triple u', z, λ will determine the same block. It is easy to see that elements in this block are indeed distinct, therefore, for fixed z and λ, we can obtain $(v - 1)/k$ blocks, which form a parallel class with singleton z. And for a fixed λ, we can obtain $v(v - 1)/k$ blocks $\mathcal{A}$, which can be partitioned into v parallel classes. Then $(X, \mathcal{A})$ is the required $AR(k, v)$. In fact, for any unordered pair $\{u, w\}$, let $x_i = (w - \lambda^i u)/(1 - \lambda^i)$, $i = 1, 2, \ldots, k-1$. Then x_i are pairwise distinct, and $\{u, w\}$ is contained exactly in the blocks B_z, $z = x_i$, $(i = 1, 2, \ldots, k-1)$ of the parallel class with singleton x_i. ∎

Notice that an $AR(5, v)$ is also a frame of type 1^v. Therefore, a frame of type 1^n exists whenever $n \in P_{1,5}$. By Lemma 6.3 we can obtain a frame of type 1^v for $v \in B(P)$ if we have an $AR(5, 6)$.

Lemma 6.5. *An* $AR(5,6)$ *exists.*

Proof: $X = Z_6, \mathcal{A} = \{Z_6 - \{i\}: i \in Z_6\}$. Every block $Z_6 - \{i\}$ form a parallel class with singleton i. ∎

Lemma 6.6. $AR(5) \supset B(P)$.

Proof: For any $v \in B(P)$ there is a (v, P)-PBD, namely, a P-GDD(1^v). Give every point weight one and apply Lemma 6.3. Since a frame of type 1^k exists for $k \in P$, we obtain a frame of type 1^v. Then, $v \in AR(5)$. ∎

To further reduce the possible exceptions of $AR(5,v)$ in Table 1, we need an $AR(5,21)$, an $AR(5,26)$ and an $AR(5,36)$, where the first two designs were pointed out to us by Alan Hartman and Zvi Yehudai.

Lemma 6.7. $AR(5) \supset \{21, 26\}$.

Proof: Develop the following parallel class (with hole $\{20\}$) modulo 21 to construct an $AR(5,21)$:

$$\{0,1,2,4,9\}, \{3,7,10,14,15\}, \{5,11,13,16,19\}, \{6,8,12,17,18\}.$$

Develop the following parallel class (with hole $\{25\}$) modulo 26 to construct an $AR(5,26)$:

$$\{0,10,13,14,17\}, \{1,3,7,12,22\}, \{2,8,20,21,23\},$$
$$\{4,5,9,11,19\}, \{6,15,16,18,24\}.$$

∎

Lemma 6.8. *An* $AR(5,36)$ *exists.*

Proof: Let $X = Z_{12}$ and $\mathcal{G} = \{G_i: 0 \leq i \leq 5\}$ where $G_i = \{i, i+6\}$. Let $\mathcal{A} = \{\{i, i+1, i+4, i+9, i+11\}: i \in Z_{12}\}$. It is readily checked that $(X, \mathcal{G}, \mathcal{A})$ is a $(\{5\}, 2)$-GDD(2^6). Take $\{i+10, i+11, i+2, i+7, i+9\}$ and $\{i+4, i+5, i+8, i+1, i+3\}$ as a parallel class with hole $\{i, i+6\}$. The GDD is also a $(\{5\}, 2)$-frame of type 2^6. From the proof of Theorem 3.11 in [5] we have an $RTD(6,3)$ of Index 2. Deleting one group yields an $RTD(5,3)$ of Index 2. A modification of Lemma 6.1 will produce a $(\{5\}, 4)$-frame of type 6^6. Since a frame of type 1^6 exists from Lemma 6.4, applying Lemma 6.2 with $\varepsilon = 0$, we obtain a frame of type 1^{36}. Therefore, $36 \in AR(5)$. ∎

Lemma 6.9. *Suppose* $u \in AR(5)$ *and* $m + \varepsilon \in AR(5)$, $\varepsilon \in \{0,1\}$. *If* $N(m) \geq 4$, *then* $um + \varepsilon \in AR(5)$.

Proof: Applying Lemma 6.1 with a frame of type 1^u, we have a frame of type m^u. Further applying Lemma 6.2 we obtain a frame of type $1^{um+\varepsilon}$. ∎

Lemma 6.10. $AR(5) \supset \{56, 166, 206, 221, 231, 276, 286, 316, 321, 336, 356, 386, 416, 441, 561, 1046\}$.

Proof: Apply Lemma 6.9 with the following expressions:

$56 = 11.5 + 1$,	$166 = 11.15 + 1$,	$206 = 41.5 + 1$,	$221 = 11.20 + 1$,
$231 = 11.21$,	$276 = 11.25 + 1$,	$286 = 26.11$,	$316 = 21.15 + 1$,
$321 = 16.20 + 1$,	$336 = 6.56$,	$356 = 71.5 + 1$,	$386 = 11.35 + 1$,
$416 = 26.16$,	$441 = 11.40 + 1$,	$561 = 16.35 + 1$,	$1046 = 11.59 + 1$.

