# A greedy search for maximal partial spreads in $PG(3,7)$

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In this short note we present maximal partial spreads in $PG(3,7)$ consisting of 23, 24 and 25 lines. This result extends some of the results of ours. In [3] and [4] we gave a construction of maximal partial spreads in $PG(3,q)$ that in the case $q = 7$ produced maximal partial spreads consisting of $n$ lines for any integer $n$ in the interval $26 \leq n \leq 44$.

Throughout this note we use the same notation as in [3]: $PG(3,7)$ is viewed as a direct product $V = GF(49) \times GF(49)$ where $GF(49)$ is considered as a vector space over the finite field $GF(7)$. Lines of $PG(3,7)$ corresponds to 2-dimensional subspaces of $V$. $L_\infty$ denotes the line $\{0\} \times GF(49)$ and $[a,b]$ denotes the line $\{(x, ax + bx^7) \mid x \in GF(49)\}$. Let $i$ be a primitive element of $GF(49)$ and let $H$ denote the multiplicative subgroup $\{i^0, i^6, i^{12}, \ldots\}$ of $GF(49)$. By Lemma 2.2 of [3] a line $L$ of $V$ is skew to $L_\infty$ if and only if $L = [a,b]$ for some elements $a$ and $b$ of $GF(49)$ and two lines $[a,b]$ and $[c,d]$ intersect if and only if $a - c = (b - d)h$ for some element $h$ of $H$. We enumerate the 2401 lines of $PG(3,7)$ skew to $L_\infty$ in the following way by using the tables of Bussey [1]: We give the line $[a,b]$ the number $343\beta + 49\alpha + 7\delta + \chi + 1$ if $a = i^\lambda = \alpha i + \beta$ and $b = i^{\lambda'} = \chi i + \delta$. (If $a = 0$ or $b = 0$ then $(\alpha,\beta) = (0,0)$ respectively $(\chi,\delta) = (0,0)$.)

The following three sets of lines will together with the line $L_\infty$ constitute the maximal partial spreads mentioned above:

$$S_{23} = \{1861, 388, 1803, 222, 358, 479, 1676, 998, 2021, 1986, 884, 262, 2114, 2282, 64, 1369, 1290, 771, 446, 1335, 1009, 159\}$$

$$S_{24} = \{233, 115, 437, 1378, 300, 1403, 1310, 2175, 979, 1927, 189, 1240, 762, 2096, 1487, 1957, 1749, 1623, 1904, 325, 476, 202, 2357\}$$

$$S_{25} = \{429, 790, 1936, 1697, 921, 2103, 757, 1268, 1899, 2083, 1344, 388, 2385, 915, 1272, 948, 11, 2291, 1970, 317, 2329, 951, 912, 981\}.$$

We found these maximal partial spreads by repeating the following greedy algorithm 2080 times:

1. Enumerate, as above, the 2401 lines of $PG(3,7)$ skew to $L_\infty$.
2. Choose at random five mutually skew lines of these 2401 lines. Denote this set of lines by $C$. Complete $C$ with the line $L_\infty$.
   Let $R(C)$ denote the set of lines that do not intersect any of the lines of $C$. For any $L \in R(C)$, $n(C, L)$ denotes the number of lines of $R(C)$ that intersect $L$.
3. Choose a line $L^{op}$ of $R(C)$ such that $n(C, L^{op}) \geq n(C, L')$ for any $L' \in R(C)$ and if $n(C, L^{op}) = n(C, L)$ for some line $L$ of $R(C)$ then $L^{op} \geq L$.
4. Let $C$ be completed with $L^{op}$ and go to 3.
   The algorithm stops when R(C) is empty.

Certainly, the result of these 2080 trials very much depends on the way the random numbers are produced. We used a preprogrammed random process of a Lightspeed compiler for a Macintosh Plus computer. Anyhow, the result of these trials were as follows:

| no. of lines | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| frequence | 3 | 70 | 495 | 700 | 521 | 206 | 49 | 12 | 6 | 0 | 0 | 2 |

| 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 0 | 1 | 0 | 4 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 4 |

Remark 1: We also made similar trials by choosing another amount of random lines. Anyone is very welcome to write to us for further information. We also made trials where step 3 in the algorithm above was changed to the following:

3′. Choose a line $L^{op}$ of $R(C)$ such that $n(C, L^{op}) \leq n(C, L')$ for any $L' \in R(C)$ and if $n(C, L^{op}) = n(C, L)$ for some line $L$ of $R(C)$ then $L^{op} \geq L$.

With 5 randomly chosen lines and 221 trials the result was as follows:

| number of lines | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| frequence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 7 | 7 | 20 | 21 | 25 | 22 | 24 | 23 | 13 | 0 | 0 | 0 | 0 | 0 | 59 |

Remark 2: Glynn has proved [2] that any maximal partial spread in $PG(3,q)$ contains at least $2q$ lines. As far as we know nobody has presented a maximal partial spread in $PG(3,7)$ with $n$ lines for any $n$ in the interval $14 \leq n \leq 22$.

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## References

1. W.H. Bussey, *Galois field tables for* $p^n \leq 169$, Bull. Amer. Math. **12** (1905–1906), 22–38.
2. D.G. Glynn, *A lower bound for maximal partial spreads in* $PG(3, q)$, Ars Combinatoria **13** (1982), 39–40.
3. 0. Heden, *Maximal partial spreads and the n-queen problem.* (To appear in Discrete Mathematics).
4. 0. Heden, *Maximal partial spreads and the n-queen problem II.*