EXTENSIONS OF GRACEFUL VALUATIONS OF 2-REGULAR GRAPHS CONSISTING OF 4-GONS

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A graceful valuation (numbering) of a graph G with m vertices and n edges is a one-to-one mapping ψ of the set $V(G)$ into the set $\{0,1,\ldots,n\}$ with the following property: If we define, for any edge $e \in E(G)$ with the end vertices u,v, the value $\overline{\psi}(e)$ of the edge e by $\overline{\psi}(e) = |\psi(u)-\psi(v)|$ then $\overline{\psi}$ is a one-to-one mapping of $E(G)$ onto the set $\{1,2,\ldots,n\}$. A graph is called graceful if it has a graceful valuation.

An α-valuation ψ of a graph G is a graceful valuation of G which has the following additional property: There exists a number γ $(0 \leq \gamma < |E(G)|)$ such that for any edge $e \in E(G)$ with the end vertices u,v, it is

$$\min[\psi(u),\psi(v)] \leq \gamma < \max[\psi(u),\psi(v)].$$

The concept of a graceful valuation (under the name β-valuation) and of an α-valuation was introduced by A. Rosa [9]. The term "graceful valuation" was introduced by S.W. Golomb [5].

Rosa [9] proved the following theorem: If an eulerian graph G is graceful then $|E(G)| \equiv 0$ or 3 (mod 4). This implies that $|E(G)| \equiv 0$ (mod 4) for any eulerian graph G which has an α-valuation. (In this theorem, an eulerian graph G is any graph in which the degree of each vertex is positive and even; G does not have to be connected.)

It is well known that the condition of the above theorem is also sufficient for cycles (Kotzig [6], Rosa [9]) and for 2-regular graphs with two isomorphic components (Kotzig [7] proved that a 2-regular graph consisting of two s-cycles (s even) has an α-valuation). A partial extension for 3 components can be found in the same paper. In this case, the condition of the above theorem is not always sufficient.

The following results proved in [2] will be useful in this paper: If G is a graceful 2-regular graph on $4r$ vertices then exactly one number $x \in \{1,2,\ldots,4r\}$ will not be used to label any vertex of G. This number satisfies the inequalities $r \leq x \leq 3r$. The given graceful valuation of G is an α-valuation if and only if either $x=r$ or $x=3r$. This number x is called the missing value of the given graceful valuation. Let us also observe that if G has an α-valuation ψ with one of the two possible missing values (r or $3r$), it also has an α-valuation ϕ with the other possible missing value. To see this, it suffices to put $\phi(v)=4r-\psi(v)$ for every $v \in V(G)$.

More recently, it has been proved in [3] that the number of graceful valuations of $(4k+3)$-cycles and the number of α-valuations of $4k$-cycles grow exponentially with k.

For the special case of 2-regular graphs consisting of 4-gons it is known that such a graph consisting of k 4-gons has an α-valuation for $1 \le k \le 10$, $k \ne 3$; for $k=3$, this graph is graceful but it does not have an α-valuation (see [7]). In this paper, the 2-regular graph consisting of k 4-gons will be denoted by A_k.

Our first result is given in

Theorem 1. Let k be a positive integer. If the graph A_k has an α-valuation then A_{4k+1} also has an α-valuation.

Corollary. The sequence $\{A_k\}_{k=1}^{\infty}$ contains infinitely many graphs which have α-valuations.

Proof of Theorem 1. If A_k has an α-valuation then the vertices of A_k are labeled by $4k$ values from the set $\{0,1,...,4k\}$. Then we have two possibilities concerning the number γ from the definition of an α valuation and the missing value:

A. $\gamma=2k$ and the missing value is k.

B. $\gamma=2k-1$ and the missing value is $3k$.

In each case, the numbers $\le\gamma$ will be referred to as the "small values (numbers)". The numbers $>\gamma$ will be called the "large values (numbers)". For our considerations, A_{4k+1} will be decomposed into five subgraphs consisting of 4-gons; each of the first four subgraphs will consist of k 4-gons, the fifth subgraph will contain only one 4-gon. We will construct an α-valuation of A_{4k+1} by describing the values of the vertices and of the edges of the 4-gons in each subgraph. The values of the vertices in each 4-gon in each of the first four subgraphs will be derived directly from the given α-valuation of A_k.