Lemma 6.11. $146, 171 \in AR(5)$.

Proof: From Wilson [15, Lemma 5.2], $q^3 + q^2 - q + 1 \in B(q+1, q^2 - q + 1, q^2 + 1)$. Taking $q = 5$ produces $146 \in B(6, 21, 26)$. So, $146 \in AR(5)$. By Mullin [9, Lemma 3.13], there exists a $TD(6, 28)$-$TD(6, 3)$. Adding three new points to groups and applying SIP produces a $(171, \{6, 21\})$-PBD. So, $171 \in AR(5)$. ∎

Lemma 6.12. $651, 686, 706, 716, 746, 771, 776 \in AR(5)$.

Proof: Give weight one to each point in a $TD(26, 25)$ except m points in some group, to which we give weight six each. Using a $(31, 6, \{6\})$-IPBD as input designs we obtain a $\{6, 26\}$-GDD of type $25^{25}(25 + m)^1$. Adding one new point to the GDD produces a $(651 + 5m, \{6, 26, 26 + 5m\}, 1)$-PBD.

Taking suitable m as shown in Table 7 we know that $26 + 5m \in AR(5)$ and $651 + 5m \in AR(5)$. ∎

Table 7

$651 + 5m$	$26 + 5m$	m
651	26	0
686	61	7
706	81	11
716	91	13
746	121	19
771	146	24
776	151	25

Lemma 6.13. $806, 831, 886, 896, 926, 946, 956 \in AR(5)$.

Proof: Give weight six to m points in a group of a $TD(26, 31)$ and weight one to other points. Using a $(31, 6, \{6\})$-IPBD as input designs produces a $\{6, 26\}$-GDD of type $31^{25}(31 + 5m)^1$. Hence, $806 + 5m \in B(6, 26, 31, 31 + 5m)$. Taking suitable m as shown in Table 8, such that $31 + 5m \in AR(5)$, we know that $806 + 5m \in AR(5)$. ∎

Table 8

$806+5m$	$31+5m$	m
806	31	0
831	56	5
866	91	12
896	121	18
926	151	24
946	171	28
956	181	30

Lemma 6.14. $1106, 1121, 1156, 1196 \in AR(5)$.

Proof: Start with a $TD(26,41)$ and do the similar construction as in Lemma 6.13. Taking $m = 8, 11, 18$ and 26, since $81, 96, 131, 171 \in AR(5)$, we have $1066 + 5m = 1106, 1121, 1156, 1196 \in AR(5)$. ∎

Lemma 6.15. $AR(5) \supset \{446, 536, 566, 621, 626, 741, 801, 1016, 1076\}$.

Proof: We use SIP construction to show that $n \in B(6, 21, 26, 56)$. Therefore, $n \in AR(5)$ since $6, 21, 26, 56 \in AR(5)$. We give the parameters in Table 9. From RBIBDs we have $81 \in IB_{16}(6)$ and $106 \in IB_{21}(6)$. $176 \in IB_{11}(P)$ comes from Lemma 5.2. For the incomplete TDs, numbers 73 and 168 come from Lemma 2.12 and others from Lemma 2.11. ∎

Table 9 Applications of SIP

$n = u(v-a)+a$	w	(PBD with flat)	incomplete TD	$u(w-a)+a$
$446 = 6(81-8)+8$	16	$81 \in IB_{16}(6)$	$73 = 7.9+2+8$	56
$536 = 6(91-2)+2$	6	$91 \in B(6)$	$89 = 7.11+8+4$	26
$566 = 6(106-14)+14$	21	$106 \in IB_{21}(6)$	$92 = 7.11+8+7$	56
$621 = 6(106-3)+3$	6	$106 \in B(6)$	$103 = 7.13+9+3$	21
$626 = 6(106-2)+2$	6	$106 \in B(6)$	$104 = 7.13+9+4$	26
$741 = 6(126-3)+3$	6	$126 \in B(6)$	$123 = 7.16+8+3$	21
$801 = 6(136-3)+3$	6	$136 \in B(6)$	$133 = 7.17+11+3$	21
$1016 = 6(176-8)+8$	11	$176 \in IB_{11}(P)$	$168 = 7.23+4+3$	26
$1076 = 6(181-2)+2$	6	$181 \in B(6)$	$179 = 7.25+4$	26

Theorem 6.16. *$AR(5)$ contains all positive integers $v \equiv 1 \pmod 5$ except possibly when $v \le 986$ and v is one of the 26 numbers not underlined in Table 1.*

Proof: Combining Lemmas 6.5, 6.7 – 6.8, 6.10 – 6.15. ∎

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