Subgraph 1. We start with an α-valuation of A_k, with the missing value k. We increase the large numbers labeling the vertices of this subgraph by $12k+4$ and leave the small numbers unchanged. The large values will become $14k+5,...,16k+4$, the small values will be $0,...,2k$, with the value k missing, and the values of the edges will be $12k+5,...,16k+4$.

Subgraph 2. We proceed from an α-valuation of A_k with the missing value $3k$. The large values (including the missing value) will be increased by $10k+4$, the small values will be increased by $2k+1$. The new large values of this subgraph will be $12k+4,\ldots,14k+4$ (with the value $13k+4$ missing), the small values will be $2k+1,\ldots,4k$; the values of the edges will be $8k+4,\ldots,12k+3$.

Subgraph 3. We take an α-valuation of A_k with the missing value k, increase the large values by $8k+3$, the small values by $4k+2$. We have now the large values $10k+4,\ldots,12k+3$, the small values $4k+2,\ldots,6k+2$, (with the value $5k+2$ missing), and the values of the edges $4k+2,\ldots,8k+1$.

Subgraph 4. We take an α-valuation of A_k with the missing value $3k$. The large values and the small values will be increased by $6k+3$ to yield the values $8k+3,\ldots,10k+3$ (with $9k+3$ missing) and $6k+3,\ldots,8k+2$. The values of the edges will be $1,\ldots,4k$.

Subgraph 5 consists of one 4-gon; its vertices will be labeled by the 4 missing values from the first four subgraphs: $13k+4$, $5k+2$, $9k+3$, k (in cyclic order), the values of the edges will be $12k+4$, $8k+3$, $8k+2$, $4k+1$.

The reader will observe that one number has not been used between the values of the second and third subgraphs. This number $(4k+1)$ is the missing value of the new α-valuation of A_{4k+1}.

Example 1. A_2 has an α-valuation (0,8,1,6), (3,7,4,5) (values of the vertices are given in cyclic order) with the missing value 2. It also has the α-valuation (8,0,7,2), (5,1,4,3) with the missing value 6. From this, we can construct an α-valuation of A_9.

Subgraph 1:	(0,36,1,34),	(3,35,4,33)
Subgraph 2:	(32,5,31,7),	(29,6,28,8)
Subgraph 3:	(10,27,11,25),	(13,26,14,24)
Subgraph 4:	(23,15,22,17),	(20,16,19,18)
Subgraph 5:	(2,30,12,21).	
Missing value:	9	

Another possible extension of a known α-valuation is given in

Theorem 2. Let k be a positive integer. If the graph A_k has an α-valuation then A_{5k+1} also has an α-valuation.

The structure of the proof is the same as in Theorem 1. We decompose A_{5k+1} into six subgraphs. Each of the first five subgraphs will consist of k 4-gons, the sixth subgraph will be one 4-gon. For each of the subgraphs, we describe the construction of the labels of the vertices; for subgraphs No. 1,...,5 this valuation is based on an α-valuation of A_k.

The details are given in the following table:

Subgraph No.	Missing value	Increase in large values	Increase in small values	Transformed missing value
1	k	$16k+4$	0	k
2	$3k$	$14k+4$	$2k+1$	$17k+4$
3	k	$12k+3$	$4k+1$	$5k+1$
4	$3k$	$10k+3$	$6k+2$	$13k+3$
5	k	$8k+2$	$8k+2$	$9k+2$
6	the values of the vertices are $17k+4$, $9k+2$, $13k+3$, k			

The missing value of A_{5k+1} is $5k+1$.

Example 2. Proceeding from the two α-valuations of A_2 given in Example 1, we will construct an α-valuation of A_{11}:

Subgraph 1:	(0,44,1,42),	(3,43,4,41)
Subgraph 2:	(40,5,39,7),	(37,6,36,8)
Subgraph 3:	(9,35,10,33),	(12,34,13,32)
Subgraph 4:	(31,14,30,16),	(28,15,27,17)
Subgraph 5:	(18,26,19,24),	(21,25,22,23)
Subgraph 6:	(38,20,29,2).	

The missing value is 11.

Theorem 3. If A_k has an α-valuation then A_{9k+2} has an α-valuation.

Proof. We will decompose A_{9k+2} into ten subgraphs. A valuation of each of the first nine subgraphs will again be constructed from an α-valuation of A_k. The tenth subgraph will consist of two 4-gons; the values of their vertices will be obtained from the missing values of the first nine subgraphs. The details are given in the following table:

Subgraph No.	Missing value	Increase in large values	Increase in small values	Transformed missing value
1	k	$32k+8$	0	k
2	$3k$	$30k+8$	$2k+1$	$33k+8$
3	k	$28k+7$	$4k+1$	$5k+1$
4	$3k$	$26k+7$	$6k+2$	$29k+7$
5	k	$24k+6$	$8k+2$	$9k+2$
6	$3k$	$22k+6$	$10k+3$	$25k+6$
7	k	$20k+5$	$12k+3$	$13k+3$
8	$3k$	$18k+5$	$14k+4$	$21k+5$
9	k	$16k+4$	$16k+4$	$17k+4$
10	consists of two 4-gons: $(k, 33k+8, 5k+1, 25k+6)$ and $(13k+3, 29k+7, 17k+4, 21k+5)$.			

Missing value of A_{9k+2}: $9k+2$.

Example 3. Proceeding from the α-valuations of A_2 given in Example 1 we will construct an α-valuation of A_{20}.

Subgraph 1:	(0,80,1,78),	(3,79,4,77)
Subgraph 2:	(76,5,75,7),	(73,6,72,8)
Subgraph 3:	(9,71,10,69),	(12,70,13,68)
Subgraph 4:	(67,14,66,16),	(64,15,63,17)
Subgraph 5:	(18,62,19,60),	(21,61,22,59)
Subgraph 6:	(58,23,57,25),	(55,24,54,26)
Subgraph 7:	(27,53,28,51),	(30,52,31,50)
Subgraph 8:	(49,32,48,34),	(46,33,45,35)
Subgraph 9:	(36,44,37,42),	(39,43,40,41)
Subgraph 10:	(2,74,11,56),	(29,65,38,47)
Missing value:	20.	

Remark. The reader may observe that the valuation of the 10th subgraph in A_{9k+2} is also derived from an α-valuation of A_2. If we subtract k from the value of each vertex in subgraph 10 and then, in the resulting valuation, divide the value of each vertex by $4k+1$, we will obtain one of the α-valuations of A_2. A similar remark could be made about subgraph 6 in the construction of an α-valuation of A_{5k+1}: The valuation of subgraph 6 in A_{5k+1} is derived from an α-valuation of A_1.

At this moment, it would be tempting to try to prove the theorem stating that if A_k has an α-valuation, so does A_{13k+3}. This theorem might be true but it cannot be proved by the method used in the proofs of Theorems 2 and 3; the reason is that the last subgraph would consist of three 4-gons and their valuation would have to be constructed from an α-valuation of A_3 -- but A_3 does not have an α-valuation. However, our method can be used to prove that the existence of an α-valuation of A_k implies the existence of an α-valuation of A_{17k+4}, and, we are convinced that it can be extended to prove the following

Conjecture. If A_r, A_s have α-valuations then $A_{4rs+r+s}$ also has an α-valuation.

The above results, together with the fact that A_k has an α-valuation for $1 \le k \le 10$, $k \ne 3$, and with the results obtained in [4] (where it is shown that, for every $n \ge 1$, A_{n^2} and A_{n^2+n} have α-valuations) show that the set of all k for which A_k has an α-valuation is fairly dense.

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